NIKOLAY TZVETKOV

Sur le problème de Cauchy pour des équations de type KP

Thèses d'Orsay, 1999

<http://www.numdam.org/item?id=BJHTUP11_1999__0565__P0_0>

ORSAY
n^0 d'ordre : 5933

UNIVERSITÉ DE PARIS-SUD

CENTRE D'ORSAY

THÈSE

présentée
pour obtenir

Le GRADE de DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS XI ORSAY

Spécialité : Mathématiques

PAR

Nikolay TZVETKOV

Sujet : **Sur le problème de Cauchy pour des équations de type KP**

Soutenue le : 18 novembre 1999 devant la Commission d'examen

MM. Jean-Marc DELORT
Vladimir GEORGIEV
Patrick GERARD
Jean GINIBRE
Frank MERLE
Jean-Claude SAUT

Remerciements

Je tiens à exprimer ma gratitude la plus profonde envers mon directeur de recherche Jean-Claude Saut, qui guide avec beaucoup de gentillesse mon travail depuis que je suis arrivé en France. Une grande partie de ma formation en mathématiques est un résultat des nombreuses discussions que nous avons eues.

Je remercie également Vladimir Georgiev qui m'a incité à choisir l'analyse comme domaine de recherche et qui a guidé avec beaucoup de patience mes premiers pas dans la recherche alors que j'étais élève à la Faculté de Mathématiques de l'Université de Sofia.

Une partie de ce travail a été réalisée en collaboration avec Hideo Takaoka. Je le remercie vivement.

Je suis très reconnaissant à Frank Merle et Luis Vega d'avoir accepté d'être les rapporteurs.

Je remercie Jean-Marc Delort, Patrick Gérard, Jean Ginibre et Frank Merle qui ont bien voulu faire partie du jury de ma thèse.

Je remercie tous ceux qui m'ont prêté de leur temps car les entretiens que nous avons eus ont largement contribué à enrichir ma culture mathématique. En particulier, je pense à Vestislav Apostolov, Jim Colliander, Patrick Gérard, Jean Ginibre, Luis Vega.

Je remercie Danièle Le Meur pour sa gentillesse et sa disponibilité pendant ces deux années.

Abstract

We study the Cauchy problem for KP type equations by harmonic analysis techniques developed in the last decade mainly by J. Bourgain, C. Kenig, G. Ponce and L. Vega.

First we consider the Cauchy problem for the KP-II equation. We prove the local well-posedness for the KP-II equation in the anisotropic Sobolev spaces $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ for $s_1 > -1/3$, $s_2 \geq 0$. On the other hand we prove that the crucial bilinear estimate needed for the local well-posedness fails for $s_1 < -1/3$, $s_2 = 0$. The global well-posedness of the KP-II equation in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ for $s_1 > -1/310$, $s_2 \geq 0$ is also proved. The main new technical ingredient is a generalization of the bilinear estimate providing the local well-posedness.

The fifth order KP equations are also considered. We prove local and global well-posedness results for both KP-I and KP-II type equations. In the case of fifth order KP-I equation we prove the global well-posedness in the energy space despite the "bad sign" in the algebraic relation related to the symbol of the linearized operator.

Key words: Kadomtsev Petviashvili equations, Cauchy problem.

AMS subject classification: 35Q53, 35Q51, 35A07.

Résumé [1]

Dans cette thèse nous étudierons le problème de Cauchy pour les équations de Kadomtsev-Petviashvili (KP). Les équations de KP apparaissent naturellement dans plusieurs contextes physiques comme un modèle "universel" pour la propagation des longues ondes dispersives faiblement non linéaires qui sont essentiellement uni-dimensionnelles avec des effets transverses faibles. Par ailleurs, elles sont un des plus importants exemples d'un système Hamiltonien bi-dimensionnel intégrable. Dans cette thèse on utilisera des techniques d'analyse harmonique, développées ces dix dernières années par J. Bourgain, C. Kenig, G. Ponce, L. Vega pour étudier les équations de KP. En effet les équations de KP sont des généralisations en dimension deux de l'équation de Korteweg - de Vries (KdV)

$$u_t + u_{xxx} + uu_x = 0 \tag{1}$$

L'équation (1) a été obtenue par Korteweg - de Vries comme un modèle pour les longues ondes uni-dimensionnelles qui se propagent dans un canal. Parmi les solutions de (1), un rôle spécial est joué par les ondes solitaires

$$u(t,x) = 3c \,\mathrm{sech}^2 \frac{1}{2}\sqrt{c}(x-ct), \tag{2}$$

où c est la vitesse de propagation. En étudiant la stabilité des ondes solitaires (2) par rapport à des perturbations bi-dimensionnelles, Kadomtsev et Petviashvili [**33**] ont obtenu les équations suivantes en 2D

$$(u_t + \sigma u_{xxx} + uu_x)_x + u_{yy} = 0. \tag{3}$$

Le rôle du paramètre σ devient transparent si on considère (3) dans le contexte des "water waves". Le signe de σ est positif (équation KP-II) quand la tension de surface est petite ou absente (le nombre de Bond $< 1/3$). Le signe de σ est négatif (équation KP-I) quand la tension de surface domine comme dans une eau très peu profonde (le nombre de Bond $> 1/3$). Dans le cas critique quand le nombre de Bond est près de $1/3$ des termes d'ordres superieurs apparaissent dans (3). On doit dans ce cas considérer l'équation de KP d'ordre 5 (voir Kawahara [**38**])

$$(u_t + \sigma u_{xxx} + \sigma_1 u_{xxxxx} + uu_x)_x + u_{yy} = 0. \tag{4}$$

Le principal but de cette thèse est l'étude mathématique des problèmes de Cauchy associés à (3) (avec $\sigma > 0$) et à (4) pour σ et σ_1 arbitraires ($\sigma_1 \neq 0$).

Décrivons maintenant la composition de la thèse.

Dans le Chapitre 1 on presente le cadre général d'une méthode pour étudier le problème de Cauchy pour des équations d'évolution non linéaires, développée par J. Bourgain

[1]Ce résumé est essentiellement une traduction en français du Chapter 0 ci-dessous

(voir [**12, 13**]), simplifiée et améliorée par C. Kenig, G. Ponce and L. Vega (voir [**42**]). On commence par introduire les espaces de Bourgain dans une situation générale. Ensuite on démontre les estimations linéaires qui sont en fait uni-dimensionnelles et ne dépendent pas du choix d'une équation particulière. Après on montre l'estimation non linéaire dans un cas simple - l'équation de KdV avec donnée dans L^2. Finalement on présente une idée récente de J. Bourgain (voir [**16**]) pour prolonger les solutions locales globalement en temps. La méthode est basée sur une décomposition des données en basses et hautes fréquences. On utilise cette idée dans le Chapitre 4 dans le contexte de l'équation de KP-II.

Dans les Chapitres 2,3,4 et 5 on étudie le problème de Cauchy pour l'équation KP-II

$$\left\{ \begin{array}{rcl} (u_t + u_{xxx} + uu_x)_x + u_{yy} &=& 0 \\ u(0,x,y) &=& \phi, \quad (x,y) \in \mathbf{R}^2 \end{array} \right. \tag{5}$$

avec des données dans des espaces de Sobolev d'indice négatif. Dans le Chapitre 3 des généralisations en dimension trois de (5) sont aussi considérées. Dans [14], J. Bourgain démontre que le problème de Cauchy pour l'équation KP-II est localement bien posé pour des données dans L^2. La preuve est présentée surtout dans le cas des données périodiques mais peut être adaptée au cas continu. La démonstration de [14] utilise une décomposition dyadique associée au symbole de l'équation linéarisée. Dans cette thèse on présentera une approche différente qui utilise des techniques dues à C. Kenig, G. Ponce, L. Vega (cf. [**42**]), développées dans un premier temps dans le contexte de l'équation de KdV et les inégalités de Strichartz pour l'équation de KP injectées dans le cadre des espaces de Bourgain associés à l'équation de KP.

