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Étude mathématique d'équations aux dérivées partielles de la physique à valeurs dans une variété

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Alain SOYEUR

Sujet :

Etude mathématique d'équations aux dérivées partielles de la physique à valeurs dans une variété

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Abstract: This thesis contains two parts, which both concern the study of Partial Differential Equations whose unknown takes values into a manifold. The first part is motivated by the mathematical study of liquid crystals. *Harmonic maps* are a simplified model. First, we study the Dirichlet problem for harmonic maps between the disc and the sphere. We answer completely to the question of the multiplicity of solutions with respect to the boudary datum. Then, we consider the heat flow associated to harmonic maps between two riemannian manifolds and we prove the existence of a global smooth solution to the Cauchy problem for "small" initial data. Eventually, we construct weak solutions for this flow and for the flow associated to relaxed energies . A non-uniqueness result is also given.

The second part is devoted to the study of the Cauchy problem for the Ishimori equations that describe the evolution of a system of spins in ferromagnetism. We prove the existence of a global strong solution for "small" initial data.

Key words: Harmonic maps, Heat flow for harmonic maps, Ishimori equations.

A.M.S. Classification: 58 G 11, 58 E 20, 35 Q 20

Table des matières

Résumé

RESUME

0. Introduction.

Cette thèse est constituée de deux parties dont le point commun est l'étude d'équations aux dérivées partielles où l'inconnue est assujettie à prendre ses valeurs dans une variété. La première partie est motivée par l'étude mathématique des cristaux liquides et concerne les applications harmoniques qui sont un modèle simplifié. La seconde partie traite des équations d'Ishimori en ferromagnétisme classique.

1. Applications harmoniques.

Un cristal liquide est constitué de molécules allongées dont l'orientation moyenne au point x peut être représentée par un vecteur unitaire u(x).
Ainsi une configuration du cristal contenu dans un récipient Ω est modélisée par une application
$u:\Omega\to R^3$, $|u|=1$. On lui associe une énergie libre dite d'Oseen-Franck:

$$\widetilde{E}(u)=\frac{1}{2}\int_\Omega\left[K_1(\mathrm{div}u)^2+K_2(u.\mathrm{rot}u)^2+K_3(u\wedge\mathrm{rot}u)^2\right]dx$$

Une configuration physique du cristal correspond à un point critique de cette fonctionnelle d'énergie. Dans le cas où les trois constantes sont égales, et si on fixe la donnée au bord, minimiser cette énergie revient à minimiser l'intégrale de Dirichlet:

$$E(u)=\int_\Omega|\nabla u|^2dx\,.$$

Une application u, point critique de cette énergie est dite *harmonique* et vérifie au sens faible l'équation d'Euler suivante:

$$-\Delta u=u|\nabla u|^2.$$

1.1. Multiplicité des applications harmoniques du disque dans la sphère.

Considérons le problème de Dirichlet suivant:

$$(1.1) \qquad \begin{cases} -\Delta u = u|\nabla u|^2 \\ u_{\partial\Omega} = \gamma \end{cases}$$

où $\Omega = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 = 1\}$, $u : \mathbb{R}^2 \to \mathbb{R}^3$, $|u| = 1$ et $\gamma : \partial\Omega \to S^2$.

Nous étudions la multiplicité des solutions de (1.1.) en fonction de la donnée au bord γ. Concernant ce problème, H.Brézis, J.M.Coron [1] et J.Jost [2] ont montré indépendamment que lorsque γ n'est pas constante, il existe au moins deux applications harmoniques distinctes solutions du système (1.1). D'autre part, on sait que la seule application harmonique correspondant à une donnée au bord constante est l'application constante.

Nous donnons une description complète du nombre de solutions en fonction de l'application γ. Notons π la projection stéréographique du pôle nord de la sphère sur le plan équatorial; $\pi o \gamma$ est alors une application du bord du disque dans le plan complexe. Notre résultat est le suivant:

Théorème: Si $\pi o \gamma\ (e^{i\theta})$ n'est pas une fraction rationnelle, il y a une infinité de solutions au problème (1.1).

Si $\pi o \gamma\ (e^{i\theta})$ est une fraction rationnelle de la forme $P(e^{i\theta})/Q(e^{i\theta})$, où P et Q sont deux polynômes premiers entre eux, il y a exactement n applications harmoniques minimisantes dans leur classe d'homotopie, où n=Max(degP,degQ).

1.2. Un résultat d'existence globale pour le flot associé aux applications harmoniques.

La notion d'applications harmoniques s'étend à des applications allant d'une variété riemannienne M dans une autre variété N plongée isométriquement dons $\mathbb{R}^n$. On définit l'énergie d'une telle application: $E(u) = \frac{1}{2}\int_M g^{ij}(x)\frac{\partial u}{\partial x^i}.\frac{\partial u}{\partial x^j}dM$, et on s'intéresse dans ce chapitre au flot L^2 associé à cette énergie:

$$(1.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_M u = g^{ij} A(u) \left(\dfrac{\partial u}{\partial x^i}, \dfrac{\partial u}{\partial x^j}\right) \\ u(x,0) = u_0(x) \end{cases}$$

où A est la deuxième forme fondamentale de N. J.Eells et H.Sampson ont utilisé les premiers ce flot pour déformer continûment une application u_0 en une application harmonique u_∞ dans le cas où la courbure sectionnelle de la variété N est négative (c.f. [3]).

Lorsque la courbure sectionnelle n'est pas partout négative (par exemple dans le cas où N est une sphère euclidienne), le problème de l'existence d'une solution définie pour tout temps est plus délicat. M.Struwe et Y.Chen ([4],[5],[6]) ont récemment construit des solutions faibles et ont décrit les singularités éventuelles. J.M.Coron, J.M.Ghidaglia [7] ainsi que Y.Chen, W.Y.Ding ([8],[9]) ont montré qu'il existe des données initiales pour lesquelles la solution de l'équation (1.2) explose en temps fini. Notre résultat principal dans ce chapitre, qui se rapproche de celui de J.Jost [10] est le suivant:

__Théorème:__ il existe une constante ε telle que si la donnée initiale u_0 vérifie $u_0 : \mathbb{R}^m \to N$, $\nabla u_0 \in L_p$, et $|\nabla u_0|_{L^m} \leq \varepsilon$, alors il existe une solution régulière u(t) au problème (1.2) définie pour tout temps, qui converge vers une application constante lorsque $t \to +\infty$.

__1.3. Construction de solutions faibles pour le flot de la chaleur associé à l'énergie de Dirichlet ainsi qu'aux énergies relaxées.__

F.Bethuel, H.Brézis, et J.M.Coron ont introduit la notion d'énergie relaxée (c.f. [11]) qui leur a permis de construire une infinité de solutions au problème de Dirichlet de la boule unité de $\mathbb{R}^3$ dans la sphère S^2. Le but de ce chapitre est de construire une solution faible pour le flot associé aux énergies relaxées E_λ ($\lambda \in [0,1]$, $\lambda=0$ correspondant à l'intégrale de Dirichlet), c'est-à-dire une application u vérifiant:

$$(1.3)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = u|\nabla u|^2 \\ u(x,0) = u_0(x) \\ u(x,t) = \gamma(x) \quad x \in \partial B^3 \end{cases}$$

où $u \in H^1(B^3 \times (0,T);S^2)$ et $E_\lambda(u(t)) + \int_0^t \int \left|\frac{\partial u}{\partial t}\right|^2 \leq E_\lambda(u_0)$, $\forall t \geq 0$.

Nous utilisons pour cela une discrétisation du problème en temps qui revient à résoudre à chaque pas de temps $n\Delta t$ un problème de minimisation. Nous démontrons également qu'il existe une infinité de solutions de l'équation (1.3).

2. Le problème de Cauchy pour les équations d'Ishimori.

Les équations d'Ishimori décrivent l'évolution d'un système de spins dans un milieu ferromagnétique:

$$(2.1) \qquad \begin{cases} S_t = S \wedge (S_{xx} - \varepsilon S_{yy}) + b(\phi_x S_y + \phi_y S_x) \\ \phi_{xx} + \varepsilon \phi_{yy} = 2\varepsilon S.(S_x \wedge S_y) \end{cases}$$

où S est une application de $\mathbb{R}^2$ dans $\mathbb{R}^3$ vérifiant $|S|=1$, b une constante de couplage et $\varepsilon = \pm 1$. Ces équations sont intégrables par la méthode de diffusion inverse (cf.[12]). Nous nous intéressons uniquement à l'équation correspondant à $\varepsilon=+1$, et nous montrons que le problème de Cauchy est bien posé lorsque les données initiales sont suffisamment petites. Pour cela, nous utilisons la projection stéréographique de S comme nouvelle variable et nous étudions une équation de Schrödinger non- linéaire:

$$(2.2) \qquad \begin{cases} iu_t + u_{xx} - u_{yy} = F(u) + G(u) \\ \Delta\phi = 4i \dfrac{u_x \bar{u}_y - \bar{u}_x u_y}{(1+|u|^2)^2} \\ u(x,y,0) = u_0(x,y) \end{cases}$$

où $F(u) = \dfrac{2\bar{u}}{1+|u|^2}(u_x^2 - u_y^2)$, $G(u) = \phi_x u_y + \phi_y u_x$, et $u = \dfrac{S^1 + iS^2}{1+S^3}$.

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Chapitre I

Applications harmoniques

The Dirichlet problem for harmonic maps from the disc into the 2-sphere

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The Dirichlet problem for harmonic maps from the disc into the 2-sphere

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Synopsis

We consider the Dirichlet problem for harmonic maps from the disc D^2 into the sphere S^2, with prescribed boundary values γ: $\partial D^2 \to S^2$, and we prove that if γ is not a rational function, one can find infinitely many nonhomotopic harmonic maps which agree with γ on ∂D^2.

1. Introduction and notation

We consider maps in the class $\varepsilon = \{u \in H^1(D^2, \mathbb{R}^3); |u| = 1$ a.e. and $u_{|\partial D^2} = \gamma\}$, where $D^2 = \{z \in \mathbb{C}, |z| < 1\}$, $\partial D^2 = S^1 = \{z \in \mathbb{C}, |z| = 1\}$ and $\gamma \in C^1(S^1, S^2)$ is given with $S^2 = \{u \in \mathbb{R}^3 \mid |u| = 1\}$. ε is nonempty, and we define the energy of a map in ε by:

$$E(u) = \frac{1}{2} \int_{D^2} |\nabla u|^2 \, dx \, dy. \tag{1.1}$$

For convenience, a map in ε will also be considered as a map from D^2 to $\mathbb{C} \cup \{\infty\}$, the target space being endowed with the conformal metric $4(1 + |u|^2)^{-2} \, dz \, d\bar{z}$, corresponding to the stereographic projection on the equator plane from the north pole. The energy of u is then:

$$E(u) = \int_{D^2} \frac{4}{(1 + |u|^2)^2} (|u_z|^2 + |u_{\bar{z}}|^2) \, dz \, \overline{dz}. \tag{1.2}$$

The map u is harmonic if and only if it is a critical point for E. The Euler equation is:

$$-\Delta u = u \, |\nabla u|^2 \quad \text{or} \quad u_{z\bar{z}} = \frac{2\bar{u}}{1 + |u|^2} u_z u_{\bar{z}},$$

if u: $D^2 \to \cup \{\infty\}$ (see for example [2, 1.34]). Given two maps u and v in ε, we define the map $u * v$ from S^2 to S^2, gluing together u in the lower hemisphere and v in the upper one:

$$u * v(z) = \begin{cases} u(z) & \text{if } |z| \leqq 1, \\ v\left(\dfrac{1}{\bar{z}}\right) & \text{if } |z| > 1. \end{cases} \tag{1.3}$$

It is easy to see that

$$\frac{1}{2} \int_{\mathbb{C}} |\nabla w|^2 \, dx \, dy = E(u) + E(v).$$

The degree of the map $u * v$ is given by $Q(u) - Q(v)$, where

$$Q(u) = \frac{1}{4\pi} \int_{D^2} u(u_x \wedge u_y)\, dx\, dy \tag{1.4}$$

if $u: D^2 \to S^2$ (see [1]), or

$$Q(u) = \frac{1}{\pi} \int_{D^2} \frac{1}{(1+|u|^2)^2} (|u_{\bar{z}}|^2 - |u_z|^2)\, dz\, \overline{dz}, \tag{1.5}$$

if $u: D^2 \to \mathbb{C} \cup \{\infty\}$. Let $\underline{u}$ be a minimiser of the energy in ε and define $\varepsilon_k = \{u \in \varepsilon; Q(\underline{u}) - Q(u) = k\}$. As a consequence of the density theorem of [6], the ε_ks are the connected components of ε. Define also $\mathcal{I}_k = \text{Inf}\,\{E(u); u \in \varepsilon_k\}$. We say that u is a minimising harmonic map, if u is in ε_k and $E(u) = \mathcal{I}_k$. H. Brezis and J. M. Coron [1] and J. Jost [2] have shown that if γ is not constant, there exist at least two minimising harmonic maps $\underline{u}$ and $\bar{u}$ with $|Q(\underline{u}) - Q(\bar{u})| = 1$. The main difficulty in proving that $\mathcal{I}_k$ is achieved by u in ε_k is that ε_k is not closed under weak H^1 convergence. Using similar ingredients as in [1, 3, 4], and suggestions of C. Taubes and K. Uhlenbeck, we prove the following result.

THEOREM 1.1. *Assume that $\gamma(e^{i\theta})$: $\partial D^2 \to \mathbb{C} \cup \{\infty\}$ is not a rational function of $e^{i\theta}$. Then one can find infinitely many non-homotopic minimising harmonic maps in ε.*

2. Proof of Theorem 1.1

LEMMA 2.1. *If two maps u_1 and u_2 in $\varepsilon \cap C^{(1)}(\bar{D}^2; \mathbb{C} \cup \{\infty\})$ are such that $u_1(z)$ and $\overline{u_2(z)}$ are meromorphic on D^2, then $\gamma(e^{i\theta})$ is a rational function in $e^{i\theta}$.*

Proof. The map $u_1 * u_2$ is a meromorphic function from $(\mathbb{C} \cup \{\infty\}) \backslash S_1$ to $\mathbb{C} \cup \{\infty\}$ and continuous on S^1. By Morera's theorem, it is a meromorphic function from S^2 to S^2 and hence a rational function. Since γ agrees with $u_1 * u_2$ at $\{|z| = 1\}$, it is also a rational function. □

We then deduce the following alternative, if γ is not a rational function: either

$$\text{for all } u \in C^1(\bar{D}^2; \mathbb{C} \cup \{\infty\}) \text{ there exists } b \in D^2 \text{ such that } u_{\bar{z}}(b) \neq 0, \tag{2.1}$$

or

$$\text{for all } u \in C^1(\bar{D}^2; \mathbb{C} \cup \{\infty\}) \text{ there exists } b \in D^2 \text{ such that } u_z(b) \neq 0. \tag{2.2}$$

In what follows, we assume that (2.1) holds. Case (2.2) is similar.

LEMMA 2.2. *If $E(u_k) = \mathcal{I}_k$ for some u_k in ε_k, then*

$$\mathcal{I}_l < \mathcal{I}_k + 4\pi(l-k) \quad \textit{for } k < l. \tag{2.3}$$

Proof. The idea is to modify u on a small disc to make it cover S^2 one more time. We first show that $\mathcal{I}_{k+1} < \mathcal{I}_k + 4\pi$. By Morrey's regularity theorem [5], since u is a minimiser, it is C^∞ in D^2. Moreover, one can find b in D^2 such that $u_{\bar{z}}(b) \neq 0$. Without loss of generality, assume that $b = 0$ and $u(b) = 0$. Taylor's formula gives: $u(z) = az + b\bar{z} + g(z)$, with $g(z) = O(|z|^2)$. Let η be in $C^\infty(D^2, \mathbb{R})$,

$0 \leqq \eta \leqq 1$, $\eta \equiv 0$ on a neighbourhood of $\{|z| \leqq \varepsilon\}$, $\eta \equiv 1$ on a neighbourhood of $\{|z| \geqq 2\varepsilon\}$ and $|\nabla\eta| \leqq c\varepsilon^{-1}$. Define then

$$u_\varepsilon(z) = \begin{cases} az + \varepsilon^2 bz^{-1} & \text{if } |z| \leqq \varepsilon, \\ az + b\bar{z} + \eta(z)g(z) & \text{if } |z| > \varepsilon. \end{cases}$$

u_ε is continuous and $Q(u) - Q(u_\varepsilon) = 1$. So, u_ε is in ε_{k+1}. Now, compute the energy of u_ε:

$$\begin{aligned} E(u_\varepsilon) = E(u) - E_{|z|<\varepsilon}(u) + E_{|z|<\varepsilon}(az + b\varepsilon^2 z^{-1}) \\ + E_{\varepsilon \leqq |z| \leqq 2\varepsilon}(az + b\bar{z} + \eta(z)g(z)) - E_{\varepsilon \leqq |z| \leqq 2\varepsilon}(az + b\bar{z} + g(z)). \end{aligned}$$

Here, $E_{\dots}$ means that the integral must be taken on the specified domain.

Since the map $az + b\varepsilon^2 z^{-1}$ is conformal, its energy and the area of its range on S^2 are equal. Hence, $E_{|z| \leqq \varepsilon}(az + b\varepsilon^2 z^{-1}) \leqq 4\pi$. Since

$$|u_{\bar{z}}(0)| = |b| \neq 0, \quad 4(|u_z|^2 + |u_{\bar{z}}|^2)(1 + |u|^2)^{-2} \geqq \delta$$

for some positive δ and $|z|$ small enough, $E_{|z|<\varepsilon}(u) \geqq 4\pi\delta\varepsilon^2$. Using (1.2) we find that the difference of the last two terms is less than

$$\begin{aligned} c\int_{\varepsilon \leqq |z| \leqq 2\varepsilon} |\nabla((\eta - 1)g)|\, |\nabla(2az + 2b\bar{z} + (\eta - 1)g)|\, dz\, \overline{dz} \\ \leqq c\int_{\varepsilon \leqq |z| \leqq 2\varepsilon} (|\nabla\eta|\, |g| + |\eta|\, |\nabla g|)\, dz\, \overline{dz} \\ \leqq c\varepsilon^3. \end{aligned}$$

So we have: $E(u_\varepsilon) \leqq E(u) + 4\pi - 4\pi\delta\varepsilon^2 + c\varepsilon^3$. For ε small enough, $E(u_\varepsilon) < E(u) + 4\pi$. Since $u_{\bar{z}}(0) \neq 0$ in a neighbourhood of 0, we can repeat the same construction on l small discs to obtain the strict inequality (2.3).