On introduit les espaces fonctionnels pour les données intiales de (5). Soit $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 \in \mathbf{R}$, $s_2 \in \mathbf{R}$ les espaces de Sobolev anisotropes munis de la norme

$$\|\phi\|_{H^{s_1,s_2}_{x,y}} = \|(1-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2}\phi\|_{L^2_{x,y}}.$$

Les espaces $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ sont naturels pour les données de (3) car leurs versions homogènes sont invariantes par les transformations de changement d'échelle qui preservent les équations de KP. On note que $H^{0,0}_{x,y}(\mathbf{R}^2) = L^2(\mathbf{R}^2)$. On introduit aussi les espaces de Sobolev anisotropes modifiés $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, munis de la norme suivante

$$\|\phi\|_{\widetilde{H}^{s_1,s_2}_{x,y}} = \|\phi\|_{H^{s_1,s_2}_{x,y}} + \|(-\partial_x^2)^{-1/2}\phi\|_{H^{s_1,s_2}_{x,y}}.$$

De façon similaire à [**48**] on note $\widetilde{H}^s(\mathbf{R}^2)$ les espaces de Sobolev modifiés, munis de la norme

$$\|\phi\|_{\widetilde{H}^s} = \|\phi\|_{H^s} + \|(-\partial_x^2)^{-1/2}\phi\|_{H^s}.$$

Dans le Chapitre 2 on démontre que le problème de Cauchy pour (5) est localement bien posé avec des données dans $\widetilde{H}^s(\mathbf{R}^2)$, $s > 0$. On se sert de l'inégalité de Strichartz pour l'équation KP-II injectée dans le cadre des espaces de Bourgain correspondant à $\widetilde{H}^s(\mathbf{R}^2)$.

Dans le Chapitre 3 on améliore le résultat du Chapitre 2. On démontre que le problème de Cauchy pour (5) est localement bien posé pour des données dans $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/4, s_2 \geq 0$. Le fait que $\phi \in \widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ signifie que $(-\partial_x^2)^{-1/2}\phi \in H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ ce qui est (comme on le verra plus tard) une restriction inutile. L'avantage des espaces de Bourgain basés sur $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ est la simplification des cas des basses fréquences dans l'estimation bilinéaire cruciale pour la preuve. Par ailleurs la taille de l'intervalle d'existence dépend de $\|\phi\|_{\widetilde{H}^{s_1,s_2}_{x,y}}$ et donc même dans le cas $s_1 = s_2 = 0$ on ne retrouve pas complétement le résultat de [14]. Dans le Chapitre 5 on présente une amélioration significative du résultat du Chapitre 3 concernant (5). Dans le Chapitre 3 une généralisation en 3D de (5) est aussi étudiée. On démontre que le problème de Cauchy pour l'équation de KP-II en 3D est localement bien posé avec des données dans $\widetilde{H}^s(\mathbf{R}^3)$, $s > 3/2$ (les espaces $\widetilde{H}^s(\mathbf{R}^3)$ sont définis similairement à $\widetilde{H}^s(\mathbf{R}^2)$). La démonstration utilise des inégalités de Strichartz pour l'équation de KP-II en 3D. Les techniques du calcul différentiel utilisées en 2D ne semblent pas être appropriées au cas de la dimension 3.

Il y a deux ingrédients principaux dans le Chapitre 4. D'abord on introduit une approche pour traiter le cas des basses fréquences dans l'estimation bilinéaire de sorte qu'il n'est plus indispensable d'utiliser les espaces $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ comme ensemble de données initiales. En exploitant la nature asymétrique de l'estimation bilinéaire on introduit un facteur supplémentaire dans la définition de l'espace restriction de Fourier ce qui nous permet de traiter les cas des basses fréquences. Il est surprenant qu'un facteur similaire soit utilisé dans [14] mais pour des raisons complètement différentes (voir Chapitre 7, Section 6, preuve de la Proposition 5 pour une application typique de l'approche de [14]). Le deuxième ingrédient dans le Chapitre 4 est la preuve d'existence d'une solution globale de (5) pour des données dans $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/310, s_2 \geq 0$. Cela généralise le résultat d'existence globale de J. Bourgain pour des données dans L^2. L'exposant $-1/310$ est de nature technique et il n'est sûrement pas optimal. La démonstration se sert de l'idée récente de J. Bourgain de décomposer les données en petits et grands modes de Fourier. Le nouvel ingrédient technique est une généralisation de l'estimation bilinéaire cruciale (voir Chapitre 4, Theorem 4.2).

Dans le Chapitre 5 on démontre que le problème de Cauchy pour (5) est localement bien posé avec des données dans $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/3, s_2 \geq 0$. La démonstration est une version raffinée des arguments du Chapitre 4. Le résultat est probalement optimal compte tenu des exemples donnés dans la deuxième partie du Chapitre (voir Chapitre 5, Theorème 4). Dans ce sens, c'est l'analogue du résultat concernant l'équation de KdV obtenu par C. Kenig, G. Ponce et L. Vega (voir [42]).

Dans les Chapitre 6 et 7 on étudie les équation de KP d'ordre 5

$$(6)\qquad \begin{cases} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} = 0 \\ u(0,x,y) = \phi(x,y) \end{cases}$$

dans le cas de la dimension 2

$$(7)\qquad \begin{cases} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} + u_{zz} = 0 \\ u(0,x,y,z) = \phi(x,y,z) \end{cases}$$

dans le cas de la dimension 3. Evidemment à l'équation de KP usuelle correspond $\beta = 0$ et $\alpha = -1$ (KP-I) ou $\alpha = +1$ (KP-II). A l'équation de KP-II d'ordre 5 correspond $\beta < 0$ et à l'équation de KP-I d'ordre 5 correspond $\beta < 0$.

Les résultats de type "bien posé" dans le Chapitre 6 sont donnés pour l'équation de KP-II d'ordre 5. Dans le cas de 2D on démontre que le problème de Cauchy pour (6) est localement bien posé pour des données dans $\widetilde{H}_{x,y}^{s_1,s_2}(\mathbf{R}^2)$, $s_1 \geq -1/4, s_2 \geq 0$. En 3D, on démontre que le problème de Cauchy pour (7) est localement bien posé pour des données dans $\widetilde{H}_{x,y}^{s_1,s_2}(\mathbf{R}^2)$, $s_1 \geq -1/8, s_2 \geq 0$. La démonstration se sert d'un effet de régularisation globale démontré dans [**5**]. Un autre effet de régularisation est une relation arithmétique pour le symbole de l'opérateur linéarisé. Cette identité a le "bon signe" seulement dans le cas de l'équation KP-II ($\beta < 0$). La deuxième partie du Chapitre 6 concerne des résultat de type "problème mal posé" pour des équation de type KP-I. La démonstration utilise essentiellement l'existence des ondes solitaires.

Dans le Chapitre 7 on continue d'étudier le problème de Cauchy pour l'équation de KP d'ordre 5. D'abord on montre que le problème de Cauchy est localement bien posé dans l'espace d'énergie pour l'équation de KP-I d'ordre 5, malgré le "mauvais signe" dans la relation algébrique liée au symbole. C'est le premier résultat de ce genre pour une équation de type KP-I. Dans le cas des données périodiques on utilise l'approche de [**14**] pour démontrer des résultats de type bien posé pour l'équation de KP-II d'ordre 5 avec des données périodiques (voir Chapitre 7, Théorèmes 4, 5).