Remarks 2.3. (i) Lemma 2.2 is a generalisation of results in [**1**] and [**3**]. The proof has been suggested to us by C. Taubes and K. Uhlenbeck.

(ii) If the alternative (2.2) holds instead of (2.1), consider $u_\varepsilon = \varepsilon^2 a\bar{z}^{-1} + b\bar{z}$ for $|z| < \varepsilon$ and show that

$$\mathcal{I}_l < \mathcal{I}_k + 4\pi(k - l) \quad \text{for } l < k. \tag{2.4}$$

(iii) The following inequality always holds (see [**1**, p. 212]):

$$\mathcal{I}_l \leqq \mathcal{I}_k + 4\pi\, |l - k| \quad \text{for } k, l \in \mathbb{Z}. \tag{2.5}$$

Proof of Theorem 1.1. $\mathcal{I}_0 = \operatorname{Inf}\{E(u); u \in \varepsilon\}$ is achieved by some map $\underline{u}$, and by Lemma 2.2 and (2.5), $\mathcal{I}_l < \mathcal{I}_p + 4\pi(l - p)$ for $p \leqq 0 < l$. Suppose that $\mathcal{I}_{k-1}$ is achieved and that

$$\mathcal{I}_l < \mathcal{I}_p + 4\pi(l - p) \quad \text{for } p \leqq k - 1 < l. \tag{2.6}$$

Take u^n a minimising sequence in ε_k. For a subsequence (still denoted by u^n), u^n tends weakly to some map u in $H^1(D^2; \mathbb{R}^3)$. Let $v^n = u^n - u$. We then have (see [**1**, p. 208]):

$$4\pi\, |Q(u^n) - Q(u)| \leqq E(v^n) + o(1). \tag{2.7}$$

Moreover $E(u^n) = E(u) + E(v^n) + o(1)$, and therefore:

$$4\pi\,|Q(u^n) - Q(u)| + E(u) \leqq \mathscr{I}_k + o(1). \tag{2.8}$$

We now have two possibilities:

First case: $u \in \varepsilon_l$ with $l < k$. But from (2.8) we should have $4\pi(k-l) + \mathscr{I}_l \leqq \mathscr{I}_k$, a contradiction with (2.6).

Second case: $u \in \varepsilon_l$ with $l \geqq k$. We have, using (2.8) and (2.5), $4\pi(l-k) + E(u) \leqq \mathscr{I}_k \leqq \mathscr{I}_l + 4\pi(l-k)$, and therefore, $E(u) \leqq \mathscr{I}_l$, which shows that u is a new minimising harmonic map in ε_l. Moreover, using (2.3) and (2.5), we verify that

$$\mathscr{I}_q < \mathscr{I}_l + 4\pi(q-l) \leqq \mathscr{I}_p + 4\pi(q-p) \quad \text{for } p \leqq l < q.$$

This construction can go on and we find infinitely many harmonic maps in ε.

Remarks 2.4. (i) We did not prove that each $\mathscr{I}_k$ is achieved, and this is an interesting open question. P. L. Lions has already shown [4] that if $\mathscr{I}_l < \mathscr{I}_k + 4\pi\,|k-l|$ for $0 \leqq l \leqq 2k$, then $\mathscr{I}_k$ is achieved. We can now say when minimising sequences escape from ε_k: if u is the limit of a minimising sequence in ε_k, and if u is in ε_l, $l < k \Rightarrow u$ is meromorphic on D^2, and $l > k \Rightarrow \overline{u(z)}$ is meromorphic on D^2 (otherwise we would have the strict inequality (2.3) or (2.4) which is impossible from (2.8)).

(ii) When (2.1) holds we have $\mathscr{I}_0 + 4\pi(l-k) \leqq E(u) + 4\pi(l-k) \leqq \mathscr{I}_0 + 4\pi k$. This shows that $l < 2k$. So, $\mathscr{I}_1$ is achieved, $\mathscr{I}_2$ or $\mathscr{I}_3$, $\mathscr{I}_4$ or $\mathscr{I}_5$ or $\mathscr{I}_6$ are achieved, etc. When (2.2) holds, $\mathscr{I}_{-1}$, $\mathscr{I}_{-2}$ or $\mathscr{I}_{-3}$, etc., are achieved.

3. The case where γ is a rational function

We first recall a lemma from [1].

LEMMA 3.1. *Let Φ in $L^\infty(\mathbb{C}; S^2)$, with $\nabla\Phi$ in $L^2(\mathbb{C}; \mathbb{R}^3)$. We always have the following inequality*:

$$\left|\int_{\mathbb{C}} \Phi(\Phi_x \wedge \Phi_y)\,dx\,dy\right| \leqq \frac{1}{2}\int_{\mathbb{C}} |\nabla\Phi|^2\,dx\,dy, \tag{3.1}$$

and if equality holds, the stereographic projection of Φ is a rational function in z or $\bar{z}$.

Proof. The inequality is obvious since $|\Phi| = 1$. Suppose that equality holds. From [1], we know that Φ is C^∞. If $\bar{\pi}$ and $\underline{\pi}$ denote the stereographic projections from the north pole and the south pole, using (1.2) and (1.5) we find:

$$(\bar{\pi}\Phi)_z = 0 \text{ almost everywhere and } (\underline{\pi}\Phi)_{\bar{z}} = 0 \text{ almost everywhere} \tag{3.2}$$

or

$$(\bar{\pi}\Phi) = 0 \text{ almost everywhere and } (\underline{\pi}\Phi)_z = 0 \text{ almost everywhere.} \tag{3.3}$$

Assume for example that (3.2) holds. The poles of $\bar{\pi}\Phi$ correspond to the zeros of $\underline{\pi}\Phi$ which are isolated. As a consequence, $\bar{\pi}\Phi$ is a holomorphic function from S^2 into S^2 and therefore a rational function.

3.1. A simple example

We consider an instructive example similar to the one treated in [1]. Take $\gamma(e^{i\theta}) = ae^{in\theta}$ with $a^2 > 2n - 1$. Let $\underline{u}(z) = a\bar{z}^{-n}$ and $\bar{u}(z) = az^n$.

THEOREM 3.2. *$\underline{u}$ is the only absolute minimiser in ε, $\underline{u}$ is in ε_0, $\bar{u}$ is in ε_n and for $0 \leqq k \leqq n$, $\mathcal{I}_k$ is achieved. Otherwise $\mathcal{I}_k$ is not achieved.*

Proof. Using (1.2) and (1.5) we obtain

$$E(\underline{u}) = 4\pi n(1 + a^2)^{-1},$$
$$E(\bar{u}) = 4\pi n(1 - (1 + a^2)^{-1}),$$
$$Q(\underline{u}) - Q(\bar{u}) = 4\pi n.$$

Claim: $\underline{u}$ is the unique minimiser in ε.
Suppose that one can find a map v in ε such that $|Q(v) - Q(\underline{u})| \geqq 1$ and $E(v) \leqq E(\underline{u})$. Then $4\pi \leqq 4\pi\,|Q(v) - Q(\underline{u})| \leqq E(v) + E(\underline{u})$ and so $E(v) \geqq 4\pi(1 - n(1 + a^2)^{-1}) > E(\underline{u})$ (because $a^2 > 2n - 1$). If now $Q(v) = Q(\underline{u})$, $w = \bar{u} * v$ satisfies $4\pi n \leqq 4\pi\,|Q(v) - Q(\bar{u})| \leqq E(v) + E(\bar{u}) \leqq 4\pi n$.

By Lemma 3.1, w is a rational function which agrees with $e^{in\theta}$ at S^1, so $v = \underline{u}$.

Claim: If $k \notin [0, n]$ $\mathcal{I}_k$ is not achieved: if v is a minimiser in ε_{-k}, $(k > 0)$, we have

$$4\pi(k + n) = 4\pi\,|Q(v) + Q(\bar{u})| \leqq E(v) + E(\bar{u}) \leqq E(\underline{u}) + E(\bar{u}) + 4\pi k \leqq 4\pi(n + k)$$

(we use (2.5)). From Lemma 3.1, $w = \bar{u} * v$ is a rational function and hence $v = \underline{u}$, a contradiction. When $k > n$, the proof is similar (consider $w = \underline{u} * v$).

Claim: If $k \in [0, n]$, $\mathcal{I}_k$ is achieved. Remark first that if u in ε_l is a minimiser, $l \in]0, n] \Rightarrow \overline{u(z)}$ is not meromorphic, and $l \in [0, n[\Rightarrow u(z)$ is not meromorphic. Indeed, $u * \underline{u}$ (or $\bar{u} * u$) would be a rational function and $l = 0$ ($l = n$). Now take a minimising sequence in ε_k and if u is the weak limit, u cannot escape from ε_k (see Remark 2.4(i)). The proof is complete. □

3.2. The general case

$\gamma(e^{i\theta}) = p(e^{i\theta})/Q(e^{i\theta})$, where P and Q contain no common irreducible factor. Define

$$\underline{u}(z) = P\left(\frac{1}{\bar{z}}\right) \Big/ Q\left(\frac{1}{\bar{z}}\right) \quad \text{and} \quad \bar{u}(z) = \frac{P(z)}{Q(z)}.$$

A direct computation shows that $\bar{u}$ and $\underline{u}$ have finite energy, and belong to ε. Moreover, since $(P/Q)_{\bar{z}} = 0$, $E(\bar{u}) + E(\underline{u}) = 4\pi\,|Q(\bar{u}) + Q(\underline{u})|$.

THEOREM 3.3. *Let $n = |Q(\bar{u}) - Q(\underline{u})|$, and $\underline{u}$ be in ε_0. Then, if k is in $[0, n]$, $\mathcal{I}_k$ is achieved. Otherwise, $\mathcal{I}_k$ is not achieved.*

Proof. If v is a minimiser in ε_{-k} $(k > 0)$, $w = \bar{u} * v$ is analytic, since

$$4\pi(n + k) = 4\pi\,|Q(\bar{u}) - Q(v)| \leqq E(\bar{u}) + E(v) \leqq E(\bar{u}) + E(\underline{u}) + 4\pi k = 4\pi(n + k).$$

So, $v = \underline{u}$: a contradiction. If v is in ε_{n+k}, the same argument applies.

Now let u be the weak limit of a minimising sequence in ε_k ($k \in [0, n]$), such that u is in ε_l. By the same proof as in Theorem 3.2 we have, if $k \neq l$: $\mathcal{I}_k < \mathcal{I}_l + 4\pi |k-l|$. Then $4\pi|k-l| + E(u) \leqq \mathcal{I}_k < \mathcal{I}_l + 4\pi|k-l|$. But since $E(u) = \mathcal{I}_l$ we deduce that $l = k$ which concludes the proof. □

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A global existence result for the heat flow of harmonic maps

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A GLOBAL EXISTENCE RESULT FOR THE HEAT FLOW OF HARMONIC MAPS

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Abstract. We consider the heat flow for harmonic maps from $\mathcal{R}^m$ to a compact manifold $\mathcal{N}$. When the L^m norm of the gradient of the initial data is small, we prove the existence of a global solution. We prove a similar result for the boundary value problem, when the boundary of the manifold $\mathcal{M}$ maps into a point.

1. Introduction.

Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be two Riemannian manifolds of respective dimension m and n. Without restriction, we may assume that $\mathcal{N}$ is isometrically imbedded into $\mathcal{R}^N$ for some N. The energy of a map $u : \mathcal{M} \to \mathcal{N}$ is given by :

$$E(u) = \frac{1}{2} \int_{\mathcal{M}} g^{ij}(x) \frac{\partial u}{\partial x^i} \cdot \frac{\partial u}{\partial x^j} d\mathcal{M}$$

where $d\mathcal{M}$ is the volume element of $(\mathcal{M}, g)$. The critical points of E are called harmonic maps.

An important question is to know whether or not each homotopy class from $\mathcal{M}$ to $\mathcal{N}$ contains a harmonic map. The first general answer was given by Eells and Sampson [5]. They proved that if the sectional curvature of $\mathcal{N}$ is nonpositive, any smooth map u_0 can be deformed continuously to a harmonic map. They considered the heat flow associated to the energy E :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{\mathcal{M}} u = A(u)(du, du) \\ u(x,0) \qquad = u_0(x) \end{cases} \tag{1}$$

where $A(u)(du, du)$ is the orthogonal part of the Beltrami operator of u in $\mathcal{R}^n$, which is quadratic in the derivatives of u :

$$A(u)(du, du) = g^{ij} \mathcal{A}(u) \left(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right)$$

and $\mathcal{A}$ is the second fundamental form of $\mathcal{N}$. When $\mathcal{N}$ is any compact manifold, we cannot hope for such a general result. Indeed, when $\mathcal{M} = \mathcal{R}^n$ and $\mathcal{N} = S^m$, Coron and Ghidaglia [3] exhibit a class of smooth initial data for which the solution of (1) blows up in finite time. Therefore, it is necessary either to impose restrictions on the initial data or to weaken the concept of a solution of (1).

For any compact manifolds $\mathcal{M}$ and $\mathcal{N}$, Struwe and Chen [11] have recently proved the existence of a global distribution solution of (1) with $\frac{\partial u}{\partial t} \in L^2(M \times \mathcal{R}^+)$ and $E(u(t)) \leq E(u_0)$. Their solution could be singular only on a set of locally finite $m-$dimensional Haussdorf (parabolic) measure. However, because of the singularities that may occur, their solution $u(t)$ may not be homotopic to u_0, and moreover as follows from Coron [1], it may not be unique.

In [10], using a monotonicity formula, Struwe proves that for initial data $u_0 \in H^1_{loc}(\mathcal{R}^m, \mathcal{N})$ with $\| \nabla u_0 \|_\infty \leq C$, one can find $\epsilon(C) > 0$ such that $E(u_0) < \epsilon$ leads to a global smooth solution. Jost [6] gives a geometrical condition on u_0 : the image of u_0 lies in a geodesic ball on $\mathcal{N}$ of radius depending on a bound of a sectional curvature of $\mathcal{N}$. By way of contrast, Chen and Ding [2] prove a blow up of the solution in finite time, provided u_0 belongs to some nontrivial homotopy class and has sufficiently small energy.

The goal of this paper is to prove the existence of a global smooth solution for initial data whose gradient have small L^m norm. It is worthwile to note that this condition is natural in the sense that it is invariant under scaling. Our method is different from the previous ones and relies on $L^p - L^q$ estimates for the heat operator. Two cases are considered : either $\mathcal{M} = \mathcal{R}^m$ or $\mathcal{M}$ is a compact manifold with boundary and solutions map $\partial\mathcal{M}$ into a single point in $\mathcal{N}$.

2. Notations and statement of the results.

We assume throughout this paper that the Christoffel symbols of $\mathcal{N}$ and the g^{ij} are bounded, so that

$$| A(u)(du, du) | \leq C | \nabla u |^2 \tag{2}$$

First, the case where $\mathcal{M} = \mathcal{R}^m$.

Let $S(t)$ denotes the semi-group associated to the linear heat equation from $\mathcal{R}^m$ to $\mathcal{R}^N$:

$$(S(t)u_0)^\alpha = \int_{\mathcal{R}^m} \frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{m/2}} (u_0(x-y))^\alpha dy \quad 1 \leq \alpha \leq N \tag{3}$$

The operator $S(t)$ maps $[L^q(\mathcal{R}^m)]^N$ into $[L^p(\mathcal{R}^m)]^N$ for $1 \leq q \leq p \leq +\infty$, and according to the Young inequality, we have the estimates

$$\| S(t)u \|_p \leq \frac{\| u \|_q}{(4\pi t)^{m/2(1/q-1/p)}}, \tag{4}$$

$$\| \nabla S(t)u \|_p \frac{C_1 \| u \|_q}{(4\pi t)^{1/2+m/2(1/q-1/p)}}. \tag{5}$$

Given $u_0 \in L^\infty(\mathcal{R}^m, \mathcal{R}^N)$ with $\nabla u_0 \in L^p(\mathcal{R}^m, \mathcal{R}^{Nm})$, $p > m$, let

$$\mathcal{F}_p = \{v : \mathcal{R}^m \to \mathcal{R}^N, \| v \|_\infty < +\infty, \| \nabla v \|_p < +\infty\},$$

$$M = 2(\| u_0 \|_\infty + \| \nabla u_0 \|_p),$$

and

$$\mathcal{E}_p^T = \{u \in C([0,T]; \mathcal{F}_p) \ ; \ \sup_{0 \le t \le T} \{\| u(t) - S(t)u_0 \|_\infty + \| \nabla u(t) - \nabla S(t)u_0 \|_p\} \le M\}$$

which is a complete metric space for the distance

$$d(u,v) = \sup_{0 \le t \le T} \{\| u(t) - v(t) \|_\infty + \| \nabla u(t) - \nabla v(t) \|_p\}.$$

First a local estimate :

Theorem 1. *Let $u_0 \in \mathcal{F}_p$, $p > m$. There exists $T(u_0) > 0$ and $u \in \mathcal{E}_p^{T(u_0)}$ smooth on $]0, T(u_0)[\times \mathcal{R}^m$ which solves (1). If $T(u_0)$ is finite, then $\| u(t) \|_\infty + \| \nabla u(t) \|_p \to +\infty$ as $t \to T(u_0)$.*

Concerning global solutions, we have the following

Theorem 2. *There exists a constant ϵ depending only on m and $\mathcal{N}$ such that if $u_0 \in \mathcal{F}_p$, $p > m$* with *$\| \nabla u_0 \|_m < \epsilon$, then $T(u_0) = +\infty$. Moreover, the solution converges to a constant map as $t \to +\infty$ with the following decay :*

$$\| \nabla u(t) \|_p \le \frac{M_p^\infty}{t^{1/2 - m/2p}},$$

where M_p^∞ depends only on m, $\mathcal{N}$ and p.