Le Chapitre 8 est consacré à une note sur le caractère "mal posé" local de l'équation de KdV. On montre que le problème de Cauchy pour l'équation de KdV est localement malposé dans H^s, $s < -3/4$, si on impose que le flot $u(0) \longmapsto u(t)$ est de classe C^2 de $\dot{H}^s(\mathbf{R})$ dans $\dot{H}^s(\mathbf{R})$. On améliore ainsi un résultat de [**15**] où des conditions de régularité C^3 sont nécessaires. Dans [**15**] le caractère "bien posé" local n'est pas satisfait dans les directions $\phi(x)$ telles que

$$\widehat{\phi}(\xi) \sim \mathbf{1}_{[-N-\gamma,-N+\gamma]}(\xi) + \mathbf{1}_{[N-\gamma,N+\gamma]}(\xi),$$

où $\mathbf{1}_A$ est la fonction caractéristique de l'ensemble A. La relation entre la fréquence N et le paramètre γ est donnée dans [**15**] par $\gamma \sim N^{-\frac{1}{2}}$. Dans notre preuve, cette relation est remplacée par $\gamma \sim N^{-2}$, ce qui est la principale nouveauté dans notre approche.

Contents

CHAPTER 0

Motivation and description of the results

The main goal of this thesis is to study the Cauchy problem for the Kadomtsev-Petviashvili (KP) equations. The KP equations naturally occur in many physical contexts (they are "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one-directional, with weak transverse effects). On the other hand, they are one of the most important example of two dimensional Hamiltonian integrable system. In this thesis, in order to study the KP equations, we shall apply harmonic analysis techniques, developed in the last decade mainly by J. Bourgain, C. Kenig, G. Ponce, L. Vega. The KP equations are two dimensional generalizations of the celebrated Korteweg - de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0 \tag{1}$$

The equation (1) was derived by Korteweg - de Vries as a model for long waves propagating in a channel. Among the solution of (1) a special role is played by the solitary wave solution

$$u(t,x) = 3c\ \mathrm{sech}^2\frac{1}{2}\sqrt{c}(x-ct), \tag{2}$$

where c stays for the propagation speed. When studying the stability of the solitary wave solution (2) with respect to weakly two dimensional perturbation, Kadomtsev and Petviashvili [**33**] derived the following equations in 2D

$$(u_t + \sigma u_{xxx} + uu_x)_x + u_{yy} = 0. \tag{3}$$

The role of the parameter σ is transparent when considering (3) in the context of water waves. The sign of σ is positive (the KP-II equation) when the surface tension is small or absent (Bond number $< 1/3$). The sign of σ is negative (the KP-I equation) when the surface tension dominates as in very shallow water (Bond number $> 1/3$). In the critical case when the Bond number is near to 1/3 higher order terms should be taken into account in (3). Therefore in that case we are in position to consider the fifth order KP equations (cf. Kawahara [**38**])

$$(u_t + \sigma u_{xxx} + \sigma_1 u_{xxxxx} + uu_x)_x + u_{yy} = 0. \tag{4}$$

This thesis is mainly devoted to the mathematical study of the Cauchy problems associated to (3) (with $\sigma > 0$) and (4) for arbitrary σ and σ_1.

Now we describe the structure of the thesis.

In Chapter 1 we present the general framework of a method for studying the well-posedness of nonlinear evolution equations, introduced by J. Bourgain (cf. [**12, 13**]), simplified and improved by C. Kenig, G. Ponce and L. Vega (cf. [**42**]). We first introduce the

Bourgain spaces in a rather general situation. Then we prove the linear estimates, which are actually one dimensional and do not depend on the particular equation. Next we prove the needed nonlinear estimates in a simple case - the KdV equation with L^2 data. Finally we present a recent idea of J. Bourgain (cf. [16]) for extending the local solutions globally in time, based on decomposing the data into low and high Fourier modes. This idea shall be used in Chapter 4 in the context of the KP-II equation.

Chapters 2,3,4,5 are devoted to the study of the local and global well-posedness of the Cauchy problem for the KP-II equation

$$\left\{ \begin{array}{rcl} (u_t + u_{xxx} + uu_x)_x + u_{yy} & = & 0 \\ u(0,x,y) & = & \phi, \quad (x,y) \in \mathbf{R}^2 \end{array} \right. \tag{5}$$

with rough initial data (Sobolev spaces of negative indices). In Chapter 3 the three dimensional generalization of (5) is also considered. In [14], J. Bourgain proved the local well-posedness of the KP-II equation for data in L^2. The proof is mainly performed for periodic boundary conditions but could be applied to the continuous case too. The proof of [14] uses suitable dyadic decompositions associated to the symbol of the linearized equation. In this thesis we shall perform an alternative approach which uses the simple calculus techniques due to C. Kenig, G. Ponce, L. Vega (cf. [42]), first performed in the context of the KdV equation and the Strichartz inequalities for the KP equation injected into the framework of the Bourgain spaces associated to the KP equation.

In order to formulate our results we define the functional spaces where the initial data is expected to belong to. Let $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 \in \mathbf{R}$, $s_2 \in \mathbf{R}$ be the anisotropic Sobolev spaces equipped with the norm

$$\|\phi\|_{H^{s_1,s_2}_{x,y}} = \|(1-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2}\phi\|_{L^2_{x,y}}.$$

The spaces $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ are natural for the initial data of (3) since their homogeneous versions are invariant under the scale transformations preserving the KP equations. Clearly $H^{0,0}_{x,y}(\mathbf{R}^2) = L^2(\mathbf{R}^2)$. Taking into account the specific structure of the KP equations we also introduce the modified anisotropic Sobolev spaces $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, equipped with the norm

$$\|\phi\|_{\widetilde{H}^{s_1,s_2}_{x,y}} = \|\phi\|_{H^{s_1,s_2}_{x,y}} + \|(-\partial_x^2)^{-1/2}\phi\|_{H^{s_1,s_2}_{x,y}}$$

Similarly to [48] we denote by $\widetilde{H}^s(\mathbf{R}^2)$ the modified Sobolev spaces, equipped with the norm

$$\|\phi\|_{\widetilde{H}^s} = \|\phi\|_{H^s} + \|(-\partial_x^2)^{-1/2}\phi\|_{H^s}.$$

In Chapter 2 we prove the local well-posedness of (5) for data in $\widetilde{H}^s(\mathbf{R}^2)$, $s > 0$. The main tool is the Strichartz inequalities for the KP-II equations injected into the framework of the Bourgain spaces corresponding to $\widetilde{H}^s(\mathbf{R}^2)$.

In Chapter 3 we improve the result of Chapter 2, proving the local well-posedness of (5) for data in $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/4, s_2 \geq 0$. The fact that $\phi \in \widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ means that $(-\partial_x^2)^{-1/2}\phi \in H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ which is (as we will see later) an unnecessary restriction. The advantage of using the Bourgain spaces based on $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ is the trivialization of the small frequencies cases in the crucial bilinear estimate. On the other hand the size of the time interval, where the local well-posedness is obtained depends on $\|\phi\|_{\widetilde{H}^{s_1,s_2}_{x,y}}$ and therefore even in the case $s_1 = s_2 = 0$ we do not recover completely the result of [14]. In Chapter 5 we give a significant improvement of the result of Chapter 3, concerning (5). In Chapter 3 the three dimensional version of (5) is also considered. We prove the local well-posedness of the 3D KP-II equation for data in $\widetilde{H}^s(\mathbf{R}^3)$, $s > 3/2$ (the spaces $\widetilde{H}^s(\mathbf{R}^3)$ are defined similarly to $\widetilde{H}^s(\mathbf{R}^2)$). The proof uses only the Strichartz inequalities for the 3D KP-II equations. The simple calculus techniques used in 2D do not seem to be appropriated in the case of three spatial dimensions.