Let us turn now to the boundary value problem. Up to a translation, we may assume that u_0 maps the boundary of $\mathcal{M}$ into $O \in \mathcal{R}^N$. The heat flow we consider is then :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_m u = A(u)(du, du) \\ u/\partial \mathcal{M} = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{7}$$

Our result reads :

Theorem 3. *There exists* $\epsilon > 0$ *depending only on* $\mathcal{M}$ *and* $\mathcal{N}$ *such that if* $\| \nabla u_0 \|_m \leq \epsilon$, (7) *possesses a global smooth solution which converges to O as* $t \to +\infty$, *and the following decay holds :*

$$\| \nabla u(t) \|_p \leq \frac{e^{-\lambda t}}{\chi(t)^{\frac{1}{2} - \frac{m}{2p}}} M_p^\infty .$$

where $\lambda > 0$ *and* M_p^∞ *depends only on* $\mathcal{M}$, $\mathcal{N}$ *and* p, *and* $\chi(t) = Min\ (1, t)$.

3. Proof of Theorem 1.

We consider the integral equation associated to (1) : Let

$$\mathcal{F}u(t) = S(t)u_0 + \int_0^t S(t-\tau)[A(u)(du, du)(\tau)]d\tau.$$

We first check that $\mathcal{F}$ maps ξ_p^T into itself for T small. Using (2) and (4), we infer that

$$\begin{aligned} \| \mathcal{F}u(t) - S(t)u_0 \|_\infty &\leq \int_0^t \| S(t-\tau)\ A(u)(du, du) \|_\infty \, d\tau \\ &\leq \int_0^t \frac{C}{(t-\tau)^{m/p}} \| \nabla u(\tau) \|_p^2 \| u(\tau) \|_\infty \, d\tau \\ &\leq CM^3 \int_0^T \frac{d\tau}{\tau^{m/p}}. \end{aligned}$$

Similarly, using (5),

$$\| \nabla \mathcal{F}u(t) - \nabla S(t)u_0 \|_p \leq CM^3 \int_0^T \frac{d\tau}{\tau^{1/2+m/2p}}.$$

Secondly, for T small, $\mathcal{F}$ is a strict contraction in ξ_p^T : for u and v in ξ_p^T, using (4) and (5),

$$\begin{aligned} \| \mathcal{F}u(t) - \mathcal{F}v(t) \|_\infty &\leq \int_0^t \frac{C}{(t-\tau)^{m/p}} (\| \nabla u(\tau) \|_p \| u - v \|_\infty + \\ &\quad \| \nabla u(\tau) - \nabla v(\tau) \|_p\ (\| \nabla u(\tau) \|_p + \| \nabla v(\tau) \|_p)) d\tau \\ &\leq 2CM \int_0^T \frac{d\tau}{\tau^{m/p}} d(u, v). \end{aligned}$$

Consequently, by the contraction mapping principle, $\mathcal{F}$ has a unique fixed point in ξ_p^T. As $A(u)(du,du)(\tau) \in C([0,T[;L^{p/2})$, we may differentiate u and check that it satisfies (1) in the sense of distributions. The blow-up alternative follows by classical means.

4. Proof of Theorem 2.

To prove that the local solution given by Theorem 1 is a global one, it suffices to obtain an a priori bound on $\| \nabla u(t) \|_p + \| u(t) \|_\infty$. Using (2), (4) and (5) :

$$\| \nabla u(t) \|_p \leq \frac{\| \nabla u_0 \|_m}{(4\pi t)^{(m/2)(1/m-1/p)}} + C\int_0^t \frac{\| \nabla u(\tau) \|_p^2}{(t-\tau)^{1/2+(m/2p)}} d\tau.$$

We set

$$M_p(T) = \sup_{0 \leq t \leq T} \{(4\pi t)^{(m/2)(1/m-1/p)} \| \nabla u(t) \|_p\}.$$

We then have :

$$\begin{aligned} &(4\pi t)^{(m/2)(1/m-1/p)} \| \nabla u(t) \|_p \\ &\leq \| \nabla u_0 \|_m + CM_p^2(T) \int_0^t \frac{(4\pi t)^{(m/2)(1/m-1/p)}}{(t-\tau)^{1/2+(m/2p)}(4\pi\tau)^{m(1/m-1/p)}} d\tau \\ &\leq \| \nabla u_0 \|_m + CM_p^2(T) \int_0^1 \frac{dy}{y^{1-(m/p)}(1-y)^{1/2+(m/2p)}} \end{aligned}$$

Let

$$\bar{C} = C\int_0^1 \frac{dy}{y^{1-(m/p)}(1-y)^{1/2+(m/2p)}}.$$

The following inequality holds :

$$\bar{C}M_p^2(T) - M_p(T) + \| \nabla u_0 \|_m \geq 0 \tag{8}$$

for all $T \in [0, T(u_0)[$.

The equation (8) possesses two positive roots for $\| \nabla u_0 \|_m < \frac{1}{4\bar{C}} = \epsilon$. If we denote by M_p^∞ the smallest one, since $T \to M_p(T)$ is a continuous map, and $M_p(0) = 0$, $M_p(T) \leq M_p^\infty$ for all t, and therefore

$$\| \nabla u(t) \|_p \leq \frac{M_p^\infty}{(4\pi t)^{1/2-(m/2p)}}.$$

Then,

$$\| u(t) \|_\infty \leq \| u_0 \|_\infty + CMM_p^\infty \int_0^t \frac{d\tau}{(t-\tau)^{(m/p)}\tau^{1/2-(m/2p)}}.$$

Therefore $u(t)$ cannot blow up in finite time in $\mathcal{F}_p$, and we get a global solution of (1).

Remark : The method used previously to derive the a priori bound is inspired from Weissler [13] in the context of semiliner heat equations.

5. Proof of Theorem 3.

If we still denote by $S(t)$ the semi-group generated by $\Delta_{\mathcal{M}}$ in $[H_0^1(\mathcal{M})]^N$, it extends to a linear operator from $[L^q(\mathcal{M})]^N$ into $[L^p(\mathcal{M})]^N$ and, for $1 \leq q \leq p \leq +\infty$, we have the following estimates (see [8])

$$\| S(t)u \|_p \leq \frac{C_1 e^{-\lambda t} \| u \|_q}{\chi(t)^{(m/2)(1/q-1/p)}} \tag{9}$$

$$\| \nabla S(t)u \|_p \leq \frac{C_2 e^{-\lambda t} \| u \|_q}{\chi(t)^{1/2+[(m/2)(1/q-1/p)]}} \tag{10}$$

where $\lambda > 0$ and $\chi(t) = Min(1,t)$. The proof is then similar to the proof of Theorem 2.

Remarks : (i) Our condition on the L^m norm of the gradient is in a sense optimal : the conditions $\| \nabla u_0 \|_\infty \leq C$ and $\| \nabla u \|_2 \leq \epsilon$ of Struwe [10] imply that $\| \nabla u \|_m$ is small. Moreover, the quantity $\| \nabla u_0 \|_m$ is invariant under scaling $x \to \lambda x$ and is therefore a geometric quantity representing u_0.

(ii) Our method does not reconduct when $\mathcal{M}$ is a compact manifold without boundary since the estimates (4) and (5) are not valid anymore. One may ask whether Theorem 2 still holds in this case.

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Heat flows and relaxed energies for harmonic maps

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Heat flows and relaxed energies for harmonic maps.

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Abstract.

In this paper we construct weak solutions for the heat flow associated with relaxed energies for harmonic maps between B^3 and S^2. Nonuniqueness results for such solutions are also given.

O. Introduction

Recently very much attention has been drawn to the study of the heat flow for harmonic maps between Riemannian manifolds (see e.g. M. Struwe [19] and the references therein). The starting point was a result of Eells and Sampson [10] which was obtained by considering strong solutions for the heat flow of harmonic maps. Without restrictive hypotheses on the target manifold, these solutions may not exist (J.M. Coron and J.M. Ghidaglia [8], W.Y. Ding [9] and Y. Chen and D.W. Ding [5]). This has led to consider weak solutions as done first by M. Struwe [17], [18] and by Y. Chen and M. Struwe in [6]. However these solutions may not be unique as shown by J.M. Coron in [7].

Recall that the energy of a map u from $B^3 = \{x \in \mathbb{R}^3; | x |< 1\}$ into $S^2 = \partial B^3$ is

$$E_0(u) = \int | \nabla u |^2 = \int_{B^3} u^i_j u^i_j dx. \tag{0.1}$$

<u>Harmonic maps</u> are critical points of $E_0(u)$; they satisfy the Euler equation

$$-\Delta u = u \mid \nabla u \mid^2 \text{ in } B^3. \tag{0.2}$$

Conversely a map $u \in H^1(B^3; S^2)$ satisfying (0.2) in the distributional sense is termed weakly harmonic. Among the weakly harmonic maps in $H^1(B^3; S^2)$ there are the so-called E_0-stationary maps which correspond to variations on the domain. More precisely, u is E_0−stationary if u is weakly harmonic and for every smooth and compactly supported vector field $X : B^3 \to \mathbb{R}^3$,

$$\frac{d}{d\epsilon} E_0(u(\cdot + \epsilon X(\cdot)))\Big|_{\epsilon=0} = 0. \tag{0.3}$$

One sees easily that (0.3) is equivalent to

$$2(u_i u_j)_i = (u_i u_i)_j \text{ for any } j \text{ in } \{1,2,3\}. \tag{0.4}$$

In [1], F. Bethuel, H. Brezis and J.M. Coron have introduced modified energies :

$$E_\lambda(u) = E_0(u) + 8\pi\lambda L(u) \tag{0.5}$$

where $\lambda \in [0,1]$, and

$$L(u) = \operatorname*{Sup}_{\xi \in C_0^\infty(B^3, \mathbb{R}); |\nabla\xi|_{L^\infty} \leq 1} \int D(u) \cdot \nabla\xi \tag{0.6}$$

with $D(u) = (u \cdot (u_2 \times u_3), u \cdot (u_3 \times u_1), u \cdot (u_1 \times u_2))$. The scalar $L(u)$ has been defined by H. Brezis, J.M. Coron and E.H. Lieb in [3] where a more operational formula for computing this quantity is also given. As a consequence of this last formula $L(u)$ depends only on the singularities of the map u. It follows in particular that a critical point of E_λ satisfies (0.2) (see [1]). Hence the heat flow associated with E_λ is independent of λ and reads

$$\frac{\partial u}{\partial t} - \Delta u = u \mid \nabla u \mid^2 . \tag{0.7}$$

Equation (0.7) is complemented with boundary and initial conditions, namely

$$u(x,t) = \gamma(x),\ t \geq 0,\ x \in \partial B^3, \tag{0.8}$$

$$u(x,0) = u^0(x),\ x \in B^3, \tag{0.9}$$

where u^0 and $\gamma \in H^1(B^3; S^2)$ and $\gamma = u^0$ on ∂B^3.

Definition 0.1. Let u^0 and γ be given as above. A weak solution for the heat flow of E_λ is a map

$$u \in H^1(B^3 \times (0,T); S^2),\ \forall T < \infty \tag{0.10}$$

satisfying (0.7) is the weak sense and (0.8)-(0.9) in the trace sense together with the inequalities

$$E_\lambda(u(t)) + \int_0^t \int \left|\frac{\partial u}{\partial t}\right|^2 \leq E_\lambda(u^0),\ \forall t \geq 0. \tag{0.11}$$

In this paper we prove the following results.

Theorem 1. *Let u^0, $\gamma \in H^1(B^3; S^2)$ with $u^0 = \gamma$ on ∂B^3 and let $\lambda \in [0,1]$. There exists a weak solution to* (0.7)-(0.9).

Theorem 2 ($\lambda = 0$). *Let u^0 and γ^0 as in Theorem 1. Assume that u^0 is weakly harmonic but not E_0-stationnary then* (0.7)-(0.9) *has infinitely many weak solutions.*

Finally Theorem 3 is the analogue of Theorem 2 in the cases $\lambda \neq 0$, see Section 3.

Remarks 0.1. a. Theorem 1 for $\lambda = 0$ was proved by Y. Chen in [4], by J. Keller, J. Rubinstein and P. Sternberg in [14] and by Y. Chen and M. Struwe in [6]. *b.* Theorem 2 was proved in [7] but the proof given here is more elementary.

1. Proof of Theorem 1. In order to construct a weak solution to (0.7) satisfying (0.8), (0.9) and (0.11) we proceed as K. Horihata and N. Kikuchi in [13]. For $h \in (0,1)$, we define the sequence $(u^n)_{n\in\mathbf{N}}$ as follows. We assume that u^{n-1}, $n \geq 1$, is known and denote by $\mathcal{E}_n$ the functional :

$$\mathcal{E}_n(v) = E_\lambda(v) + \int \frac{|\, v - u^{n-1}\,|^2}{h}.$$

Then

$$\begin{aligned} &u^n \text{ minimizes } \mathcal{E}_n(v) \text{ under the constraint } v \in H^1_\gamma = \\ &= \{w : B^3 \to S^2;\ w \in H^1(B^3, S^2),\ w = \gamma \text{ on } \partial B^3\}. \end{aligned} \tag{1.1}$$

Problem (1.1) possesses at least one solution (see [1]) which satisfies

$$\frac{u^n - u^{n-1}}{h} - \Delta u^n = \lambda^n u^n \tag{1.2}$$

where

$$\lambda^n = |\, \nabla u^n\,|^2 + \frac{1 - u^{n-1}.u^n}{h} = |\, \nabla u^n\,|^2 + \frac{|\, u^n - u^{n-1}\,|^2}{2h}.$$

Let us now construct two mappings $\bar{u}^h : B^3 \times [0,\infty) \to S^2$ and $u^h : B^3 \times [0,\infty) \to B^3$ by setting for $(n-1)h < t \leq nh$:

$$\begin{aligned} \bar{u}^h(x,t) &= u^n(x), \\ u^h(x,t) &= \frac{t-(n-1)h}{h} u^n(x) + \frac{nh-t}{h} u^{n-1}(x). \end{aligned}$$

With these notations, (1.2) reads

$$\frac{\partial u^h}{\partial t} - \Delta \bar{u}^h = (|\, \nabla \bar{u}^h\,|^2 + \frac{h}{2}\left|\frac{\partial u^h}{\partial t}\right|^2)\bar{u}^h. \tag{1.3}$$

Now since u^h solves (1.1), $\mathcal{E}_n(u^n) \leq \mathcal{E}_n(u^{n-1})$:

$$E_\lambda(u^n) + \int \frac{|u^n - u^{n-1}|^2}{h} \leq E_\lambda(u^{n-1}); \tag{1.4}$$

adding these inequalities we obtain (as in [13])

$$E_\lambda(u^N) + \sum_{n=1}^{N} \int \frac{|u^n - u^{n-1}|^2}{h} \leq E_\lambda(u^0). \tag{1.5}$$

In terms of $\bar{u}^h$ and u^h, (1.5) reads

$$E_\lambda(\bar{u}^h(Nh)) + \int_0^{Nh} \int \left|\frac{\partial u^h}{\partial t}\right|^2 \leq E_\lambda(u^0) \tag{1.6}$$

which implies that

$$\begin{cases} \displaystyle\int_0^{+\infty} \int \left|\frac{\partial u^h}{\partial t}\right|^2 \leq E_\lambda(u^0), \\ E_\lambda(\bar{u}^h(t)) \leq E_\lambda(u^0), \ \forall t \geq 0. \end{cases} \tag{1.7}$$

Since $L(v) \geq 0$ for every v, it follows from (1.7) that the family $(u^h)_{h>0}$ (resp. $(\bar{u}_i^h)_{h>0}$) is bounded in $H^1(B^3 \times (0,T))^3$ (resp. $L^2(B^3 \times (0,T))^3$ $\forall i \in \{1,2,3\}$), $\forall T < \infty$. Therefore, up to the extraction of subfamilies, we can assume that for every $T < \infty$,

$$\begin{cases} u^h \rightharpoonup u \text{ in } H^1(B^3 \times (0,T))^3, \\ u^h \longrightarrow u \text{ in } L^2(B^3 \times (0,T))^3 \text{ and } a.e., \end{cases} \tag{1.8}$$

and

$$\bar{u}_i^h \rightharpoonup \varphi_i \text{ in } L^2(B^3 \times (0,T))^3, \ \forall i \in \{1,2,3\}, \tag{1.9}$$

where $\rightharpoonup$ stands for weak convergence, while $\longrightarrow$ stands for strong convergence.

Now our goal is to pass to the limit when $h \longrightarrow 0$ in (1.3). Proceeding as in [4], [14] and [15], we take the cross product of (1.3) with $\bar{u}^h$; this leads to

$$\frac{\partial u^h}{\partial t} \times \bar{u}^h - (\bar{u}_i^h \times \bar{u}^h)_i = 0. \tag{1.10}$$

On the other hand, according to the definition of u^h and $\bar{u}^h$, we have $| u^h - \bar{u}^h | \leq h \left|\frac{\partial u^h}{\partial t}\right|$, so that (see (1.7)):

$$\int_0^T \int | u^h - \bar{u}^h |^2 \leq h^2 E_\lambda(u^0). \tag{1.11}$$

Hence using (1.8) we find that for every $T < \infty$,

$$\bar{u}^h \longrightarrow u \text{ in } L^2(B^3 \times]0,T[)^3, \tag{1.12}$$

and therefore

$$\varphi_i = u_i. \tag{1.13}$$

It follows then from (1.8), (1.9), (1.10) and (1.13) that

$$\frac{\partial u}{\partial t} \times u - (u_i \times u)_i = 0, \tag{1.14}$$

and

$$| u |= 1 \ a.e. \tag{1.15}$$

Now from (1.14) and (1.15) (see [4], [14], [15]) we deduce that

$$\frac{\partial u}{\partial t} - \Delta u =| \nabla u |^2 u,$$

and finally using (1.6)

$$E_\lambda(u(t)) + \int_0^t \int \left|\frac{\partial u}{\partial t}\right|^2 \leq E_\lambda(u^0).$$

This completes the proof of Theorem 1.

2. Proof of Theorem 2.

We assume here that $\lambda = 0$. Let $u^0 : B^3 \to S^2$ be weakly harmonic ($-\Delta u^0 = | \nabla u^0 |^2 u^0$) which is not E_0-stationnary. That means

$$\exists j_0 \in \{1,2,3\}, \ 2(u_i^0 u_{j_0}^0)_i \neq (u_i u_i)_{j_0}. \tag{2.1}$$

In order to prove Theorem 2, it is sufficient to show that the weak solution to (0.7)-(0.9) constructed in the previous section is not constant with respect to time (see [7]).

Let X be a smooth compactly supported vector field on B^3 and let u_ϵ^n be defined by $u_\epsilon^n(x) = u^n(x + \epsilon X(x))$. Since u^n minimizes $\mathcal{E}_n$ we have $\frac{d}{d\epsilon}\mathcal{E}_n(u_\epsilon^n)_{|\epsilon=0} = 0$. Hence for every $i \in \{1,2,3\}$ we have

$$2u_i^n \frac{u^n - u^{n-1}}{h} + (u_j^n u_j^n)_i - 2(u_j^n u_i^n)_j = 0.$$

These relations read also

$$2u_i^h \frac{\partial u_i^h}{\partial t} + (\bar{u}_i^h \bar{u}_j^h)_j - 2(\bar{u}_j^h \bar{u}_i^h)_j = 0. \tag{2.2}$$

Let us assume by contradiction

$$u(x,t) = u^0(x), \ \forall x \in B^3, \ \forall t \geq 0. \tag{2.3}$$

We claim that

$$\begin{cases} \nabla \bar{u}^h \longrightarrow \nabla u \text{ in } L^2(B^3 \times (0,T))^9, \\ \dfrac{\partial u^h}{\partial t} \longrightarrow 0 \left(= \dfrac{\partial u}{\partial t}\right) \text{ in } L^2(B^3 \times (0,T))^3. \end{cases} \tag{2.4}$$

We postpone the proof of (2.4) and we complete the proof of Theorem 2. Passing to the limit in (2.2) we obtain

$$(u_j^0 u_j^0)_i = 2(u_j^0 u_i^0)_j,$$

which contradicts (2.1) and proves that (2.3) cannot hold true.

It remains to show (2.4). According to (1.6) we have, for $t = nh$,

$$E_0(\bar{u}^h(t)) + \int_0^t \int \left| \frac{\partial u^h}{\partial t} \right|^2 \leq E_0(u^0).$$

Since the left-hand side of this inequality is lower semicontinuous, we deduce, for every t,

$$E_0(u(t)) + \int_0^t \int \left| \frac{\partial u}{\partial t} \right|^2 \leq E_0(u^0). \tag{2.5}$$

But according to (2.3), (2.5) is an equality and this proves (2.4).

Remark 1.1. Let $\omega : S^2 \simeq \mathbf{C} \cup \{\infty\} \rightarrow S^2 \simeq \mathbf{C} \cup \{\infty\}$ be a rational fraction of $z \in \mathbf{C}$. The function $u^0 : u^0(x) = \omega(x / \mid x \mid)$ is weakly harmonic and (2.1) is equivalent to

$$\mathbb{R}^3 \ni \int_{S^2} \mid \nabla \omega \mid^2 \vec{s} d\sigma(\vec{s}) \neq \vec{O}.$$

3. The case $\lambda \in]0,1[$.

In order to define E_λ-stationary mappings, we define, for u in $H^1(B^3; S^2)$, the set $C(u)$ (see [11], [12]),

$$\begin{aligned} C(u) = \{&\text{one dimensional rectifiable current } c = \tau(\mathcal{C}, \theta, \varsigma) \\ &\text{such that the mass } M(c) \text{ of } c \text{ is equal to} \\ &L(u) \text{ and } -4\pi\partial c = div\ D(u) \text{ in } \mathcal{D}'(B^3)\}. \end{aligned}$$

Theorem 1 in [12] asserts that $C(u) \neq \emptyset$. We now may define E_λ-stationary mappings.

Definition 3.1. We say that u is E_λ-stationary if there exists $c \in C(u)$ such that for every $X \in \mathcal{C}_0^\infty(B^3, \mathbb{R}^3)$

$$\begin{aligned} &\int (\frac{1}{2} \mid \nabla u \mid^2 X_i^i - u_i u_j X_j^i) dx + \\ &\qquad + 4\pi\lambda \int \varsigma^i \varsigma^j X_j^i \theta d\mathcal{H}^1 = 0. \end{aligned} \tag{3.1}$$

The following result is a generalization of Theorem 2.

Theorem 3. *Let $u^0 \in H^1(B^3, S^2)$ be weakly harmonic. If u^0 is not E_λ-stationary ($\lambda \in]0,1[$) then there exist infinitely many weak solutions for the flow associated to E_λ.*

Proof of Theorem 3. As for Theorem 2, it is sufficient to show that (2.3) does not hold true. With the notations of the previous sections, there exists a sequence of currents (c^n) with $c^n = \tau(C^n, \theta^n, \varsigma^n) \in C(u^n)$ such that for any $X \in C_0^\infty(B^3; \mathbb{R}^3)$:

$$2\int \frac{u^n - u^{n-1}}{h} u_i^n X^i + \int |\nabla u^n|^2 X_i^i - 2\int u_i^n u_j^n X_j^i dx \\ + 8\pi\lambda \int \varsigma_n^i \varsigma_n^i X_j^i \theta^n d\mathcal{H}^1 = 0. \tag{3.2}$$

The proof proceeds again by contradiction : we assume that $u(t) = u^0$ i.e. we assume (2.3). We claim that (2.4) holds true again for $\lambda \in (0,1)$. The proof follows the outlines of the corresponding proof in Section 2. We need the following

Lemma 3.1. *Let λ be fixed in [0,1). If v_n weakly converges to v in H^1 and $E_\lambda(v_n)$ converge to $E_\lambda(v)$ then v_n converges strongly in H^1.*

Proof of Lemma 3.1. It is sufficient to show that $E_0(v_n)$ converges to $E_0(v)$. This follows from the fact that E_1 is weakly lower continuous on H^1 (see [1]) and the identity :

$$E_\lambda(v_n) = (1-\lambda)E_0(v_n) + \lambda E_1(v_n).$$

We now complete the proof of Theorem 3. Let us pass to the limit in (3.2). Let t_0 be in $(0,\infty)$ and h be a sequence converging to zero such that

$$\nabla \bar{u}^h(t_0) \longrightarrow \nabla u^0 \text{ in } L^2(B^3)^9, \tag{3.3}$$

$$\bar{u}^h(t_0) \longrightarrow u^0 \text{ a.e. in } B^3, \tag{3.4}$$

$$\frac{\partial u^h}{\partial t}(t_0) \longrightarrow 0 \text{ in } L^2(B^3)^3. \tag{3.5}$$

For sake of simplicity, we write u^h instead of $\bar{u}^h(t_0)$ and c^h instead of c^n (where $(n-1)h \leq t_0 < nh$). We also denote by $\mathcal{D}_1(B^3)$ the set of 1-dimensional currents on B^3 equipped with the weak topology. Let us prove that, for a subsequence of the c^h, there exists a 1-dimensional rectifiable current c on B^3 such that

$$c^h \rightharpoonup c \text{ in } \mathcal{D}_1(B^3). \tag{3.6}$$

We recall that for v and w in $H^1(B^3; S^2)$ the scalar $L(v,w)$ is defined as follows (see [1])

$$L(v,w) = \operatorname*{Sup}_{\substack{\xi \in C_0^\infty(B^3, \mathbb{R}) \\ |\nabla \xi|_{L^\infty} \leq 1}} \int (D(v) - D(w)) \cdot \nabla \xi.$$

According to [1], there exists a constant A such that

$$L(v,w) \leq A \parallel \nabla v - \nabla w \parallel_{L^2} (\parallel \nabla v \parallel_{L^2} + \parallel \nabla w \parallel_{L^2}), \quad \forall v, w \in H^1(B^3, S^2). \tag{3.7}$$

Let $C(v,w)$ be the set of 1-dimensional rectifiable currents c such that

$$4\pi \partial c = div(D(v) - D(w)) \text{ and } M(c) = L(v,w).$$

It is shown in [12] that $C(v,w) \neq \emptyset$ (in [12] it is assumed that either v or w has a finite number of singular points, however this assumption can be removed using theorem 4bis in [2]). Let $\tilde{c}^h$ be given in $C(\bar{u}^h, u^0)$ and let $c_0 \in C(u)$. We set $\bar{c}^h = c^h + \tilde{c}^h - c_0$ and compute :

$$-4\pi \partial \bar{c}^h = \text{div} D(\bar{u}^h) - \text{div} D(\bar{u}^h) + \text{div} D(u^0) - \text{div} D(u^0) = 0, \tag{3.8}$$

$$M(\bar{c}^h) \leq L(\bar{u}^h) + L(\bar{u}^h, u^0) + L(u^0) \leq 2L(u^0) + 2L(\bar{u}^h, u^0). \tag{3.9}$$

Using (3.3), (3.7) and (3.9) we infer that the sequence $M(\bar{c}^h)$ is bounded which, combined with (3.8) and a classical compactness theorem due to Federer and Fleming (see [16, Theorem 27.3]), implies that (for a subsequence) there exists a 1-dimensional integral current $\bar{c}$ such that

$$\bar{c}^h \rightharpoonup \bar{c} \text{ in } \mathcal{D}_1(B^3). \tag{3.10}$$

On the other hand, using (3.3) and (3.7), we have

$$M(\tilde{c}^h) = L(\bar{u}^h, u^0) \to 0. \tag{3.11}$$

We set $c = \bar{c} + c_0$. Clearly, c is a 1-dimensional rectifiable current which, thanks to (3.10) and (3.11), satisfies (3.6). By weak lower semicontinuity of M, (3.3), (3.6) and (3.7)

$$M(c) \leq \liminf M(c^h) = L(u^0) + \liminf (L(\bar{u}^h) - L(u^0)) \leq L(u^0) + \liminf L(u^0, \bar{u}^h) = L(u^0). \tag{3.12}$$

Since $\bar{u}^h \to u^0$ in $H^1(B^3, S^2)$ we have

$$\text{div} D(\bar{u}^h) \rightharpoonup \text{div} D(u^0) \text{ in } \mathcal{D}'(B^3). \tag{3.13}$$

From (3.13) and (3.6), we deduce

$$-4\pi \partial c = \text{div } D(u^0). \tag{3.14}$$

Next, using (3.14) and (3.12) we see that

$$c \in C(u^0). \tag{3.15}$$

We write $c = \tau(C, \theta, \varsigma)$. In order to complete the proof of Theorem 3, it suffices to check that u^0 is E_λ-stationary, which follows from (3.2), (3.3), (3.4), (3.5) and

$$\varsigma_i^h \varsigma_j^h \theta^h d\mathcal{H}^1 \rightharpoonup \varsigma_i \varsigma_j \theta d\mathcal{H}^1 \text{ in } \mathcal{D}'(B^3). \tag{3.16}$$

We now prove (3.16). Since $M(c^h)$ is bounded we have (see (3.6))

$$\varsigma_i^h \theta^h d\mathcal{H}^1 \rightharpoonup \varsigma_i \theta d\mathcal{H}^1 \text{ in } \mathcal{M}(B^3) \tag{3.17}$$

where $\mathcal{M}(B^3)$ denotes the set of Radon measures on B^3 endowed with its weak topology.

Using the fact that $M(c^h)$ is bounded, we may assume that

$$\theta^h d\mathcal{H}^1 \rightharpoonup \theta^* d\mathcal{H}^1 \text{ in } \mathcal{M}(B^3). \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\theta d\mathcal{H}^1 \leq \theta^* d\mathcal{H}^1. \tag{3.19}$$

On the other hand (3.12) and (3.15) imply

$$M(\theta^* d\mathcal{H}^1) \leq \liminf M(\theta^h d\mathcal{H}^1) = M(\theta d\mathcal{H}^1)$$

which, combined with (3.19) gives

$$\theta d\mathcal{H}^1 = \theta^* d\mathcal{H}^1. \tag{3.20}$$

Since $|\varsigma_i^h| \leq 1$, there exists a borelian function ρ_i such that

$$(\varsigma_i^h)^2 \theta^h d\mathcal{H}^1 \rightharpoonup \rho_i \theta d\mathcal{H}^1 \text{ in } \mathcal{M}(B^3). \tag{3.21}$$

From (3.21) and (3.17) we obtain easily

$$\varsigma_i^2 \theta d\mathcal{H}^1 \leq \rho_i \theta d\mathcal{H}^1. \tag{3.22}$$

Since $\varsigma_i^h \varsigma_i^h \theta^h d\mathcal{H}^1 = \theta^h d\mathcal{H}^1$, we have using (3.21) and (3.22) (for $i = 1, 2, 3$)

$$(\varsigma_i^h)^2 \theta^h d\mathcal{H}^1 \rightharpoonup \rho_i \theta d\mathcal{H}^1 = \varsigma_i^2 \theta d\mathcal{H}^1 \text{ in } \mathcal{M}(B^3). \tag{3.23}$$

Proceeding as in the proof of (3.23), we have for $i = 1,2,3$,

$$(R_{ij}\varsigma_j^h)^2\theta^h d\mathcal{H}^1 \rightharpoonup (R_{ij}\varsigma_j)^2\theta d\mathcal{H}^1 \text{ in } \mathcal{M}(B^3) \\ \text{for any } R = (R_{ij}) \text{ in } SO(3). \tag{3.24}$$

Finally (3.16) readily follows from (3.24).

Acknowledgements. We thank M. Struwe for having drawn our attention to [13].

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Chapitre II

Equations d'Ishimori

Le problème de Cauchy pour les équations d'Ishimori

(Article à paraître dans les Comptes Rendus de l'Académie des Sciences Paris)

Analyse Mathématique/ *Mathematical Analysis*[1]

Le problème de Cauchy pour l'équation d'Ishimori

Alain Soyeur

Résumé: Les équations d'Ishimori modélisent l'évolution d'une distribution continue de spins dans un milieu ferromagnétique à deux dimensions. Nous montrons que le problème de Cauchy associé est bien posé dans les espaces de Sobolev lorsque les spins sont initialement quasi-parallèles.

The Cauchy problem for the Ishimori equation

Abstract: *The Ishimori equations describe the evolution of a continuous distribution of spins in a two-dimensional ferromagnet. We prove that the associated Cauchy problem is well-posed in Sobolev spaces when initial spins are quasi-parallel.*

Abridged English Version *Y.Ishimori [1] proposed the two equations (1) ($\varepsilon = \pm 1$) in the context of ferromagnetism. When $b = 1$, these equations are integrable by inverse scattering (see B.G.Konopelchenko and B.T.Matkarimov[2]). We are interested in the Cauchy problem associated to equation (1), $\varepsilon=+1$, when the initial datum S_0 is closed to the north pole (0,0,1). It is then convenient to write equation (1) in stereographic variables*
$u = (S^1 + iS^2) / (1 + S^3)$. We find equation (2) which is a nonlinear Schrödinger equation with a non-local term $G(u)$. It is necessary to solve a Laplace equation to find ϕ, but when u is in $H^1(R^2)$, the function $f = 4i \frac{u_x\bar{u}_y - \bar{u}_x u_y}{(1+|u|^2)^2}$ is only in $L^1(R^2)$, a bad space to invert the Laplace operator. We notice that f is a Jacobian, and we use an idea of H.Wente[3] (see also H.Brezis and J.M.Coron[4]) in lemma 1 to derive ad-hoc *estimates on ϕ in terms of u. Another difficulty comes from the presence of derivatives of u in the right-hand side of equation (2). Even in the simpler case $b = 0$, we cannot apply the results of S.Klainerman, G.Ponce[7] or J.Shatah[8], since the map $\frac{\partial F}{\partial(\partial_i u)}$ is not real-valued. We overcome this difficulty by proving (proposition 3) an a-priori estimate on a non-quadratic expression of u:*

---

[1] Note présentée par:

$$[u]m = \int \frac{|u|^2}{1+|u|^2} + \sum_{1\leq|\alpha|\leq m} \int \frac{|D^\alpha u|^2}{\left(1+|u|^2\right)^2}.$$

Thanks to this estimate, we obtain the following:

Theorem 1: For $m \geq 3$, there exists two positive constants C_0 and α such that for all initial data u_0 satisfying $|u_0|_{H^m} \leq \alpha$, there exists a unique solution $u(t)$ of equation (2) with:

$$u \in L^\infty(0,T;H^m(R^2)),\ u_t \in L^\infty(0,T;H^{m-2}(R^2)),$$

$$\phi \in L^\infty(0,T;W^{m-1,\infty}),\quad D\phi \in L^\infty(0,T;H^m(R^2)),$$

and $\quad T = C_0/\alpha^2.$

Using L^p-L^q estimates for the "hyperbolic" Schrödinger equation, this local existence result can be turned into a global one, following [7], [8], and C.Bardos, C.Sulem and P.L.Sulem[9]:

Theorem 2: For $m \geq 6$, there exists a positive constant δ, such that for all initial data u_0 satisfying:

$$|u_0|_{H^m} + |u_0|_{W^{m-1,6/5}} < \delta,$$

there exists a unique solution $u(t)$ of equation (2) with:

$$u \in L^\infty(R^+; H^m(R^2) \cap W^{m-3,6}(R^2)).$$

Moreover it has the following asymptotic behaviour as $t \to +\infty$:

$$|u(t)|_{H^m} = O(1),\quad |u(t)|_{W^{m-3,6}} = O((1+t)^{-2/3}),$$

and $\quad |u(t)-\underline{u}(t)|_{H^{m-1}} = O((1+t)^{-1/3}),$

where $\underline{u}(t)$ is a solution of the linear "hyperbolic" Schrödinger equation:

(3) $\qquad i\underline{u}_t + \underline{u}_{xx} - \underline{u}_{yy} = 0.$

1. Introduction. Y.Ishimori [1] a proposé les deux équations suivantes qui modélisent l'évolution d'un système de spins (classique) dans un milieu ferromagnétique:

(1)
$$\begin{cases} S_t = S\wedge(S_{xx}-\varepsilon S_{yy})+b(\phi_x S_y+\phi_y S_x) \\ \phi_{xx}+\varepsilon\phi_{yy}=2\varepsilon S.(S_x\wedge S_y) \end{cases}$$

où $\varepsilon=\pm 1$, S: $R^2 \to R^3$, où S est un vecteur unitaire, b est une constante de couplage, et ϕ un potentiel relié à la densité de charge topologique. B.G.Konopelchenko et B.T.Matkarimov [2] ont montré que dans le cas où b=1, ces équations sont intégrables par la méthode de diffusion inverse. Nous nous intéressons au problème de Cauchy associé à l'équation (1) lorsque $\varepsilon=+1$, et pour des données initiales S_0 voisines du pôle nord (0,0,1). Dans ce cas, il est commode d'introduire la projection stéréographique , u, de S, par rapport au pôle sud , sur le plan équatorial: $u=(S^1+iS^2)/(1+S^3)$. Dans ces nouvelles variables, l'équation (1) s'écrit:

(2)
$$\begin{cases} iu_t+u_{xx}-u_{yy}=F(u)+G(u) \\ u(x,y,0)=u_0(x,y) \end{cases}$$

où $F(u)=\frac{2\bar{u}}{1+|u|^2}(u_x^2-u_y^2)$, $G(u) = \phi_x u_y + \phi_y u_x$, et

$\Delta\phi=4i\frac{u_x\bar{u}_y-\bar{u}_x u_y}{(1+|u|^2)^2}$.