There are two main ingredients in Chapter 4. First we show how to treat the small frequencies cases in the bilinear estimates and thus we are not obliged to use the spaces $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ for the initial data. Using the asymmetric nature of the crucial bilinear estimates we introduce an additional factor in the definition of the Fourier transform restriction spaces which helps us to deal with the small frequencies cases. Surprisingly a similar factor is used in [14] but for different purposes (cf. Chapter 7, Section 6, the proof of Proposition 5 for a typical application of the approach of [14]). The second main ingredient of Chapter 4 is the proof of the global well-posedness of (5) for data in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/310, s_2 \geq 0$. This generalizes the L^2 global well-posedness result of [14]. The exponent $-1/310$ is of technical nature and is surely not optimal. The proof uses an idea due to J. Bourgain of decomposing the initial data into high and low Fourier modes. The main new technical ingredient is a generalization of the bilinear estimate providing the local well-posedness (cf. Chapter 4, Theorem 4.2).

In Chapter 5 we prove the local well-posedness of (5) for data in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/3, s_2 \geq 0$. The proof is a refinement of the arguments of Chapter 4. The result seems to be optimal in the sense that the key estimate for the local well-posedness fails for $s_1 < -1/3$, $s_2 = 0$ (cf. Chapter 5, Theorem 4). In this aspect, it is the counterpart of the $H^s(\mathbf{R})$, $s > -3/4$ local well-posedness for the KdV equation proved by C. Kenig, G. Ponce and L. Vega in [42].

Chapters 6 and 7 are devoted to the study of the fifth order KP equation

$$\left\{ \begin{array}{c} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} = 0 \\ u(0,x,y) = \phi(x,y) \end{array} \right. \tag{6}$$

in the two dimensional case and

$$\left\{ \begin{array}{c} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} + u_{zz} = 0 \\ u(0,x,y,z) = \phi(x,y,z) \end{array} \right. \tag{7}$$

in the three dimensional case. Clearly "usual" KP equations correspond to $\beta = 0$ and $\alpha = -1$ (KP-I) or $\alpha = +1$ (KP-II). Further the fifth order KP-II equation corresponds to $\beta < 0$, while the fifth order KP-I corresponds to $\beta > 0$.

The well-posedness results of Chapter 6 are devoted to the fifth order KP-II equations. In 2D we prove the local well-posedness of (6) for data in $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 \geq -1/4, s_2 \geq 0$. In 3D we prove the local well-posedness of (7) for data in $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 \geq -1/8, s_2 \geq 0$. The proof makes use of a global smoothing effect for the linearized equation established in [5]. Another smoothing effect essentially used in the proof is an arithmetic identity involving the symbol of the linearized operator. This identity has "good" sign only in the case KP-II equation ($\beta < 0$). The second part of Chapter 6 is devoted to the proof of ill-posedness results for some KP-I type equations. The proof relays on the existence of solitary wave solutions.

In Chapter 7 we continue the study of the fifth order KP equations. We first prove the global well-posedness in the energy space of the fifth order KP-I equation despite the "bad sign" in the algebraic relation related to the symbol. This result is the first of this kind for KP-I type equations. The proof uses the global smoothing effect for the linearized equation from **[5]** and the simple calculus techniques due to C. Kenig, G. Ponce and L. Vega. Then we turn to the case of periodic boundary conditions. The difficulty in that case is that the free evolution does not possess any dispersive property. Here we use the approach from **[14]** in order to obtain the local well-posedness for the periodic fifth order KP-II equation with rough data (cf. Chapter 7, Theorems 4, 5).

Chapter 8 is devoted to a remark on the local ill-posedness for the KdV equation. We prove that the Cauchy problem for KdV equation is locally ill-posed in H^s, $s < -3/4$ if one asks the flow map $u(0) \longmapsto u(t)$ to be C^2 from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$. This is a slight improvement of a result from **[15]**, where C^3 regularity is needed. In **[15]** the local well-posedness fails along the directions $\phi(x)$ such that

$$\widehat{\phi}(\xi) \sim \mathbf{1}_{[-N-\gamma,-N+\gamma]}(\xi) + \mathbf{1}_{[N-\gamma,N+\gamma]}(\xi),$$

where $\mathbf{1}_A$ is the characteristic function of the set A. The relation between the frequency N and the parameter γ is $\gamma \sim N^{-\frac{1}{2}}$. In our proof the relation between γ and N in an example similar to **[15]** is $\gamma \sim N^{-2}$, which is the new point.

CHAPTER 1

Preliminaries

1. Local existence

We shall give an idea for the method developed by J. Bourgain for studying the local well-posedness of nonlinear evolution equations (we also refer to the expository paper [**23**] for a more complete presentation). The method is a rather general scheme and can be applied for equations of type

$$(1) \qquad \begin{cases} iu_t &= Au + f(u) \\ u(0) &= u_0, \end{cases}$$

where A is a self-adjoint operator acting on a Hilbert space. In most cases A is a differential operator (or a Fourier multiplier) defined on a classical Sobolev space. The term $f(u)$ represent the nonlinear interaction and u_0 is the initial data. Several examples could be injected into the previous framework:
- $A = \Delta$, where Δ is the Laplace operator in $\mathbf{R}^n$, which corresponds to a nonlinear Schrödinger equation .
- $A = i\partial_x^3 \quad x \in \mathbf{R}$, $f(u) = -iuu_x$, which corresponds to the Korteweg de Vries equation.
- $A = i\partial_x^3 \pm i\partial_x^{-1}\partial_y^2$, $\quad (x, y) \in \mathbf{R}^2$, $f(u) = -iuu_x$, which corresponds to the Kadomtsev-Petviashvili equations.
- $A = \sqrt{-\Delta}$, which corresponds to a nonlinear wave equation.
- $A = \text{diag}\{-\Delta, \sqrt{-\Delta}\}$, where diag stays for the diagonal matrix, $f(u) = (u_1u_2, \Delta|u|^2)$, which corresponds to the Zakharov system.

Actually we shall solve the integral equation corresponding to (1). Let ψ be a cut-off function such that $\psi \in C_0^\infty(\mathbf{R})$, supp $\psi \subset [-2, 2]$, $\psi = 1$ on the interval $[-1, 1]$. Consider a truncated version of the integral equation corresponding to (1)

$$(2) \qquad u(t) = \psi(t)U(t)u_0 - i\psi(t)\int_0^t U(t-t')f(u(t'))dt',$$

where $U(t) = \exp(-itA)$ define the free evolution of the system. Sometimes we change $\psi(t)$ by $\psi(t/T)$ in the integral term and we look for solutions on the time interval $[-T, T]$. Clearly to a global solution of (2) corresponds a local one of (1) on the interval $[-1, 1]$ (respectively $[-T, T]$). Now we define the Bourgain spaces. These are the spaces B equipped with the following norms $\|u\|_B = \|U(-t)u\|_X$, where X stays for a classical functional space, for example a Sobolev space. We shall solve (2) by a fixed point argument in B. Suppose that the initial data belongs to Y and the space X be of the form $X^b = Y(H_t^b)$, $b \in \mathbf{R}$. We denote by B^b the Bourgain space corresponding to X^b. Then we have the next result.