Nous rencontrons principalement deux difficultés pour montrer l'existence d'une solution de l'équation (2). La première provient du terme non-local G(u) qui nécessite la résolution d'une équation de Laplace. Lorsque u est dans $H^1(R^2)$, la fonction $f = \frac{u_x\bar{u}_y - \bar{u}_x u_y}{(1+|u|^2)^2}$ est dans $L^1(R^2)$ qui n'est pas un bon espace pour inverser le laplacien. Nous surmontons ce problème en utilisant une idée de H.Wente [3] reprise par H.Brézis et J.M.Coron [4] (voir aussi R.Coifman, P.L.Lions, Y.Meyer, S.Semmes[5] et S.Muller[6]), qui nous permet d'estimer f en fonction de u, puisque f est un jacobien (lemme 1). Nous montrons alors que le terme G(u) se comporte comme F(u) dans les estimées (lemme 2). La deuxième difficulté provient de la présence de dérivées dans le terme non-linéaire F(u). Même dans le cas simplifié où b=0, l'équation (2) ne rentre pas dans le cadre étudié par S.Klainerman, G.Ponce [7] ou J.Shatah [8]. En effet, puisque $\frac{\partial F}{\partial(\partial_i u)}$ n'est pas à valeurs réelles, nous perdons une dérivée dans les estimations a priori classiques. Nous surmontons ce problème en prouvant une estimation (proposition 3) sur une quantité non-quadratique en u:

$$[u]_m = \int \frac{|u|^2}{1+|u|^2} + \sum_{1\leq|\alpha|\leq m} \int \frac{|D^\alpha u|^2}{(1+|u|^2)^2},$$

qui est équivalente à la norme H^m lorsque u est "petit". Cela nous conduit au résultat d'existence locale suivant:

Théorème 1: Pour $m \geq 3$, il existe deux constantes positives C_0 et α telles que pour toute donnée initiale u_0 dans $H^m(R^2)$ vérifiant: $|u|_{H^m} \leq \alpha$, il existe une unique solution u(t) *de l'équation (2) avec $u \in L^\infty(0,T;H^m(R^2))$, $u_t \in L^\infty(0,T;H^{m-2}(R^2))$, $\phi \in L^\infty(0,T;W^{m-1,\infty})$, $D\phi \in L^\infty(0,T;H^m(R^2))$, et $T = C_0/\alpha^2$.*

Nous utilisons ensuite les estimées L^p-L^q pour l'équation de Schrödinger "hyperbolique" (lemme 5) et les techniques développées par S.Klainerman, G.Ponce [7], J.Shatah [8], et C.Bardos, C.Sulem, P.L.Sulem [9], pour montrer l'existence globale de cette solution:

Théorème2: Pour $m \geq 6$, il existe une constante δ positive telle que pour toute donnée initiale u_0 dans $H^m(R^2) \cap W^{m-1,6/5}(R^2)$, telle que:

$$|u_0|_{H^m} + |u_0|_{W^{m-1,6/5}} < \delta,$$

il existe une unique solution $u(t)$ de l'équation (2) vérifiant:

$$u \in L^\infty(R^+; H^m(R^2) \cap W^{m-3,6}(R^2)).$$

De plus cette solution vérifie:

$$|u(t)|_{H^m} = O(1), \quad |u(t)|_{W^{m-3,6}} = O((1+t)^{-2/3}),$$

et $$|u(t)-\underline{u}(t)|_{H^{m-1}} = O((1+t)^{-1/3}),$$

où $\underline{u}(t)$ est une solution de l'équation de Schrödinger "hyperbolique":

(3) $$i\underline{u}_t + \underline{u}_{xx} - \underline{u}_{yy} = 0.$$

2. Eléments pour la preuve du théorème 1.

Lemme 1: Soient f et g deux fonctions de $C_0^\infty(R^2,C)$. Nous définissons :

$$E(f,g) = 1/2\pi \mathrm{Log} r * (f_x g_y - f_y g_x) \qquad (r = (x^2+y^2)^{1/2}).$$

L'application E s'étend en un opérateur bilinéaire continu

$$E : H^{k+1}(R^2) \times H^{k+1}(R^2) \to W^{k,\infty}(R^2)$$

avec $$|DE(f,g)|_{H^k} + |E(f,g)|_{W^{k,\infty}} \leq C|Df|_{H^k}|Dg|_{H^k}.$$

De plus nous avons au sens des distributions:

$$\Delta E(f,g) = f_x g_y - f_y g_x.$$

Preuve: nous suivons [3] et [4], écrivons le produit de convolution définissant E(f,g) en coordonnées polaires puis intégrons par parties. Le résultat s'obtient alors grâce à l'inégalité de Poincaré.
Nous utilisons ce lemme pour définir ϕ en posant:

$$\phi = E(4iu , \frac{\overline{u}}{1+|u|^2}).$$

La fonction ϕ ainsi définie vérifie les estimées suivantes:

Lemme 2: Lorsque u est dans $H^{m+1}(R^2)$,

$|\phi|_{W^{k,\infty}} \leq C|u|^2_{H^{k+1}}$, $|D^k\phi|_2 \leq C|Du|_\infty |u|_{H^{k+1}}$, $|D\phi|_{W^{k,p}} \leq C|u|_{W^{1,\infty}} |u|_{W^{k+1,p}}$, *pour $0 \leq k \leq m$ et $1<p<\infty$.*

Preuve: La première inégalité est une conséquence immédiate du lemme 1. La deuxième inégalité s'obtient grâce à la formule de Leibnitz, puisque $|D^{k+2}\phi|_2 = |D^k\Delta\phi|_2$. Pour démontrer la troisième inégalité remarquons que:

$$\Delta\phi = 4i\left[\left(\frac{\bar{u}u_x}{1+|u|^2}\right)_y - \left(\frac{\bar{u}u_y}{1+|u|^2}\right)_x\right].$$

En utilisant les inégalités de Riesz (cf. par exemple E.Stein[10]), nous avons:

$|D^k\phi_x|_p + |D^k\phi_y|_p \leq C_p\left(\left| D^k\left(\frac{\bar{u}u_x}{1+|u|^2}\right)\right|_p + \left| D^k\left(\frac{\bar{u}u_y}{1+|u|^2}\right)\right|_p \right)$. L'inégalité cherchée s'obtient alors à l'aide de la formule de Leibnitz et des inégalités de Gagliardo-Nirenberg.

Considérons maintenant l'équation approchée (régularisation parabolique):

$$(4)\qquad \begin{cases} i(u_t - \varepsilon\Delta u) + u_{xx} - u_{yy} = F(u) + G(u) \\ u(x, y, 0) = u_0(x, y) \end{cases} \qquad 0 < \varepsilon \leq 1 .$$

Pour toute donnée initiale u_0 dans $H^m(R^2)$, il existe une unique solution $u^\varepsilon(t)$ dans $L^\infty(0,T_\varepsilon; H^m(R^2))$ (cf. par exemple D.Henry[11]). Sur cette solution nous établissons l'estimation suivante:

Proposition3: Lorsque u_0 est dans $H^m(R^2)$, la solution de l'équation (4) vérifie:

$$[u(t)]_m \leq [u_0]_m + C_1 \int_0^t \left(|u(\tau)|^2_{H^3} + |u(\tau)|^3_{H^3} \right) |u(\tau)|^2_{H^m}\, d\tau ,$$

où la constante C_1 dépend de m mais pas de $\varepsilon \in (0,1]$.

Preuve: Nous choisissons un facteur multiplicatif de la forme $\theta(|u|^2)\bar{u}$, prenons le produit scalaire L^2 avec l'équation (4), gardons la partie imaginaire et déterminons θ afin d'éliminer les dérivées de u. Nous trouvons $\theta = \frac{1}{(1+|u|^2)^2}$. La même technique s'applique pour les estimations d'ordre supérieur. Soit α un multi-indice de longueur $|\alpha| \leq m$. Nous différentions l'équation (4) à l'ordre α, prenons le produit scalaire L^2 avec $D^\alpha\bar{u} / (1+|u|^2)^2$, et gardons la partie imaginaire. Les termes comportant des dérivées d'ordre supérieur à m s'éliminent, et nous estimons les termes restants en utilisant les inégalités de Gagliardo-Nirenberg. Quant aux termes en ϕ, ils sont traités en utilisant le lemme 2.

3. Indications de preuve du théorème1: Soit C_2 la constante d'injection de Sobolev $H^m(R^2) \to L^\infty(R^2)$. Posons $\alpha = \mathrm{Min}\,(1, 1/C_2)$. Notons $T^*_\varepsilon = \mathrm{Sup}\,\{\, T>0,\ |u^\varepsilon(t)|_{H^m} < 2\alpha \quad \text{pour } 0\leq t\leq T \,\}$. Si $T^*_\varepsilon = \infty$, $T^*_\varepsilon \geq C_0/\alpha^2$. Sinon, par continuité, $|\, u(T^*_\varepsilon)\,|_{H^m} = 2\alpha$, et la proposition 3 donne alors:

$4\alpha^2 \leq (1+4C_2{}^2\alpha^2)\,\alpha^2\,\mathrm{Exp}(C_3\alpha^2 T^*_\varepsilon)$.

On en déduit une borne inférieure pour le temps d'existence des solutions u^ε (indépendante de ε):

$T^*_\varepsilon \geq C_0/\alpha^2 \quad (=T)$.

Puisque u^ε est dans un borné de $L^\infty(0,T;\ H^m(R^2))$, on peut extraire une sous-suite qui converge vers u(t) faible étoile, et u(t) vérifie l'équation (2). Remarquons que pour démontrer l'unicité de la solution (ainsi que la dépendance continue par rapport à la donnée initiale), nous rencontrons le même problème que lors de la démonstration de l'existence. Si (u,ϕ) et (v,ψ) sont deux solutions, et que nous posons $w = u-v$, $\xi = \phi - \psi$, la fonction w vérifie l'équation suivante:

$$
\begin{aligned}
iw_t + w_{xx} - w_{yy} &= \left(\frac{\bar{u}}{1+|u|^2} + \frac{\bar{v}}{1+|v|^2}\right)\left((u_x+v_x)\,w_x - (u_y+v_y)\,w_y\right) \\
&+ \left(\frac{\bar{u}}{1+|u|^2} - \frac{\bar{v}}{1+|v|^2}\right)\left(u_x^2 - u_y^2 + v_x^2 - v_y^2\right) \qquad (5)\\
&+ ib\left(\phi_x w_y + v_y\xi_x + \phi_y w_x + v_x\xi_y\right).
\end{aligned}
$$

Sur cette équation, en multipliant par $\overline{w}$, intégrant et prenant la partie imaginaire, la présence de dérivées de w ne permet pas de conclure classiquement par le lemme de Gronwall. Nous utilisons une fois encore un facteur multiplicatif $\eta = \frac{1}{\left(1+\left|\frac{u+v}{2}\right|^2\right)^2}$ afin d'éliminer ces dérivées.

4. Preuve du théorème 2.

Lemme4: On dispose des estimées suivantes:

$$|F(u)|_{W^{m+2,6/5}} + |G(u)|_{W^{m+2,6/5}} \leq C|u|^2_{W^{[m/2]+2,6}} |u|_{H^{m+3}}.$$

$$|F(u)|_{H^{m-1}} + |G(u)|_{H^{m-1}} \leq C|u|^2_{W^{2,6}} |u|_{H^m}.$$

Preuve: Comme dans [9], nous utilisons la formule de Leibnitz et les inégalités de Hölder pour démontrer les inégalités sur F(u). Les inégalités sur G(u) sont similaires en utilisant le lemme 2.

Lemme 5: Soit $\Sigma(t)$ le semi-groupe associé à l'équation de Schrödinger "hyperbolique" (4). Il vérifie:

$$|\Sigma(t)|_{W^{k,q}} \leq \frac{C}{(1+t)^{1-\frac{2}{q}}} |u|_{W^{k+N_p,p}},$$

où $1/p+1/q = 1$, $q \geq 2$, et $N_p \geq 2(2-p)/p$.

Preuve: cf. par exemple J.M.Ghidaglia, J.C.Saut[12].

Ecrivons maintenant l'équation (2) sous forme intégrale:

$$\text{(8)} \qquad u(t) = \Sigma(t)\, u_0 + \int_0^t \Sigma(t-\tau)\, \big(F(u(\tau)) + G(u(\tau))\big)\, d\tau .$$

En introduisant $M(T) = \mathrm{Sup}\left\{ (1+t)^{2/3} \,|u(t)|_{W^{m-3,6}} ,\quad 0 \leq t \leq T \right\}$, on peut montrer en utilisant (comme dans [9]) le lemme 5 que M(T) reste borné. On en déduit que la norme H^m de u(t) est contrôlée et on peut réappliquer le théorème 1. Le comportement asymptotique de u(t) s'obtient par les mêmes arguments.

Les résultats présentés dans cette note sont détaillés dans une publication à venir [13].

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The cauchy problem for the Ishimori equations

(version détaillée)

(Article à paraître dans Journal of fonctional analysis)

THE CAUCHY PROBLEM FOR THE ISHIMORI EQUATIONS

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Abstract : In 1984, Y. Ishimori derived a nonlinear wave equation which is a two-dimensional analogue of the classical isotropic spin-chain in ferromagnetism. In this paper we show that the Cauchy problem associated to this equation is well-posed in Sobolev spaces when initial spins are almost parallel.

0. Introduction.

Y. Ishimori was looking for two-dimensional generalizations of the Heisenberg equation in ferromagnetism. He proposed the following equations ([5]) :

$$\begin{cases} S_t = S \wedge (S_{xx} - S_{yy}) + b(\phi_x S_y + \phi_y S_x) \\ \phi_{xx} + \phi_{yy} = 2S \cdot (S_x \wedge S_y) \\ S(x,0) = S_0(x) \end{cases} \tag{0.1}$$

$$\begin{cases} S_t = S \wedge (S_{xx} + S_{yy}) + b(\phi_x S_y + \phi_y S_x) \\ \phi_{xx} - \phi_{yy} = -2S \cdot (S_x \wedge S_y) \\ S(x,0) = S_0(x) \end{cases} \tag{0.2}$$

where $S(\cdot,t) \ : \ \mathbb{R}^2 \to \mathbb{R}^3$, $| S |^2 = 1$, $S \underset{|x| \to +\infty}{\longrightarrow} (0,0,1)$ and $\wedge$ denotes the wedge product in $\mathbb{R}^3$. The coupling potential ϕ is a scalar unknown related to the topological charge density $q(S) = 2S \cdot (S_x \wedge S_y)$. The total topological charge is defined by :

$$Q(S) = \frac{1}{4\pi} \int_{\mathbb{R}^2} S \cdot (S_x \wedge S_y),$$

which is the degree of the mapping $S \ : \ S^2 (\cong \mathbb{R}^2 \cup \{\infty\}) \to S^2$. The real b is a coupling constant. When $b = 0$, equation (0.2) reduces to the two-dimensional Heisenberg equation, which was studied by C. Bardos, C. Sulem and P.L. Sulem ([1]). When $b = 1$, the main interest of equations (0.1) and (0.2) is that they are completely integrable by inverse scattering (see [6]). It is interesting to note that in this case, taking formally the L^2-scalar product of equation (0.1) with ΔS, integrating by parts and using the fact that $| S |^2 = 1$ leads to the conservation of energy : $\frac{d}{dt} \int | \nabla S |^2 = 0$. By way of contrast, equation (0.2) leads to the conservation of the quantity $\int | S_x |^2 - | S_y |^2$. Therefore, equation (0.2) looks more problematic than equation (0.1), the more so as we have to solve a wave equation for ϕ, and we loose some regularity. It is worthwhile to note that the same kind of difficulties arise in Davey-Stewartson systems studied by J.M. Ghidaglia and J.C. Saut ([3]), another example of two-dimensional equations integrable by inverse scattering.

In the following we consider only equation (0.1), and in order to get rid of the constraint $|S|^2 = 1$, we rewrite it in stereographic variables $u : \mathbb{R}^2 \to \mathbb{C}$ (see the Appendix), where

$$u = \frac{S^1 + iS^2}{1 + S^3}.$$

Equation (0.1) becomes

$$\begin{cases} iu_t + u_{xx} - u_{yy} = \dfrac{2\bar{u}}{1 + |u|^2}(u_x^2 - u_y^2) + ib(\phi_x u_y + \phi_y u_x) & (0.3.) \\ \phi_{xx} + \phi_{yy} = 4i\dfrac{u_x \bar{u}_y - \bar{u}_x u_y}{(1 + |u|^2)^2} & (0.4) \\ u(\bar{x}, 0) = u_0(\bar{x}) \quad (\bar{x} = (x, y)) \end{cases}$$

and the condition at infinity is : $u(\bar{x}) \xrightarrow[|\bar{x}| \to +\infty]{} 0$. Equation (0.3)-(0.4) is a nonlinear Schrödinger equation of the form

$$iu_t + u_{xx} - u_{yy} = F(u, \partial_i u).$$

Apart from the fact that the linear equation is "hyperbolic", it does not enter in the framework of J. Shatah ([9]) or S. Klainerman and G. Ponce ([7]). The reason is that $\frac{\partial F}{\partial(\partial_i u)}$ is not a real-valued function, and classical a priori estimates are not available, even to prove the local existence of a solution. There are two difficulties in equation (0.3)-(0.4). The first one comes from the presence of the term $(u_x^2 - u_y^2)$ in the right-hand side : at first sight we loose derivatives in a priori estimates. The second difficulty is the coupling potential ϕ that introduces a nonlocal term. We roughly show that this nonlocal term behaves like the nonlinear term $\frac{2\bar{u}}{1+|u|^2}(u_x^2 - u_y^2)$. Our methods do not rely on inverse scattering techniques, but on classical analysis, and thus work for any coupling constant b. The results of this paper were announced in [10].

1. The coupling potential.

We need to give a sense to the coupling potential ϕ in equation (0.3). Indeed, when u belongs to $H^1(\mathbb{R}^2)$, $f = \dfrac{u_x \bar{u}_y - u_y \bar{u}_x}{(1 + |u|^2)^2}$ only belongs to $L^1(\mathbb{R}^2)$, a "bad" space for inverting the Laplace operator (see [11]). We take profit of the special form of f which is Jacobian, to express ϕ in terms of u and to replace system (0.3) by a single equation with a nonlocal term. This term has roughly the same regularity as $(Du)^2$. In the following lemma, we use an idea of H. Wente [12] (cf. also H. Brezis, J.M. Coron [2]).

LEMMA 1.1. Let f and g in $C_0^\infty(\mathbb{R}^2; \mathbb{C})$. We define

$$E(f, g) = \frac{1}{2\pi} \ln r \, * \, (f_x g_y - f_y g_x) \quad (r = (x^2 + y^2)^{1/2}) \tag{1.1}$$

Then (in the distribution sense) :

$$\Delta E(f,g) = f_x g_y - f_y g_x.$$

The map E extends to a continuous bilinear operator

$$E \ : \ H^{k+1}(\mathbb{R}^2) \times H^{k+1}(\mathbb{R}^2) \to W^{k,\infty}(\mathbb{R}^2)$$

with the following bound :

$$| DE(f,g) |_{H^k} + | E(f,g) |_{W^{k,\infty}} \leq C \mid Df |_{H^k} | Dg |_{H^k} . \tag{1.2}$$

Proof. We follow closely [2], write the explicit integral which defines $E(f,g)$, and use polar coordinates to estimate $E(f,g)(0)$:

$$E(f,g)(0) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} lnr \frac{(f_r g_\theta - f_\theta g_r)}{r} r dr d\theta$$
$$= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} lnr[(fg_\theta)_r - (fg_r)_\theta] dr d\theta.$$

The second term in the integral vanishes, and we perform an integration by parts on the first term. The boundary terms vanish since f is compactly supported, and $\ell n\epsilon \, fg_\theta(\epsilon\cos\theta, \epsilon\sin\theta)$ $= \epsilon \ell n \epsilon \left(\frac{\partial g}{\partial x}\cos\theta + \frac{\partial g}{\partial y}\sin\theta \right)$ .Hence,

$$E(f,g)(0) = -\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{fg_\theta}{r} dr d\theta.$$

Introducing the circular mean value of f :

$$\bar{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta) d\theta,$$

we write

$$E(f,g)(0) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{(f - \bar{f})}{r} \frac{g_\theta}{r} r dr d\theta.$$

Thanks to Cauchy-Schwartz and Poincaré's inequality we bound :

$$| E(f,g)(0) | \leq C \int_0^{+\infty} \left(\int_0^{2\pi} \frac{| f_\theta |^2}{r^2} d\theta \right)^{1/2} \left(\int_0^{2\pi} \frac{| g_\theta |^2}{r^2} d\theta \right)^{1/2} r dr$$
$$\leq C \left(\int_0^{+\infty} \int_0^{2\pi} \frac{| f_\theta |^2}{r^2} r dr d\theta \right)^{1/2} \left(\int_0^{+\infty} \int_0^{2\pi} \frac{| g_\theta |^2}{r^2} r dr d\theta \right)^{1/2}$$
$$\leq C \mid Df |_2 | Dg |_2 .$$

Since $E(f,g)(\xi) = E(f \circ \tau_\xi, g \circ \tau_\xi)$, where $\tau_\xi(\bar{x}) = \xi - \bar{x}$, we obtain the same bound for $| E(f,g) |_\infty$. If we now differentiate equality (1.1) up to the order α, (α a multiindex of length $\leq k$), we find

$$D^\alpha E(f,g) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \frac{1}{2\pi} \ln r \ * \ [(D^\beta f)_x (D^\gamma g)_y - (D^\beta f)_y (D^\gamma g)_x]$$

Each term in the above sum has the structure of a Jacobian, and we can apply the same technique to obtain the desired bound.

Let now f, g be in $H^{k+1}(\mathbb{R}^2)$. We approximate f and g by smooth functions f^n, g^n in $C_0^\infty(\mathbb{R}^2)$:

$$f^n \to f,\ g^n \to g \ \text{ in } H^{k+1}(\mathbb{R}^2).$$

We check that

$$| E(f^n,g^n)-E(f^m,g^m) |_{W^{k,\infty}} \leq C\left(| Df^n |_{H^k} | D(g^n-g) |_{H^k} + | Dg^m |_{H^k} | D(f^n-f) |_{H^k}\right)$$

Therefore the Cauchy sequence $E(f^n,g^n)$ converges to a map $E(f,g)$ in $W^{k,\infty}(\mathbb{R}^2)$. Since $\frac{1}{2\pi}\ln r$ is the fundamental solution of the Laplace operator in $\mathbb{R}^2$, and f^n, g^n are compactly supported, there holds :

$$\Delta E(f^n,g^n) = f^n_x g^n_y - f^n_y g^n_x \ \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

Multiplying this equation by $E(f^n,g^n)$ and integrating by parts leads to the H^k bound, and equality (1.2) is checked by the passage to the limit when $n \to +\infty$.

We notice that when u is in $H^m(\mathbb{R}^2)$ $(m \geq 1)$, the following equality holds in $\mathcal{D}'(\mathbb{R}^2)$:

$$4i\frac{u_x\bar{u}_y - \bar{u}_x u_y}{(1+|u|^2)^2} = (4iu)_x\left(\frac{\bar{u}}{1+|u|^2}\right)_y - (4iu)_y\left(\frac{\bar{u}}{1+|u|^2}\right)_x .$$

And thus, if we let

$$\phi = E\left(4iu, \frac{\bar{u}}{1+|u|^2}\right), \tag{1.3}$$

ϕ satisfies the Laplace equation (0.4). Before listing the estimates we need on ϕ, we state two calculus inequalities from J. Moser [8], whose proofs are consequence of Gagliardo-Nirenberg inequalities.

LEMMA 1.2. *Let η be a C^∞ function whose all derivatives are uniformly bounded, and u in $H^k(\mathbb{R}^2)$. There holds, for $1 \leq p < +\infty$:*

$$| \eta(u) |_{W^{k,p}} \leq C_p | u |_{W^{k,p}} \tag{1.4}$$

LEMMA 1.3. *Let $u_1, ..., u_k$ in $W^{m,p}(\mathbb{R}^n)$ and $\alpha_1, ..., \alpha_k$ integers satisfying $\alpha_1 + ... + \alpha_k = m$. For any $1 \leq p \leq +\infty$ there holds :*

$$| D^{\alpha_1}u_1 ... D^{\alpha_k}u_k |_p \leq C | D^m u_1 |_p^{\frac{\alpha_1}{m}} ... | D^m u_k |_p^{\frac{\alpha_k}{m}} | u_1 |_\infty^{\frac{m-\alpha_1}{m}} ... | u_k |_\infty^{\frac{m-\alpha_k}{m}} \tag{1.5}$$

In particular if u, v are in $W^{m,p}(\mathbb{R}^n)$ there holds :

$$| D^m(uv) |_p \leq C\left(| D^m u |_p | v |_\infty + | u |_\infty | D^m v |_p\right). \tag{1.6}$$

<u>Proof</u>. By Hölder's inequality, we have

$$| D^{\alpha_1}u_1 ... D^{\alpha_k}u_k |_p \leq | D^{\alpha_1}u_1 |_{\frac{mp}{\alpha_1}} ... | D^{\alpha_k}u_k |_{\frac{mp}{\alpha_k}} .$$

But from Gagliardo-Nirenberg inequality we bound from above

$$| D^{\alpha_i} u_i |_{\frac{mp}{\alpha_i}} \leq C \, | D^m u_i |_p^{\frac{\alpha_i}{m}} \, | u_i |_\infty^{\frac{m-\alpha_i}{m}}$$

Inequality (1.6) is a direct consequence of (1.5) and Young's inequality.

■

The next lemma shows that ϕ behaves like u^2 in $W^{k,p}$ norms, for $p \in]1, \infty[$.

LEMMA 1.4 *For u in $H^{m+1}(\mathbb{R}^2)$, and ϕ defined as above, we have the following bounds :*

$$| \phi |_{W^{k,\infty}} \leq C \, | u |_{H^{k+1}}^2 \tag{1.7}$$

$$| D^{k+2}\phi |_2 \leq C \, | Du |_\infty | u |_{H^{k+1}} \tag{1.8}$$

$$| D\phi |_{W^{k,p}} \leq C \, | u |_{W^{1,\infty}} | u |_{W^{k+1,p}} \tag{1.9}$$

for $0 \leq k \leq m$ and $1 < p < +\infty$.

Proof. The first bound is an immediate consequence of (1.2) and (1.4). In order to get inequality (1.8), remark that, by integration par parts,

$$| D^k \Delta\phi |_2 = | D^{k+2}\phi |_2 \, .$$

Using equation (0.4),

$$| D^{k+2}\phi |_2 \leq \left| D^k u_x \left(\frac{4i\bar{u}}{1+ | u |^2} \right)_y \right|_2 + \left| D^k u_y \left(\frac{4i\bar{u}}{1+ | u |^2} \right)_x \right|_2$$

And by (1.5) and lemma 1.2, one finds the desired bound.

Concerning (1.9), we notice that the following equality makes sense when u belongs to $H^m(\mathbb{R}^2)$ $(m \geq 2)$:

$$\Delta\phi = 4i \left[\left(\frac{\bar{u}u_x}{1+ | u |^2} \right)_y - \left(\frac{\bar{u}u_y}{1+ | u |^2} \right)_x \right] \text{ in } S'(\mathbb{R}^2).$$

Taking the Fourier transform gives :

$$\widehat{D^k\phi_x}(\xi) = 4i \left[\frac{\xi_1\xi_2}{\xi_1^2+\xi_2^2} \left(\widehat{D^k \left(\frac{\bar{u}u_x}{1+ | u |^2} \right)} \right) - \frac{\xi_1^2}{\xi_1^2+\xi_2^2} \left(\widehat{D^k \left(\frac{\bar{u}u_y}{1+ | u |^2} \right)} \right) \right]$$

$$\widehat{D^k\phi_y}(\xi) = 4i \left[\frac{\xi_2^2}{\xi_1^2+\xi_2^2} \left(\widehat{D^k \left(\frac{\bar{u}u_x}{1+ | u |^2} \right)} \right) - \frac{\xi_1\xi_2}{\xi_1^2+\xi_2^2} \left(\widehat{D^k \left(\frac{\bar{u}u_y}{1+ | u |^2} \right)} \right) \right]$$

The multiplying factors $\frac{\xi_i\xi_j}{\xi_i^2+\xi_j^2}$ are homogeneous functions of degree zero, and thanks to Riesz's nequalities (see e.g. [11]) we obtain the following L^p bounds $(1 < p < +\infty)$:

$$| D^k\phi_x |_p \, | D^k\phi_y |_p \leq C_p \left[\left| D^k \left(\frac{\bar{u}u_x}{1+ | u |^2} \right) \right|_p + \left| D^k \left(\frac{\bar{u}u_y}{1+ | u |^2} \right) \right|_p \right].$$

Inequality (1.9) follows readily from lemmas (1.2) and (1.3).

■

We are now able to replace equation (0.3) by a single equation in u. For u in $H^{m+1}(\mathbb{R}^2)$, we define the following local and non-local mappings as follows :

$$F(u) = \frac{2\bar{u}}{1+\mid u\mid^2}(u_x^2 - u_y^2), \quad G(u) = ib(\phi_x u_y + \phi_y u_x)$$

where ϕ is given by formula (1.3). Equation (0.3) now reads :

$$iu_t + u_{xx} - u_{yy} = F(u) + G(u). \tag{1.10}$$

We achieve this section with technical estimates on F and G that will be used in the next section.

LEMMA 1.5. *F and G are Lipschitz maps on the bounded sets of $H^{m+2}(\mathbb{R}^2)$ with values into $H^m(\mathbb{R}^2)$, and moreover they satisfy the following inequality :*

$$\mid F(u)\mid_{W^{m+2,6/5}} + \mid G(u)\mid_{W^{m+2,6/5}} \leq C \mid u\mid^2_{W^{[m/2]+2,6}} \mid u\mid_{H^{m+3}} \tag{1.11}$$

$$\mid F(u)\mid_{H^{m-1}} + \mid G(u)\mid_{H^{m-1}} \leq C \mid u\mid^2_{W^{2,6}} \mid u\mid_{H^m} \tag{1.12}$$

Proof. Let u, v in $H^{m+2}(\mathbb{R}^2)$ and ϕ, ψ the corresponding potentials given by formula (1.3). We estimate the difference :

$$\begin{aligned}|G(u) - G(v)|_{H^m} &\leq C\left(|D\phi D(u-v)|_{H^m} + |D(\phi-\psi)Dv|_{H^m}\right)\\ &\leq C\left(|D\phi|_{W^{m,\infty}}|u-v|_{H^{m+1}} + |Dv|_{H^{m+1}}|D(\phi-\psi)|_{W^{m,\infty}}\right)\\ &\leq C|u|_{H^{m+1}}|u-v|_{H^{m+2}}.\end{aligned}$$

which indeed shows the first assertion. In order to prove inequality (1.11), for F we proceed as in [1] :

$$|F(u)|_{W^{m+2,6/5}} \leq C \sum_{i+j+k\leq m+2} \left| D^i\left(\frac{u}{1+\mid u\mid^2}\right) D^{j+1}uD^{k+1}u\right|_{6/5}$$

For $j+k\leq m-1$, with Hölder inequality,

$$\begin{aligned}&\sum_{\substack{i+j+k\leq m+2\\ j+k\leq m-1}} \left| D^i\left(\frac{u}{1+\mid u\mid^2}\right) D^{j+1}uD^{k+1}u\right|_{6/5}\\ &\leq \sum_{\substack{i+j+k\leq m+2\\ j+k\leq m-1}} \left| D^i\left(\frac{u}{1+\mid u\mid^2}\right)\right|_2 \mid D^{j+1}u\mid_6 \mid D^{k+1}u\mid_6\\ &\leq C\mid u\mid^2_{W^{m,6}}\mid u\mid_{H^{m+2}} \quad \text{(by 1.4)}\end{aligned}$$

While for $j+k>m-1$, $i\le 3$ and similarly :

$$\sum_{\substack{i+j+k\le m+2\\ i\le 3}} \left| D^i\left(\frac{u}{1+|u|^2}\right) D^{j+1}uD^{k+1}u\right|_{6/5}$$

$$\le \sum_{\substack{i+j+k\le m+2\\ i\le 3\\ j\le [m/2]+1}} \left| D^i\left(\frac{u}{1+|u|^2}\right)\right|_6 | D^{j+1}u|_6 | D^{k+1}u|_2$$

$$\le C\,|u|^2_{W^{[m/2]+1,6}} |u|_{H^{m+3}}$$

We proceed in a similar way for G :

$$|G(u)|_{W^{m+2,6/5}} \le C \sum_{i+j\le m+2} | D^{i+1}\phi D^{j+1}u|_{6/5}$$

For $0\le i\le \left[\frac{m}{2}\right]+1$ we estimate the terms in the sum as follows :

$$\sum_{\substack{i+j\le m+2\\ i\le [m/2]+1}} | D^{i+1}\phi|_3 | D^{j+1}u|_2$$

$$\le \sum_{\substack{j+k+\ell\le m+2\\ i\le [m/2]+1}} \left| D^k\left(\frac{u}{1+|u|^2}\right)\right|_6 | D^{\ell+1}u|_6 | D^{j+1}u|_2$$

$$\le C\,|u|^2_{W^{[m/2]+2,6}} |u|_{H^{m+3}}$$

and for $0\le j\le [m/2]+1$,

$$\sum_{\substack{i+j\le m+2\\ i\le [m/2]+1}} | D^{i+1}\phi|_3 | D^{j+1}u|_6$$

$$\le \sum_{\substack{j+k+\ell\le m+2\\ j\le [m/2]+1}} \left| D^k\left(\frac{u}{1+|u|^2}\right) D^{\ell+1}u\right|_3 | D^{j+1}u|_6$$

$$\le C \sum_{\substack{j+k+\ell\le m+2\\ k,j\le [m/2]+1}} \left| D^k\left(\frac{u}{1+|u|^2}\right)\right|_6 | D^{\ell+1}u|_2 | D^{j+1}u|_6$$

$$+C \sum_{\substack{j+k+\ell\le m+2\\ \ell,j\le [m/2]+1}} \left| D^k\left(\frac{u}{1+|u|^2}\right)\right|_6 | D^{\ell+1}u|_6 | D^{j+1}u|_6$$

$$\le C\,|u|^2_{W^{[m/2]+2,6}} |u|_{H^{m+3}}$$

Putting together the above inequalities, we arrive at (1.11). The proof of inequality (1.12) readily results from (1.6).

■

2. The main estimate and a local existence result.

In this section, we overcome the difficulty due to the presence of derivatives of u in F and G by proving a "high energies" estimate reminiscent to the one in [7] and [9]. If we classicaly differentiate equation (0.3), take formally the L^2 scalar product with $D\bar{u}$ and look for the imaginary part, no energy estimate is available since we cannot get rid of the higher derivatives coming from $DF(u)$ and $DG(u)$. On the other hand, the technique used in [1] which consists in working on the initial equation on S, differentiating it with respect to time does not work here even for the case $b = 0$. Indeed, we could only hope for estimates on $\int |Sx|^2 - |Sy|^2$ which does not imply bounds in Sobolev spaces.

Our idea comes from the Heisenberg equation $S_t = S \wedge \Delta S$. Formally, the energy $\int |\nabla S|^2$ is conserved, and in the stereographic variables, the corresponding quantity is $\int \frac{|\nabla u|^2}{(1+|u|^2)^2}$. Therefore the *ad hoc* expressions to look for seem to be non-quadratic in u.

Consider the following approximate equation (parabolic regularization) for $0 < \epsilon \leq 1$:

$$\begin{cases} i(u_t - \epsilon \Delta u) + u_{xx} - u_{yy} = F(u) + G(u) \\ u(0) = u_0 \end{cases} \tag{2.1$_\epsilon$}$$

The following lemma is classical.