THEOREM 1. (linear estimates)

$$\|\psi(t)U(t)\phi\|_{B^b} \lesssim \|\phi\|_Y, \quad \phi \in Y. \tag{3}$$

$$\left\|\psi(t)\int_0^t U(t-t')f(u(t'))dt'\right\|_{B^b} \lesssim \|f(u)\|_{B^{-b'}}, \tag{4}$$

where $b'+b \le 1,\ b > 1/2,\ b' \ge 0.$

Proof. The estimates are in fact one dimensional (with respect to time) and do not depend on a particular choice of the unitary group $U(t)$. The proof of (3) is a direct consequence of the definition. It suffices to note that $\psi(t)$ and $U(t)$ commute since the operator $U(t)$ is a multiplier operator with respect to time. In order to prove (4), we note that it is equivalent to

$$\left\|\psi(t)\int_0^t v(t')dt'\right\|_{H^b_t} \lesssim \|v\|_{H^{-b'}}. \tag{5}$$

Since $b+b' \le 1$, the inequality (5) affirms that "by integrating we gain a derivative". Using Fourier transform we obtain

$$\begin{aligned}
\psi(t)\int_0^t v(t')dt' &= \psi(t)\int_{-\infty}^{\infty} \frac{e^{it\tau}-1}{i\tau}\widehat{v}(\tau)d\tau \\
&= \psi(t)\int_{|\tau|\le 1} \frac{e^{it\tau}-1}{i\tau}\widehat{v}(\tau)d\tau + \psi(t)\int_{|\tau|\ge 1}\frac{e^{it\tau}}{i\tau}\widehat{v}(\tau)d\tau - \psi(t)\int_{|\tau|\ge 1}\frac{\widehat{v}(\tau)}{i\tau}d\tau \\
&:= I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

We shall now estimate the three terms. First

$$\|I_1(t)\|_{H^b_t} \le \sum_{k\ge 1}\frac{1}{k!}\|t^k\psi(t)\|_{H^b_t}\|v\|_{H^{-b'}}\left\{\int_{|\tau|\le 1}\langle\tau\rangle^{2b'}d\tau\right\}^{\frac{1}{2}} \lesssim \|v\|_{H^{-b'}}.$$

Further we have

$$\|I_2(t)\|_{H^b_t} \lesssim \left\|\int_{|\tau|\ge 1}\frac{e^{it\tau}}{i\tau}\widehat{v}(\tau)d\tau\right\|_{H^b_{t,loc}} \lesssim \left\|\int_{|\tau|\ge 1}\frac{e^{it\tau}}{i\tau}\widehat{v}(\tau)d\tau\right\|_{H^b_t} \lesssim \|v\|_{H^{b-1}} \lesssim \|v\|_{H^{-b'}}$$

It remains to estimate $I_3(t)$. Using Cauchy-Schwarz inequality, we obtain for $b > 1/2$

$$\begin{aligned}
\|I_3(t)\|_{H^b_t} &\le \|\psi(t)\|_{H^b_t}\int_{-\infty}^{\infty}\frac{|\widehat{v}(\tau)|}{\langle\tau\rangle}d\tau \\
&\le \|\psi(t)\|_{H^b_t}\left\{\frac{d\tau}{\langle\tau\rangle^{2b}}\right\}^{\frac{1}{2}}\left\{\frac{|\widehat{v}(\tau)|^2d\tau}{\langle\tau\rangle^{2(1-b)}}\right\}^{\frac{1}{2}} \\
&\lesssim \|\psi(t)\|_{H^b_t}\|v\|_{H^{b-1}} \\
&\lesssim \|v\|_{H^{-b'}}, \quad \text{since } b+b' \le 1.
\end{aligned}$$

This completes the proof of Theorem 1.
There are different variants of the estimate (4). For example if we wish to take into account the factor T of the cut-off function then in **[23]** the following estimate is proven

$$\|\psi(t/T)\int_0^t U(t-t')f(u(t'))dt'\|_{B^b} \lesssim T^{1-b-b'}\|f(u)\|_{B^{-b'}}, \quad b+b' \leq 1,\ b' < 1/2.$$

Sometimes one needs to take $b = 1/2$ in the definition of the Bourgain's spaces. For example, this is the case of the periodic KdV equation. In this case the terms $I_1(t)$ and $I_2(t)$ can be treated in the same fashion as above. A logarithmic singularity appears in the estimate for $I_3(t)$. Using the above arguments one can easily obtain

$$\left\|\psi(t)\int_0^t U(t-t')f(u(t'))dt'\right\|_{B^{\frac{1}{2}}} \lesssim \left(\|f(u)\|_{B^{-\frac{1}{2}}} + \|f(u)\|_{Z^{-\frac{1}{2}}}\right),$$

where the space $Z^{-\frac{1}{2}}$ is equipped with the norm

$$\|u\|_{Z^{-\frac{1}{2}}} = \|U(-t)u\|_{Y\left(W_t^{1,-\frac{1}{2}}\right)}$$

and $W_t^{1,-\frac{1}{2}}$ stays for the Sobolev space with respect to time based on L^1 and measuring regularity of order $-1/2$. For more details we refer to **[23, 24]**.

In order to apply a fixed point argument we have to prove the following estimate

$$\|f(u)\|_{B^{-b'}} \lesssim \|u\|_{B^b}^{\lambda}, \tag{6}$$

where λ is the order of the nonlinearity. The estimate (6) is called "nonlinear estimate" or "bilinear estimate" in the case of a quadratic nonlinearity.

Here we shall consider a simple case when an estimate of type (6) holds. More precisely we shall prove (6) when

$$f(u) = iuu_x, \quad Y = L^2(\mathbf{R}), \quad A = i\partial_x^3, \quad b = \frac{1}{2}+ \quad b' = \frac{1}{2}-. \tag{7}$$

From this version of (6) together with Theorem 1 and a Picard fixed point argument will result the local well-posedness of the KdV equation for data in $L^2(\mathbf{R})$. Thanks to the conservation of the L^2 norm one extends the solutions globally in time. Note that assuming (7), the norm of the Bourgain spaces B^b has the form

$$\|u\|_{B^b} = \|\langle\tau+\xi^3\rangle^b\widehat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}$$

A duality argument shows that when (7) holds the estimate (6) has the form

$$J \lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}, \tag{8}$$

where

$$J = \left|\int\int \frac{|\xi|f(\tau_1,\xi_1)g(\tau-\tau_1,\xi-\xi_1)h(\tau,\xi)}{\langle\tau+\xi^3\rangle^{\frac{1}{2}-}\langle\tau_1+\xi_1^3\rangle^{\frac{1}{2}+}\langle\tau-\tau_1+(\xi-\xi_1)^3\rangle^{\frac{1}{2}+}}d\tau d\xi d\tau_1 d\xi_1\right|$$

We can assume that f, g, h are positive. A use of the Cauchy-Schwarz inequality yields

$$J \leq \int I(\tau,\xi)\left\{\int\int |f(\tau_1,\xi_1)g(\tau-\tau_1,\xi-\xi_1)|^2 d\tau_1 d\xi_1\right\}^{1/2} h(\tau,\xi)d\tau d\xi,$$

where

$$I(\tau,\xi)=\frac{|\xi|}{\langle\tau+\xi^3\rangle^{\frac12-}}\left(\int\frac{d\tau_1 d\xi_1}{\langle\tau_1+\xi_1^3\rangle^{1+}\langle\tau-\tau_1+(\xi-\xi_1)^3\rangle^{1+}}\right)^{1/2}.$$

Further we easily obtain

$$I(\tau,\xi)\lesssim\frac{|\xi|}{\langle\tau+\xi^3\rangle^{\frac12-}}\left(\int\frac{d\xi_1}{\langle\tau+\xi_1^3+(\xi-\xi_1)^3\rangle^{1+}}\right)^{1/2}.$$