LEMMA 2.1. *For any u_0 in $H^m(\mathbb{R}^2)$ $(m > 1)$, equation $(2.1)_\epsilon$ possesses a unique solution $u^\epsilon \in C([0, T_\epsilon[; H^m)$. If the maximal existence time T_ϵ is finite, we have the blow-up condition :*

$$\limsup_{t \to T_\epsilon} | u^\epsilon(t) |_{H^m} = +\infty.$$

<u>Proof</u>. For ϵ positive, the operator $A = -\epsilon\Delta - i\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)$ is sectorial on the Banach space $X = H^m(\mathbb{R}^2)$ with domain $D(A) = H^{m+2}(\mathbb{R}^2)$. By lemma (1.5), F and G are Lipschitz maps from $X^\alpha = H^{m+2}$ into $X = H^m$ $(\alpha \in [0,1[)$. The conclusion follows for example from [4, Theorem 3.3].

■

We introduce the following notation :

DEFINITION 2.2. For u in $H^m(\mathbb{R}^2)$, let

$$[u]_m = \int \frac{|u|^2}{1+|u|^2} + \sum_{1 \leq |\alpha| \leq m} \int \frac{|D^\alpha u|^2}{(1+|u|^2)^2} \tag{2.2}$$

Of course, when u is small $[u]_m$ is equivalent to $| u |^2_{H^m}$:

$$(1+ | u |^2_\infty)^{-2} | u |^2_{H^m} \le [u]_m \le | u |^2_{H^m} . \tag{2.3}$$

We prove the following estimate on $[u^\epsilon(t)]_m$:

PROPOSITION 2.3. *Let u_0 in $H^m(\mathbb{R}^2)$ $(m \ge 3)$. The solution $u^\epsilon(t)$ of equation $(2.1)_\epsilon$ satisfies, for $0 \le t < T_\epsilon$:*

$$[u^\epsilon(t)]_m \le [u_0]_m + C_1 \int (| u(\tau) |^2_{H^3} + | u(\tau) |^3_{H^3}) | u(\tau) |^2_{H^m} \, d\tau \tag{2.4}$$

$$[u^\epsilon(t)]_m \le [u_0]_m + C_1 \int (| u(\tau) |^2_{W^{3,6}} + | u(\tau) |^3_{W^{3,6}}) | u(\tau) |^2_{H^m} \, d\tau \tag{2.5}$$

where C_1 and C_2 are constants that depend on m, but not on u_0 and $\epsilon \in]0,1]$.

Proof. We choose a multiplying factor of the form $\theta(| u |^2)\bar{u}$, take the L^2 scalar product with equation $(2.1)_\epsilon$ keep the imaginary part and determine θ in order to cancel higher derivatives. After some computations we find

$$\theta(| u |^2) = \frac{1}{(1+ | u |^2)^2}.$$

Indeed :

$$Re \int \frac{\bar{u} u_t}{(1+ | u |^2)^2} - \epsilon Re \int \frac{\Delta u \bar{u}}{(1+ | u |^2)^2} + Im \int (u_{xx} - u_{yy}) \frac{\bar{u}}{(1+ | u |^2)^2}$$
$$= Im \int \frac{2\bar{u}^2}{(1+ | u |^2)^3}(u_x^2 - u_y^2) + b \; Re \int \frac{\bar{u}}{(1+ | u |^2)^2}(u_x \phi_y + u_y \phi_x) \tag{2.6}$$

Let us develop the first term in the right-hand side of (2.6) :

$$Im \int \frac{2(\bar{u}u_x)^2 - 2(\bar{u}u_y)^2}{(1+ | u |^2)^3}$$
$$= 4 \int \frac{Re(\bar{u}u_x)Im(\bar{u}u_x) - Re(\bar{u}u_y)Im(\bar{u}u_y)}{(1+ | u |^2)^3}$$
$$= 2 \int \frac{| u |^2_x}{(1+ | u |^2)^3} Im(\bar{u}u_x) - 2 \int \frac{| u |^2_y}{(1+ | u |^2)^3} Im(\bar{u}u_y)$$
$$= \int \frac{Im[(\bar{u}u_x)_x - (\bar{u}u_y)_y]}{(1+ | u |^2)^2}$$
$$= Im \int \frac{\bar{u}}{(1+ | u |^2)^2}(u_{xx} - u_{yy})$$

We recover the last term in the left-hand side of (2.6). We estimate the remaining terms as follows :

$$\epsilon \; Re \int \Delta u \frac{\bar{u}}{(1+ | u |^2)^2}$$
$$= -\epsilon \int \frac{| u_x |^2 + | u_y |^2}{(1+ | u |^2)^2} + 2\epsilon \; Re \int \frac{\bar{u}}{(1+ | u |^2)^3}(u_x | u |^2_x + u_y | u |^2_y)$$
$$\le C | Du |^2_\infty \int \frac{| u |^2}{1+ | u |^2}$$

$$
\begin{aligned}
&Re \int \frac{\bar{u}}{(1+|u|^2)^2}(u_x\phi_y + u_y\phi_x) \\
&= -\int \phi_{xy}\frac{|u|^2}{1+|u|^2} \\
&\leq |D^2\phi|_\infty \int \frac{|u|^2}{1+|u|^2}.
\end{aligned}
$$

Putting together the preceeding inequalities in (2.6) we find :

$$
\frac{1}{2}\frac{d}{dt}\int \frac{|u|^2}{1+|u|^2} \leq C(|Du|_\infty^2 + |D^2\phi|_\infty)\int \frac{|u|^2}{1+|u|^2}.
$$

We follow the same line to derive higher order estimates. Let α be a multiindex of length $|\alpha| \leq m$. Differentiate equation $(2.1)_\epsilon$ up to the order α, take the L^2 scalar product with $\frac{D^\alpha \bar{u}}{(1+|u|^2)^2}$ and keep the imaginary part :

$$
\begin{aligned}
&\frac{1}{2}\int \frac{|D^\alpha u|_t^2}{(1+|u|^2)^2} - \epsilon\, Re\int \frac{\Delta D^\alpha u D^\alpha \bar{u}}{(1+|u|^2)^2} + Im\int (D^\alpha u_{xx} - D^\alpha u_{yy})\frac{D^\alpha \bar{u}}{(1+|u|^2)^2} \\
&= Im\int D^\alpha\left[\left(\frac{2\bar{u}}{1+|u|^2}\right)(u_x^2 - u_y^2)\right]\frac{D^\alpha \bar{u}}{(1+|u|^2)^2} \\
&\quad +b\, Re\int D^\alpha(u_x\phi_y + u_y\phi_x)\frac{D^\alpha \bar{u}}{(1+|u|^2)^2}.
\end{aligned}
\tag{2.7}
$$

First isolate the first term in the right-hand side and apply Leibnitz's formula :

$$
\begin{aligned}
&Im\int D^\alpha\left[\left(\frac{2\bar{u}}{1+|u|^2}\right)(u_x^2 - u_y^2)\right]\frac{D^\alpha \bar{u}}{(1+|u|^2)^2} \\
&= \sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta} Im\int D^\beta\left(\frac{2\bar{u}}{1+|u|^2}\right)D^\gamma(u_x^2 - u_y^2)\frac{D^\alpha \bar{u}}{(1+|u|^2)^2}
\end{aligned}
\tag{2.8}
$$

We handle the term where appear derivatives of order higher than α in the expression (2.8), to recover the last term in the left-hand side of (2.7).

$$
\begin{aligned}
&Im\int \frac{\bar{u}}{(1+|u|^2)^3}(4u_x D^\alpha u_x - 4u_y D^\alpha u_y)D^\alpha \bar{u} \\
&= 2\int \frac{|u|_x^2}{(1+|u|^2)^3} Im(D^\alpha u_x D^\alpha \bar{u}) - 2\int \frac{|u|_y^2}{(1+|u|^2)^3} Im(D^\alpha u_y D^\alpha \bar{u}) \\
&\quad +2\int \frac{|D^\alpha u|_x^2}{(1+|u|^2)^3} Im(\bar{u}u_x) - 2\int \frac{|D^\alpha u|_y^2}{(1+|u|^2)^3} Im(\bar{u}u_y) \\
&= Im\int (D^\alpha u_{xx} - D^\alpha u_{yy})\frac{D^\alpha \bar{u}}{(1+|u|^2)^2} + 2\int \frac{|D^\alpha u|_x^2}{(1+|u|^2)^3} Im(\bar{u}u_x) \\
&\quad -2\int \frac{|D^\alpha u|_y^2}{(1+|u|^2)^3} Im(\bar{u}u_y)
\end{aligned}
\tag{2.9}
$$

We estimate the two last terms in (2.9) with an integration by parts :

$$\left|2\int \frac{|D^\alpha u|_x^2}{(1+|u|^2)^3} Im(\bar{u}u_x) - 2\int \frac{|D^\alpha u|_y^2}{(1+|u|^2)^3} Im(\bar{u}u_y)\right|$$
$$\leq C\,|u|_{W^{2,\infty}}^2 \int \frac{|D^\alpha u|^2}{(1+|u|^2)^2}.$$

The remaining terms in (2.8) are bounded from above by :

$$\sum_{\substack{\beta+\gamma+\nu\leq m\\ \gamma,\nu\leq m-1}} \left|\int D^\beta\left(\frac{2\bar{u}}{1+|u|^2}\right) D^\gamma(Du)D^\nu(Du)\frac{D^\alpha \bar{u}}{(1+|u|^2)^2}\right|$$
$$\leq \sum_{\substack{\beta+\gamma+\nu\leq m\\ \gamma,\nu\leq m-1}} \left|D^{\beta-1}\left(D\left(\frac{2\bar{u}}{1+|u|^2}\right)\right) D^\gamma(Du)D^\nu(Du)\right|_2 |u|_{H^m}.$$

But from (1.5),

$$\leq C\left|D^m\left(\frac{2\bar{u}}{1+|u|^2}\right)\right|_2^{\frac{\beta-1}{m-1}} |D^m u|_2^{\frac{\gamma+\nu}{m-1}} \left|D^m\left(\frac{2\bar{u}}{1+|u|^2}\right)\right|_\infty^{\frac{m-1-\beta+1}{m-1}} |Du|_\infty^{\frac{m-1-\gamma}{m-1}} |Du|_\infty^{\frac{m-1-\nu}{m-1}}$$
$$\leq |u|_{H^m}^2 |Du|_\infty^2.$$

With an integration by parts, we get rid of the higher derivatives in the ϵ terms.

$$\epsilon\, Re\int \Delta D^\alpha u \frac{D^\alpha \bar{u}}{(1+|u|^2)^2}$$
$$= -\epsilon\int \frac{|\nabla D^\alpha u|^2}{1+|u|^2} + 2\epsilon Re\left(\int D^\alpha u_x D^\alpha \bar{u}\frac{|u|_x^2}{(1+|u|^2)^3} + \int D^\alpha u_y D^\alpha \bar{u}\frac{|u|_y^2}{(1+|u|^2)^3}\right)$$
$$\leq C\,|u|_{H^m}^2 |u|_{W^{2,\infty}}^2.$$

We now deal with terms involving ϕ :

$$\begin{aligned} & Re\int D^\alpha(\phi_x u_y + \phi_y u_x)\frac{D^\alpha \bar{u}}{(1+|u|^2)^2} \\ & \leq Re\int \phi_x D^\alpha u_y \frac{D^\alpha \bar{u}}{(1+|u|^2)^2} + \phi_y D^\alpha u_x \frac{D^\alpha \bar{u}}{(1+|u|^2)^2} \\ & + \sum_{\substack{\beta+\gamma=m\\ \gamma\leq m-1}} \left|\int (D^\beta\phi_x D^\gamma u_y + D^\beta \phi_y D^\gamma u_x)\frac{D^\alpha \bar{u}}{(1+|u|^2)^2}\right| \end{aligned} \qquad (2.10)$$

We treat the first terms using an integration by parts :

$$\int \frac{\phi_x}{2}\frac{|D^\alpha u|_y^2}{(1+|u|^2)^2} + \frac{\phi_y}{2}\frac{|D^\alpha u|_x^2}{(1+|u|^2)^2}$$

$$= -\int \frac{|D^\alpha u|^2}{2}\left(\left(\frac{\phi_x}{(1+|u|^2)^2}\right)_y + \left(\frac{\phi_y}{(1+|u|^2)^2}\right)_x\right)$$

$$\leq C\,|D\phi|_\infty |Du|_\infty + |D^2\phi|_\infty)\,|u|_{H^m}^2 .$$

By (1.7) and Sobolev inequality, we can bound the above terms by

$$C\left(|u|_{H^3}^2 + |u|_{H^3}^3\right)|u|_{H^m}^2$$

or in turn, we can also use (1.9) to infer

$$|D\phi|_{W^{1,\infty}} \leq |D\phi|_{W^{2,6}} \leq |u|_{W^{1,\infty}}|u|_{W^{3,6}} \leq |u|_{W^{3,6}}^2$$

and therefore the above terms are also bounded from above by

$$C\left(|u|_{W^{3,6}}^2 + |u|_{W^{3,6}}^3\right)|u|_{H^m}^2 .$$

We use inequality (1.6) to estimate the other terms in (2.10) :

$$I = \sum_{\substack{\beta+\gamma=m\\ 1\leq\beta\\ \gamma\leq m-1}} \left|\int (D^\beta\phi_x D^\gamma u_x + D^\beta\phi_y D^\gamma u_y)\frac{D^\alpha\bar{u}}{(1+|u|^2)^2}\right|$$

$$\leq C \sum_{\substack{\beta+\gamma=m\\ 1\leq\beta\\ \gamma\leq m-1}} |D^{\beta-1}(D^2\phi)D^\gamma(Du)|_2 |u|_{H^m}$$

$$\leq C\,|D^{m+1}\phi|_2^{\frac{\beta-1}{m-1}}\,|D^m u|_2^{\frac{\gamma}{m-1}}\,|D^2\phi|_\infty^{\frac{m-\beta}{m-1}}\,|Du|_\infty^{\frac{m-1-\gamma}{m-1}}$$

But from (1.7) we deduce

$$|D^{m+1}\phi|_2^{\frac{\beta-1}{m-1}} \leq C|Du|_\infty^{2\frac{\beta-1}{m-1}}|u|_{H^m}^{\frac{\beta-1}{m-1}}$$

$$|D^2\phi|_\infty \leq C\,|u|_{H^3}^2 \text{ or } C\,|u|_{W^{1,\infty}}|u|_{W^{3,6}} .$$

Therefore we have

$$I \leq C\,|u|_{H^3}^2|u|_{H^m}^2$$

and

$$I \leq C\,|u|_{W^{3,6}}^2|u|_{H^m}^2 .$$

Eventually we notice that

$$\frac{d}{dt}\int \frac{|D^\alpha u|^2}{(1+|u|^2)^2} = \int \frac{|D^\alpha u|_t^2}{(1+|u|^2)^2} - 2\int \frac{|D^\alpha u|^2}{(1+|u|^2)^2}|u|_t^2$$

and by equation $(2.1)_\epsilon$:

$$|u|_t^2 = 2Im\left(\frac{2\bar{u}}{1+|u|^2}(u_x^2 - u_y^2) - \bar{u}(u_{xx} + u_{yy})\right) + 2Re(\bar{u}(\epsilon\Delta u + b(u_x\phi_y + u_y\phi_x)).$$

Therefore,

$$\frac{d}{dt}\int \frac{|D^\alpha u|^2}{(1+|u|^2)^2} \leq \int \frac{|D^\alpha u|_t^2}{(1+|u|^2)^2} + C\left(|Du|_\infty^2 + |u|_\infty |D^2u|_\infty + |Du|_\infty |D\phi|_\infty\right)|u|_{H^m}^2$$

Putting together the above inequalities for all multiindex $|\alpha| \leq m$, we obtain the following differential inequalities :

$$\frac{d}{dt}[u]_m \leq C_1\left(|u|_{H^3}^2 + |u|_{H^3}^3\right)|u|_{H^m}^2$$
$$\frac{d}{dt}[u]_m \leq C_2\left(|u|_{W^{3,\infty}}^2 + |u|_{W^{3,\infty}}^3\right)|u|_{H^m}^2 .$$

They immediately lead to inequalities (2.4) and (2.5).

■

We use these estimates to prove the following local existence result for small initial data :

THEOREM 2.4. *For $m \geq 3$, there exists two positive constants C_0, α_0 such that for all u_0 in $H^m(\mathbb{R}^2)$ satisfying $|u_0|_{H^m} \leq \alpha_0$, there exists a solution u of equation* (0.3) *with :*

$$u \in L^\infty(0,T;H^m(\mathbb{R}^2)), \qquad u_t \in L^\infty(0,T;H^{m-2}(\mathbb{R}^2))$$
$$\phi \in L^\infty(0,T;W^{m-1,\infty}(\mathbb{R}^2)), \quad D\phi \in L^\infty(0,T;H^m(\mathbb{R}^2))$$

where $T = \dfrac{C_0}{\alpha_0^2}$. Moreover this solution satisfies inequalties (2.4) *and* (2.5).

Proof. Denote by C_2 the Sobolev imbedding constant $H^m(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, and let $\alpha_0 = \text{Min}\left(\dfrac{1}{2C_2}\right)$. For $0 < \epsilon \leq 1$ and $|u_0|_{H^m} < \alpha_0$ denote by

$$T_\epsilon^* = \text{Sup}\,\{T > 0\ ;\ |u^\epsilon(t)|_{H^m} < 2\alpha_0 \text{ for } 0 \leq t \leq T\}.$$

By continuity, $|u(T_\epsilon^*)|_{H^m} = 2\alpha_0$.