Performing the change of the variable $\xi_1\longmapsto\mu$, where $\mu=\tau+\xi_1^3+(\xi-\xi_1)^3$ we arrive at

$$\begin{aligned}I(\tau,\xi) &\lesssim \frac{|\xi|^{3/4}}{\langle\tau+\xi^3\rangle^{\frac12-}}\left(\int\frac{d\mu}{\langle\mu\rangle^{1+}|4\tau+\xi^3-4\mu|^{\frac12}}\right)^{1/2}\\ &\lesssim \frac{|\xi|^{3/4}}{\langle\tau+\xi^3\rangle^{\frac12-}\langle4\tau+\xi^3\rangle^{\frac14}}\\ &\lesssim \text{const.}\end{aligned}$$

Hence $I(\tau,\xi)$ is bounded and another use of the Cauchy-Schwarz inequality completes the proof of (8). Now it remains to use a standard fixed point argument in order to prove that the initial value problem for the KdV equation is locally well-posed for data in L^2. The above arguments are further developed in **[42]** where the bilinear estimate is proven for the Bourgain spaces based on $H^s(\mathbf{R})$, $s>-3/4$ yielding the local well-posedness for the Cauchy problem for KdV equation for data in $H^s(\mathbf{R})$, $s>-3/4$. This result is sharp due to the counter examples constructed in **[42]**.

2. Global existence

We restrict our considerations to the following case

$$\text{(9)}\qquad \begin{cases} iu_t &= p(D)u+f(u)\\ u(0) &= \phi,\end{cases}$$

where $p(D)$ is a Fourier multiplier acting on the classical Sobolev spaces H^s. Once we have a local well-posedness in a suitable space we can extend the solutions globally by the aid of the conservation laws due to the symmetries of (9). Here we shall give an idea for the method developed by J. Bourgain to prove the global well-posedness in cases when the conservation laws are not directly available. Let $H(u(t))=H(\phi)$ be a conservation law. Actually we shall consider only the case $H(\phi)=\|\phi\|_{L^2}$. We shall try to construct global solutions for data below L^2. Note that there is no conservation of the Sobolev norms below L^2 for (9). We suppose that a local well-posedness below L^2 could be proven. Let $\phi\in H^{-s}$, with $s\geq0$. Consider the following decomposition

$$\phi=\phi_0+\eta_0,$$

where ϕ_0 is the spectral projector on the interval $[-N,N]$. Here $N\gg1$. Clearly

$$\|\phi\|_{L^2}\sim N^s,\quad \|\eta_0\|_{H^{-\sigma}}\sim N^{s-\sigma}.$$

Let $B^{b,-s}$ be the Bourgain space associated to $H^{-s}(H_t^b)$. Consider the following system

$$(10) \qquad \begin{cases} iv_t &= p(D)v + f(v) \\ iw_t &= p(D)w + f(v+w) - f(v) \\ (v(0), w(0)) &= (\phi_0, \eta_0) \in L^2 \times H^{-\sigma} \end{cases}$$

The next Theorem could be proven by using the techniques described in the previous section.

THEOREM 2. (local existence) *There exists* $(u,v) \in B^{\frac{1}{2}+,0} \times B^{\frac{1}{2}+,-\sigma}$, *a unique local solution of (10) on the time interval* $[-\delta, \delta]$, *where* $\delta \sim N^{-\alpha s}$, $\alpha > 0$ *and in addition*

$$\|v\|_{B^{\frac{1}{2}+,0}} \leq cN^{\beta}, \quad \beta > 0,$$

$$\|w\|_{B^{\frac{1}{2}+,-s}} \leq cN^{-\gamma}, \quad \gamma > 0.$$

Write

$$w(t) = U(t)\eta_0 + r(t).$$

The new estimate is the following.

THEOREM 3. *For* $t \in [0, \delta]$, *we have*

$$\begin{aligned} \|r(t)\|_{L^2} &\leq c\left(\|v\|_{B^{\frac{1}{2}+,0}}\|w\|_{B^{\frac{1}{2}+,-s}} + \|w\|^2_{B^{\frac{1}{2}+,-s}}\right) \\ &\leq cN^{\beta-\gamma} \end{aligned}$$

We can easily obtain the following corollary.

THEOREM 4. *Suppose that Theorem 2 and Theorem 3 are proven and* $s+\gamma-\beta-\alpha s > 0$. *Then (9) is globally well-posed for data in* H^{-s}.

Proof. Let δ be the size of the existence interval from Theorem 2. Set

$$\phi_1 = u(\delta) + r(\delta), \quad \eta_1 = U(\delta)\eta_0.$$

Since $U(t)$ is an isometry on the classical Sobolev spaces we have

$$\|\eta_1\|_{H^{-\sigma}} = \|\eta_0\|_{H^{-\sigma}}.$$

Further we have

$$\begin{aligned} \|\phi_1\|_{L^2} &\leq \|u(\delta)\|_{L^2} + \|r(\delta)\|_{L^2} \\ &\leq \|\phi_0\|_{L^2} + \|r(\delta)\|_{L^2} \\ &\leq \|\phi_0\|_{L^2} + cN^{\beta-\gamma}. \end{aligned}$$

Hence the L^2 norm increase as $N^{\beta-\gamma}$. We would like to iterate the process $M(N)$ times. We need that

$$M(N)N^{\beta-\gamma} \sim N^s, \text{ i.e. } M(N) \sim N^{s-\beta+\gamma}.$$

Thus we obtain an interval of size

$$M(N)\delta \sim N^{s+\gamma-\beta-\alpha s}.$$

Since $s+\gamma-\beta-\alpha s > 0$ and $N >> 1$, we can finally achieve any interval.

CHAPTER 2

Remarque sur la régularité locale de l'équation de Kadomtsev-Petviashvili-II

This Chapter essentially contains the paper [**56**] *(Remarque sur la régularité locale de l'équation de Kadomtsev-Petviashvili, C.R. Acad. Sci. Paris, t. 326, Série I, p. 709-712, 1998).*

Résumé. On donne une démonstration courte d'une partie d'un résultat de Bourgain [**14**]. Plus précisement, on démontre que le problème de Cauchy pour l'équation de Kadomtsev-Petviashvili-II est localement bien posé dans des espaces de Sobolev d'indice strictement positif.

Abstract. We give a short proof of a part of the Bourgain's result from [**14**]. More precisely we prove that the Cauchy problem for the Kadomtsev-Petviashvili equation is locally well posed in Sobolev spaces with positive index.

1. Introduction

On considère le problème de Cauchy pour l'équation de Kadomtsev-Petviashvili-II

$$(1)\qquad \begin{cases} (u_t + u_{xxx} + uu_x)_x + u_{yy} = 0, \\ u(0,x,y) = \phi. \end{cases}$$

En utilisant une méthode initialement appliquée aux équations de Schrödinger non-linéaire et KdV ([**12, 13**]), Bourgain [**14**] démontre que le problème (1) est localement bien posé pour des données initiales dans $H^s(\mathbf{T}^2)$, $s \geq 0$, $\mathbf{T}^2 = \mathbf{R}^2|\mathbf{Z}^2$. La preuve peut être adaptée aux données dans $\mathbf{R}^2$. Bourgain [**14**] utilise une décomposition dyadique associée à la structure d'équation (1). La preuve n'utilise pas les propriétés dispersives de l'équation linéarisée dans le cas des données dans $\mathbf{R}^2$. Ici, on donne une démonstration courte d'une partie du résultat de Bourgain qui est basée sur les propriétés dispersives de l'équation (1). On écrit (1) sous la forme

$$(2)\qquad \begin{cases} i\partial_t u = p(D_1, D_2)u - iuu_x, \\ u(0,x,y) = \phi, \end{cases}$$

où $D_1 = -i\partial_x$, $D_2 = -i\partial_y$ et l' opérateur $p(D_1, D_2)$ est défini par la transformation de Fourier $\mathcal{F}(p(D_1,D_2)u)(\zeta) = p(\zeta)\hat{u}(\zeta)$. On note par $\zeta = (\xi, \eta)$ les variables conjuguées de Fourier et $p(\zeta) = -\xi^3 + \eta^2/\xi$. On note $H^{b,s}(\mathbf{R}^3)$ l'espace de Sobolev classique muni de la norme

$$\|u; H^{b,s}(\mathbf{R}^3)\| = \|\langle\tau\rangle^b\langle\zeta\rangle^s\hat{u}(\tau,\zeta); L^2_{(\tau,\zeta)}\|,$$

où $\langle.\rangle = 1 + |.|$. On note $\widetilde{H}^{b,s}(\mathbf{R}^3)$ l'espace de Sobolev modifié (voir [**48**]) muni de la norme

$$\|u; \widetilde{H}^{b,s}(\mathbf{R}^3)\| = \|u; H^{b,s}(\mathbf{R}^3)\| + \||D_1|^{-1}u; H^{b,s}(\mathbf{R}^3)\|.$$

On definit de même manière l'espace $\widetilde{H}^s(\mathbf{R}^2)$ et on note $\widetilde{L}^2$ l'espace $\widetilde{H}^0$. On peut remarquer que quelque soit $u \in \tilde{H}^{b,s_1,s_2}$, on a $\int u(t,x,y)dx = 0$. Soit $U(t) = \exp(-itp(D_1,D_2))$ le groupe unitaire qui définit l'évolution libre de (2). On va résoudre (2) dans l'espace de Bourgain $B^{b,s}$ muni de la norme $\|u; B^{b,s}\| = \|U(-t)u; \widetilde{H}^{b,s}\|$. Comme $\mathcal{F}(U(-t)u)(\tau,\zeta) = \hat{u}(\tau - p(\zeta),\zeta)$, on obtient que la norme dans $B^{b,s}$ s'écrit sous la forme

$$\|u; B^{b,s}\|^2 = \int \langle \tau + p(\zeta)\rangle^{2b}(1+|\xi|^{-1})^2\langle\zeta\rangle^{2s}|\hat{u}(\tau,\zeta)|^2 d\tau d\zeta.$$

On a le théorème suivant.

THÉORÈME 1. *Soit $s > 0$. Alors quelque soit $\phi \in \widetilde{H}^s$, il existe $T = T(\|\phi\|_{\widetilde{H}^s})$ positive et $u(t,x,y)$, une solution unique de* (1) *dans l'intervalle* $[-T,T]$ *telle que* $u \in C([-T,T]; \widetilde{H}^s(\mathbf{R}^2)) \cap B^{1/2,s}$.

On va résoudre une version tronquée de l'équation (2)

$$u(t) = \psi(t)U(t)u(0) - \frac{1}{2}\psi(t/T)\int_0^t U(t-t')\partial_x(\psi^2(t'/2T)u^2(t'))dt'. \tag{3}$$

où ψ est une troncature telle que $\psi \in C_0^\infty(\mathbf{R})$, supp $\psi \subset [-2,2]$, $\psi = 1$ sur l'intervalle $[-1,1]$. Aux solutions globales en temps de (3) correspondent des solutions locales de (1) sur l'intervalle $[-T,T]$. On a l'estimation suivante pour l'évolution libre

$$\|\psi(t)U(t)\phi; B^{b,s}\| \le c\|\phi; \widetilde{H}^s\|.$$

Pour estimer le terme intégral de (3), on utilise la Proposition suivante.

PROPOSITION 1. *(voir* [24]*, Lemme 2.1). Il existe $c > 0$ telle que*

$$\left\|\psi(t/T)\int_0^t U(t-t')\partial_x(u^2(t'))dt'; B^{1/2,s}\right\| \le c\{\|uu_x; B^{-1/2,s}\| + \|uu_x; Y^s\|\},$$

où Y^s est muni de la norme

$$\|u; Y^s\| = \|\langle\tau\rangle^{-1}\langle\zeta\rangle^s(1+|\xi|^{-1})\mathcal{F}(U(-t)u); L^2_\zeta(L^1_\tau)\|.$$

L'estimation non-linéaire est la suivante.

PROPOSITION 2. *Soit $s > 0$ et supp $u \subset \{(t,x) : |t| \le cT\}$. Alors il existe $\gamma > 0$ tel que on a les inégalités suivantes*

$$\|uu_x; B^{-1/2,s}\| \le cT^\gamma\|u; B^{1/2,s}\|^2, \tag{4}$$

$$\|uu_x; Y^s\| \le cT^\gamma\|u; B^{1/2,s}\|^2. \tag{5}$$

On va essentiellement utiliser l'estimation suivante.

PROPOSITION 3. *(*inégalité de Strichartz, voir [24], Lemme 3.1 et [48], Proposition 2.3*). Soit*

$$0 \le \gamma \le 1, \quad 0 < \theta \le 1, \quad \epsilon_1 > 0, \quad supp\ u \subset \{(t,x) : |t| \le cT\}.$$

Alors, quelque soit $u \in \widetilde{L}^2(\mathbf{R}^3)$, on a l'inégalité suivante

$$\|\mathcal{F}^{-1}(\langle\tau + p(\zeta)\rangle^{-1/2}|\hat{u}(\tau,\zeta)|); L^q_t(L^r_{(x,y)})\| \le cT^{\gamma^\star/2}\|u\|_{L^2}, \tag{6}$$

où $\gamma^\star = \gamma - 0$ *et* q, r *sont telles que*

$$\frac{2}{q} = 1 - \frac{\theta(1-\gamma)}{1+\epsilon_1}, \quad \delta(r) := 2(\frac{1}{2} - \frac{1}{r}) = \frac{(1-\theta)(1-\gamma)}{1+\epsilon_1}.$$

2. Démontration de l'estimation non-linéaire

On pose $\sigma = \sigma(\tau,\zeta) = \tau + p(\zeta)$, $\sigma_1 = \sigma(\tau_1,\zeta_1)$, $\sigma_2 = \sigma(\tau-\tau_1,\zeta-\zeta_1)$ et

$$\hat{w}(\tau,\zeta) = \langle \tau + p(\zeta)\rangle^{1/2}\langle\zeta\rangle^s \hat{u}(\tau,\zeta).$$

Par dualité, on écrit (4) sous la forme

$$\left|\int\!\!\int\!\!\int K(\tau,\zeta,\tau_1,\zeta_1)\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)\hat{v}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta\right| \leq$$

$$cT^\gamma\{\|w\|_{L^2}^2 + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|_{L^2}^2\}\|v\|_{L^2}, \tag{7}$$

où
$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{\langle\xi\rangle\langle\zeta\rangle^s\langle\zeta_1\rangle^{-s}\langle\zeta-\zeta_1\rangle^{-s}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\sigma\rangle^{1/2}}.$$

On peut supposer que $\hat{w} \geq 0$ et $\hat{v} \geq 0$. On a la relation suivante

$$\sigma_1 + \sigma_2 - \sigma = 3\xi_1\xi(\xi-\xi_1) + \frac{(\xi_1\eta - \xi\eta_1)^2}{\xi_1\xi(\xi-\xi_1)}.$$

Donc $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \geq |\xi_1\xi(\xi-\xi_1)|$. Par symétrie, on suppose que $|\sigma_1| \geq |\sigma_2|$. On note J le membre de gauche de l'inégalité (7). On considère trois cas principaux.