On the other hand, using inequalities (2.3), (2.4) and the definition of C_2, for $0 \leq t \leq T_\epsilon^*$ we obtain :

$$|u^\epsilon(t)|_{H^m}^2 \leq \left(1 + 4C_2^2\alpha_0^2\right)^2\left[|u_0|_{H^m}^2 + C_1\int_0^t (4\alpha_0^2 + 8\alpha_0^3)\,|u^\epsilon(\tau)|_{H^m}^2\,d\tau\right]$$

and by Gronwall's lemma,

$$4\alpha_0^2 = | u^\epsilon(T_\epsilon^*) |_{H^m}^2 \leq \left(1 + 4C_2^2\alpha_0^2\right)^2 \alpha_0^2 \exp\left(C_3\alpha_0^2 T_\epsilon^*\right).$$

This inequality gives the following lower bound for the existence time of solutions of equation $(2.1)_\epsilon$:

$$T_\epsilon^* \geq \frac{C_0}{\alpha_0^2} \quad (: = T).$$

Since

$$u^\epsilon \text{ is in a bounded set of } L^\infty(0,T;H^m(\mathbb{R}^2))$$
$$u_t^\epsilon \text{ is in a bounded set of } L^\infty(0,T;H^{m-2}(\mathbb{R}^2)),$$

up to a subsequence we can pass to the limit in equation $(2.1)_\epsilon$ to find a solution u of equation (0.3) on $[0,T]$. Inequalities (2.4) and (2.5) are satisfied by the passage to the limit.

■

3. Global existence.

Theorem 2.4 can be turned into a global existence result, using $L^p - L^q$ estimates on the linear "hyperbolic" Schrödinger equation.

LEMMA 3.1. *Let $\Sigma(t)$ be the semi-group associated to the linear equation*

$$\begin{cases} iu_t + u_{xx} - u_{yy} \\ u(0) = u_0 \end{cases} \tag{3.1}$$

The following inequality holds :

$$| \Sigma(t)u |_{W^{k,q}} \leq \frac{C_4}{(1+t)^{1-(2/q)}} | u |_{W^{k+N_p,q}} \tag{3.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $q \geq 2$ and $N_p \geq 2\frac{2-p}{p}$.

Proof. It can be found in [3]. The idea is that the fundamental solution of equation (3.1) splits into the product of two one-dimensional integrals and from the fact that the one-dimensional semi-group satisfies $| \Sigma(t)u |_\infty \leq \frac{C}{t} | u |_1$. Since it is also unitary in $L^2(\mathbb{R})$, the conclusion follows by interpolation and Sobolev inequalities.

■

Our proof then follows the classical lines of [1], [7] and [9].

THEOREM 3.1. *Let $m \geq 6$. There exists a constant $\delta > 0$ such that for all $u_0 \in H^m(\mathbb{R}^2) \cap W^{m-1,6/5}$ satisfying :*

$$| u_0 |_{H^m} + | u_0 |_{W^{m-1,6/5}} < \delta,$$

the solution $u(t)$ of Theorem 2.4 is global with the following asymptotic behaviour :

$$| u(t) |_{W^{m-3,6}} \leq \frac{C}{(1+t)^{2/3}}, \quad | u(t) |_{H^m} \leq C,$$
$$| u(t) - \underline{u}(t) |_{H^{m-1}} \leq \frac{C}{(1+t)^{1/3}},$$

where $\underline{u}(t)$ is a solution of the linear "hyperbolic" Schrödinger equation (3.1).

Proof. The solution $u(t)$ given by Theorem 2.4 satisfies the integral equation :

$$u(t) = \Sigma(t)u_0 + \int_0^t \Sigma(t-\tau)(F(u(\tau)) + G(u(\tau))d\tau.$$

We estimate

$$| u(t) |_{W^{m-3,6}} \leq \frac{C}{(1+t)^{2/3}} | u_0 |_{W^{m-1,6/5}} + \int_0^t \frac{| F(u(\tau)) |_{W^{m-1,6/5}} + | G(u(\tau)) |_{W^{m-1,6/5}}}{(1+t-\tau)^{2/3}} \tag{3.3}$$

But from (1.11) we have :

$$| F(u) + G(u) |_{W^{m-1,6/5}} \leq C | u |^2_{W^{[\frac{m-1}{2}]+1,6}} | u |_{H^m}$$
$$\leq C | u |^2_{W^{m-3,6}} | u |_{H^m}$$

Let $M(t) = \sup_{0 \leq t \leq T} ((1+t)^{2/3} | u |_{W^{m-3,6}})$. The estimate (2.5) gives :

$$| u(t) |^2_{H^m} \leq (1 + C_2^2 | u |^2_\infty)^2 \left(| u_0 |^2_{H^m} + C_1 \int_0^t \left(| u |^2_{W^{3,6}} + | u |^3_{W^{3,6}} \right) | u |^2_{H^m} d\tau \right)$$
$$\leq (1 + CM^2(t))^2 \left(| u_0 |^2_{H^m} + C_1 \int_0^t \left(\frac{M^2(t)}{(1+\tau)^{4/3}} + \frac{M^3(t)}{(1+\tau)^2} \right) | u |^2_{H^m} d\tau \right)$$

and by Gronwall's lemma, for $0 \leq t \leq T$

$$| u(t)_{H^m} \leq (1 + C_3 M^2(T)) | u_0 |_{H^m} \ \exp C_4(M^2(T) + M^3(T)) \tag{3.4}$$

(the above integrals are bounded). Therefore we obtain the bound

$$| F(u) + G(u) |_{W^{m-1,6/5}}$$
$$\leq C_5(1 + C_3 M^2(T)) | u_0 |_{H^m} \ \exp (C_4(M^2(T) + M^3(T)) | u |^2_{W^{m-3,6}} .$$

And replacing in (3.3) gives :

$$M(t) \leq C_6 | u_0 |_{W^{m-1,6/5}} +$$
$$+ C_5(1 + C_3 M^2(T)) | u_0 |_{H^m} e^{C_4(M^2(T)+M^3(T))} \int_0^t \frac{(1+t)^{2/3} M^2(T)}{(1+t-\tau)^{2/3}(1+\tau)^{4/3}} d\tau.$$

The above integral is bounded, and if we define the function

$$f(x) = \delta\left(C_6 + C_7(1 + C_3x^2)e^{C_4(x^2+x^3)}e^{C_4(x^2+x^3)}\right) - x,$$

$(C_7 = C_5 \underset{t \geq 0}{\text{Sup}} \int_0^t \frac{(1+t)^{2/3}}{(1+t-\tau)^{2/3}(1+\tau)^{4/3}} d\tau)$, for δ sufficiently small,

$$\begin{aligned} f(x) &\geq 0 \text{ if } x \in [0, K] \\ f(x) &< 0 \text{ if } x \in [0, K + \epsilon[. \end{aligned}$$

Since $M(0) = 0$ and $f(M(T)) \geq 0$, we deduce that $M(T) \leq K$, and this bound implies that the solution $u(t)$ is global : from (3.4), if we choose δ sufficiently small, we can apply again the local existence theorem 2.4..

To study the asymptotic behaviour of the solution $u(t)$, define

$$\begin{aligned} \underline{u}(t) &= \Sigma(t)u_0 + \int_0^{+\infty} \Sigma(t-\tau)(F(u(\tau)) + G(u(\tau))d\tau \\ &= \Sigma(t)\left(u_0 + \int_0^{+\infty} \Sigma(-\tau)(F(u(\tau)) + G(u(\tau)))d\tau\right) \end{aligned}$$

$\underline{u}(t)$ satisfies the linear equation (3.1) and we have

$$\begin{aligned} | u(t) - \underline{u}(t) |_{H^{m-1}} &\leq \int_0^{+\infty} | \Sigma(t-\tau)(F(u(\tau)) + G(u(\tau))) |_{H^{m-1}} d\tau \\ &\leq \int_0^{+\infty} | F(u(\tau)) + G(u(\tau)) |_{H^{m-1}} d\tau \end{aligned}$$

But from (1.12),

$$\begin{aligned} | F(u(\tau)) + G(u(\tau)) |_{H^{m-1}} &\leq C \, | u |^2_{W^{m-3,6}} | u |_{H^m} \\ &\leq \frac{C}{(1+t)^{4/3}}. \end{aligned}$$

■

4. Uniqueness of solutions.

THEOREM 4.1. *Let u and v two solutions of equation* (0.3) *satisfying*

$$u, v \in L^\infty(0, T; H^3(\mathbb{R}^2)) \cap L^\infty(0, T; W^{2,\infty}(\mathbb{R}^2)).$$

If $u(0) = v(0)$, *then* $u(t) = v(t)$ *for* $0 \leq t < T$.

Proof. Let $w = u - v$, ϕ, ψ the corresponding potentials given by formula (1.3), and $\xi = \phi - \psi$. The map w satisfies the following equation :

$$
\begin{aligned}
iw_t + w_{xx} - w_{yy} = &\left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)((u_x + v_x)w_x - (u_y + v_y)w_y) \\
&+ \left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)(u_x^2 - u_y^2 + v_x^2 - v_y^2) + ib(\phi_x w_y + v_y \xi_x \\
&+ \phi_x w_y + v_y \xi_x + \phi_y w_x + v_x \xi_y)
\end{aligned} \tag{4.1}
$$

We have the same difficulty as for the proof of existence. If we multiply equation (4.1) by $\bar w$, integrate over $\mathbb{R}^2$ and take the imaginary part, the presence of derivatives in the right-hand side of equation (4.1) does not allow us to conclude by Gronwall's lemma as usual.

We therefore use the same idea as in the proof of proposition (2.3). We take the L^2 scalar product of equation (4.1) with a well-chosen multiplying factor of the form $\theta(u,v)\bar w$ to cancel these derivatives.
Let $\theta = \dfrac{1}{\left(1 + \left|\frac{u+v}{2}\right|^2\right)^2}$, then $\theta_x = -\dfrac{\theta}{2}\dfrac{|u+v|^2}{1+\left|\frac{u+v}{2}\right|^2}$. Integrating by parts the last term in the left-hand member, we find :

$$
\begin{aligned}
&Im \int (w_{xx} - w_{yy})\bar w \theta \\
&= -Im \int \bar w (w_x \theta_x - w_y \theta_y) \\
&= \frac{1}{2}\int \theta \left(Im(\bar w w_x)\frac{|u+v|_x^2}{1+\left|\frac{u+v}{2}\right|^2} - Im(\bar w w_y)\frac{|u+v|_y^2}{1+\left|\frac{u+v}{2}\right|^2}\right)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&Im \int \left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)((u_x + v_x)w_x - (u_y + v_y)w_y)\theta \bar w \\
&= \int \theta \left[Re\left(\left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)(u_x + v_x)\right) Im\ (\bar w w_x) \right. \\
&\left. -Re\left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)(u_y + v_y) Im(\bar w w_y)\right] - \\
&- \int \frac{\theta |w|^2}{2}\left(Im\left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)(u_x + v_x)\right)_x \\
&+ \int \frac{\theta |w|^2}{2}\left(Im\left(\frac{\bar u}{1+|u|^2} + \frac{\bar v}{1+|v|^2}\right)(u_y + v_y)\right)_y .
\end{aligned}
$$

Concerning the terms with potentials we have :

$$Re\int(\phi_x w_y + v_y\xi_x + \phi_y w_x + v_x\xi_y)\bar{w}\theta$$

$$= -2\int \phi_{xy} \mid w \mid^2 \theta - \int(\phi_x\theta_y + \theta_x\phi_y) \mid w \mid^2 + Re\int \theta\bar{w}(v_y\xi_x + v_x\xi_y).$$

And putting together the above inequalities, we deduce that

$$\frac{d}{dt}\int \theta \mid w \mid^2 \leq C\left(\int \mid w \mid^2 + \mid D\xi \mid_2 \mid w \mid_2\right).$$

But we cannot bound from above $\mid D\xi \mid_2$ by $\mid w \mid_2$. Thus we differentiate equation (4.1) and multiply it by $\theta D\bar{w}$. Once more the terms with derivatives of higher order cancel, and eventually we find

$$\frac{d}{dt}\int \theta(\mid w \mid^2 + \mid Dw \mid^2) \leq C\left(\int(\mid w \mid^2 + \mid Dw \mid^2 + \mid D^2\xi \mid_2 \mid Dw \mid_2 + \mid D\xi \mid_2 \mid w \mid_2\right)$$

But ξ satisfies the following Laplace equation :

$$\Delta\xi = (f_x w_y - f_y w_x) + (v_y(f-g)_x - v_x(f-g)_y)$$

where $f = 4i\frac{\bar{u}}{1+|u|^2}$ and $g = 4i\frac{\bar{v}}{1+|v|^2}$. Therefore $\mid D^2\xi \mid_2 = \mid \Delta\xi \mid_2 \leq C \mid Dw \mid_2$ and from Lemma 1.4 we also have $\mid D\xi \mid_2 \leq C \mid Dw \mid_2$. We conclude by the usual Gronwall's lemma.

■

COROLLARY 4.2. *The Cauchy problem* (0.3) *is well-posed in* $H^m(\mathbb{R}^2)$ *for* $m \geq 4$.

The proof of continuous dependence is similar to the proof of uniqueness.

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Appendix.

We derive here equations (0.1) and (0.2) in stereographic variables. We use stereographic projection from the south pole to the equator plane :

$$u = \frac{S^1 + iS^2}{1+S^3} \qquad S = \frac{1}{1+\mid u \mid^2}\left|\begin{matrix} u+\bar{u} \\ -i(u-\bar{u}) \\ 1-\mid u \mid^2 \end{matrix}\right.$$

Compute :

$$\begin{cases} S_t^1 = \dfrac{1}{(1+\mid u \mid^2)^2}[(1-\bar{u}^2)u_t + (1-u^2)\bar{u}_t] \\ S_t^2 = \dfrac{-i}{(1+\mid u \mid^2)^2}[(1+\bar{u}^2)u_t - (1+u^2)\bar{u}_t] \\ S_t^3 = \dfrac{-2}{(1+\mid u \mid^2)^2}[\bar{u}u_t + u\bar{u}_t] \end{cases}$$

$$\begin{cases} S^1_{xx} = \dfrac{1}{(1+|u|^2)^2}[(1-\bar{u}^2)u_{xx} + (1-u^2)\bar{u}_{xx} - 2u_x\bar{u}_x(u+\bar{u}) \\ \quad -2\dfrac{(u\bar{u}_x + \bar{u}u_x)}{(1+|u|^2)^3}[(1-\bar{u}^2)u_x + (1-u^2)\bar{u}_x] \\ S^2_{xx} = \dfrac{-i}{(1+|u|^2)^2}[(1+\bar{u}^2)u_{xx} - (1+u^2)\bar{u}_{xx} + 2u_x\bar{u}_x(\bar{u}-u) \\ \quad +2i\dfrac{(u\bar{u}_x + \bar{u}u_x)}{(1+|u|^2)^3}[(1+\bar{u}^2)u_x - (1+u^2)\bar{u}_x] \\ S^3_{xx} = \dfrac{-2}{(1+|u|^2)^2}[\bar{u}u_{xx} + u\bar{u}_{xx} + 2u_x\bar{u}_x] + \dfrac{4}{(1+|u|^2)^3}(\bar{u}u_x + u\bar{u}_x)^2 \end{cases}$$

$$\begin{aligned} S\cdot(S_x \wedge S_y) &= \frac{2i(u_x\bar{u}_y - \bar{u}_x u_y}{(1+|u|^2)^5}[(1-\bar{u}^2)u + (1-u^2)\bar{u})(u+\bar{u}) + \\ &+ (\bar{u}(1-u^2) - u(1-\bar{u}^2))(u-\bar{u}) + (1-|u|^2)(1-|u|^4)] \\ &= \frac{2i(u_x\bar{u}_y - \bar{u}_x u_y)}{(1+|u|^2)^2}. \end{aligned}$$

The equation satisfied by the components of S is :

$$\begin{cases} S^1_t = S^2(S^3_{xx} - S^3_{yy}) - S^3(S^2_{xx} - S^2_{yy}) + b(\phi_y S^1_x + \phi_x S^1_y) \\ S^2_t = S^3(S^1_{xx} - S^1_{yy}) - S^1(S^3_{xx} - S^3_{yy}) + b(\phi_y S^2_x + \phi_x S^2_y) \\ S^3_t = S^1(S^2_{xx} - S^2_{yy}) - S^2(S^1_{xx} - S^1_{yy}) + b(\phi_y S^3_x + \phi_x S^3_y) \end{cases}$$

If we express for example the second component using u, we find :

$$\begin{aligned} &\frac{-i}{(1+|u|^2)^2}[(1+\bar{u}^2)u_t - (1+u^2)\bar{u}_t] = \frac{1-|u|^2}{(1+|u|^2)^4}[(1+|u|^2\,(1-\bar{u}^2)u_{xx} - u_{yy}) \\ &+(1-u^2)(\bar{u}_{xx} - \bar{u}_{yy})] - 2[2(u+\bar{u})(u_x\bar{u}_x - u_y\bar{u}_y) + \bar{u}(1-\bar{u}^2)(u_x^2 - u_y^2) \\ &+u(1-u^2)(\bar{u}_x^2 - \bar{u}_y^2)] + 2\frac{(u+\bar{u})}{(1+|u|^2)^4}[\bar{u}(1+|u|^2)(u_{xx} - u_{yy}) + \\ &+u(1+|u|^2\,(\bar{u}_{xx} - \bar{u}_{yy}) + 2(1-|u|^2)(|u_x|^2 - |u_y|^2) - 2\bar{u}^2(u_x^2 - y)_y^2 \\ &-2u^2(\bar{u}_x^2 - \bar{u}_y^2)] - ib\phi_x\left\{\frac{(1+\bar{u}^2)u_y - (1+u^2)\bar{u}_y}{(1+|u|^2)^2}\right\} \\ &-ib\phi_y\left\{\frac{(1+\bar{u}^2)u_x - (1+u^2)\bar{u}_x}{(1+|u|^2)^2}\right\}. \end{aligned}$$

Writing the corresponding equation for the first component, and eliminating $\bar{u}_t$, we find

$$\begin{cases} iu_t + u_{xx} - u_{yy} = \dfrac{2\bar{u}}{1+|u|^2}(u_x^2 - u_y^2) + ib(u_x\phi_y + u_y\phi_x) \\ \phi_{xx} + \phi_{yy} = 4i\dfrac{(u_x\bar{u}_y - \bar{u}_x u_y)}{(1+|u|^2)^2} \end{cases}$$

and for equation (0.2) :

$$\begin{cases} iu_t + u_{xx} - u_{yy} = \dfrac{2\bar{u}}{1+\mid u\mid^2}(u_x^2 + u_y^2) + ib(u_x\phi_y + u_y\phi_x) \\ \phi_{xx} + \phi_{yy} = -4i\dfrac{(u_x\bar{u}_y - \bar{u}_x u_y)}{(1+\mid u\mid^2)^2} \end{cases}$$

$$u_0 = \frac{S_0^1 + iS_0^2}{1+S_0^3}.$$

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