Cas 1. $|\sigma| \geq |\sigma_1|, |\xi| \geq 2$. On note J_1 la restriction de J sur cette région. On peut supposer que $|\zeta| \leq 2|\zeta_1|$. Si $|\zeta| \geq 2|\zeta_1|$, alors $|\zeta| \leq 2|\zeta - \zeta_1|$ et les arguments sont les mêmes. Si $|\xi_1| \geq 1$ et $|\xi - \xi_1| \geq 1$, alors $|\xi|\langle\sigma\rangle^{-1/2} \leq c^{te}$ et on obtient en utilisant les inégalités de Hölder, Strichartz et Sobolev

$$\begin{aligned}
J_1 &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-1/2}\hat{w}(\tau_1,\zeta_1)); L_t^4(L_{(x,y)}^{r_1})\| \\
&\quad \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2}\langle\zeta-\zeta_1\rangle^{-s}\hat{w}(\tau-\tau_1,\zeta-\zeta_1)); L_t^4(L_{(x,y)}^{r_2})\|\|v\|_{L^2} \\
&\leq cT^{\gamma^\star/2}\|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1,\zeta-\zeta_1)); L_t^4(L_{(x,y)}^{r_2'})\|\|w\|_{L^2}\|v\|_{L^2} \\
&\leq cT^{\gamma^\star}\|w\|_{L^2}^2\|v\|_{L^2},
\end{aligned}$$

à condition que $\delta(r_1) = \delta(r_2') = \frac{1-\gamma}{1+\epsilon_1} - \frac{1}{2}$, $\delta(r_1) + \delta(r_2) = 1$, $s \geq \delta(r_2) - \delta(r_2')$. On pose $\gamma = \epsilon$ et la restriction de l'injection de Sobolev devient $s \geq \frac{2\epsilon+2\epsilon_1}{1+\epsilon_1}$, qui est satisfaite pour $s > 0$ et ϵ, ϵ_1 suffisamment petits. Si $|\xi_1| \leq 1$, alors $|\xi|\langle\sigma\rangle^{-1/2} \leq c^{te}|\xi_1|^{-1}$ et les arguments sont les mêmes en utilisant $\||D_1|^{-1}w\|_{L^2}$ à la place de $\|w\|_{L^2}$ où cela est nécessaire.

Cas 2. $|\sigma_1| \geq |\sigma|, |\xi| \geq 2$. On note J_2 la restriction de J sur cette région. Si $|\zeta| \leq 2|\zeta_1|$, alors on peut appliquer les arguments du cas 1. Soit $|\zeta| \geq 2|\zeta_1|$. Si $|\xi_1| \geq 1$ et $|\xi - \xi_1| \geq 1$, alors $|\xi|\langle\sigma_1\rangle^{-1/2} \leq c^{te}$ et en utilisant les inégalités de Hölder, Strichartz et Sobolev, on obtient

$$\begin{aligned}
J_2 &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2}\hat{v}(\tau,\zeta)); L_t^4(L_{(x,y)}^{r_1})\|\|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1,\zeta-\zeta_1)); L_t^4(L_{(x,y)}^{r_1})\| \\
&\quad \|\mathcal{F}^{-1}(\langle\zeta_1\rangle^{-s}\hat{w}(\tau_1,\zeta_1)); L_t^2(L_{(x,y)}^{r_2})\| \\
&\leq cT^{\gamma^\star}\|w\|_{L^2}^2\|v\|_{L^2},
\end{aligned}$$

à condition que $\delta(r_1) = \frac{1-\gamma}{1+\epsilon_1} - \frac{1}{2}$, $2\delta(r_1) + \delta(r_2) = 1$, $s \geq \delta(r_2)$. On pose $\gamma = \epsilon$ et la restriction de l'injection de Sobolev devient $s \geq \frac{2\epsilon+2\epsilon_1}{1+\epsilon_1}$, qui est satisfaite pour $s > 0$ et ϵ, ϵ_1 suffisamment petits. Si $|\xi_1| \leq 1$, alors $|\xi|\langle\sigma_1\rangle^{-1/2} \leq c^{te}|\xi_1|^{-1}$ et les arguments sont les mêmes.

Cas 3. $|\xi| \leq 2$. Dans ce cas, on a $|\xi|\langle\sigma\rangle^{-1/2} \leq c^{te}$ et on peut utiliser les arguments du cas 1.

On remarque que $w \in \widetilde{L}^2$ et on peut donc utiliser l'inégalité de Strichartz avec w dans tout les cas précédents. On a utilisé l'inégalité de Strichartz avec v seulement si $|\xi| \geq 2$, car dans ce cas on peut supposer que la fonction v est telle que $\hat{v}(\tau, 0, \eta) = 0$ près de $\xi = 0$ et donc qu'elle appartient à $\widetilde{L}^2$. Par dualité, on écrit (5) sous la forme

$$|\int\int K_1(\tau,\zeta,\tau_1,\zeta_1)\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)\hat{v}(\zeta)d\tau_1 d\zeta_1 d\tau d\zeta| \leq$$

$$cT^{\gamma}\{\|w\|^2_{L^2} + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|^2_{L^2}\}\|v\|_{L^2}, \tag{8}$$

où
$$K_1(\tau,\zeta,\tau_1,\zeta_1) = \frac{\langle\xi\rangle\langle\zeta\rangle^s\langle\zeta_1\rangle^{-s}\langle\zeta-\zeta_1\rangle^{-s}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\sigma\rangle}.$$

Si $|\xi| \leq 2$, alors on utilise les mêmes arguments que dans la preuve de (4) dans le cas 1 en remplacant $\hat{v}(\tau,\zeta)$ par $\langle\sigma\rangle^{-1}\hat{v}(\zeta) \in L^2_{(\tau,\zeta)}$. Si $|\sigma_1| \geq |\sigma|$ on utilise les arguments de la preuve de (4) en remplacant $\hat{v}(\tau,\zeta)$ avec $\langle\sigma\rangle^{-1/2-\epsilon}\hat{v}(\zeta) \in L^2_{(\tau,\zeta)}$ pour ϵ suffisamment petit. Si $|\sigma| \geq |\sigma_1|$ et $|\xi| \geq 2$, on a alors que $|\sigma| \geq c|\xi|^2|\xi_1|$, lorsque $|\xi_1| \leq 1$ et $|\sigma| \geq c|\xi|^2$, lorsque $|\xi_1| \geq 1$. Il reste à remarquer que

$$\frac{\langle\xi\rangle\hat{v}(\zeta)}{\langle\sigma\rangle + |\xi|^2} \in L^2_{(\tau,\zeta)}, \quad \text{avec} \quad \left\|\frac{\langle\xi\rangle\hat{v}(\zeta)}{\langle\sigma\rangle + |\xi|^2}; L^2_{(\tau,\zeta)}\right\| \leq c\|v\|_{L^2},$$

et à utiliser les arguments de la preuve de (4). Ceci termine la preuve de la Proposition 2. Des arguments standards de point fixe terminent la preuve du Théorème 1.

Remarques. L'idée de ce travail vient de [**32**], où est donnée une démonstration différente du Théorème 1. Elle n'utilise pas les propriétés dispersives de (1), mais se sert de la méthode utilisée dans [**42**] pour l'équation de KdV. Dans [**57**], nous démontrons que le problème (1) est bien posé dans des espaces de Sobolev nonisotropes d'indice négatif par rapport à x (en particulier dans L^2)[1].

[1]cf. Chapters 3,4,5

CHAPTER 3

On the Cauchy problem for Kadomtsev-Petviashvili equation

This Chapter essentially contains the paper [**57**] *(On the Cauchy problem for Kadomtsev-Petviashvili equation, Communications in Partial Differential Equations, 24 (1999) 1367-1397).*

Abstract.

Using the method of Bourgain we prove local well-posedness of the Cauchy problem for KP-II equation in non-isotropic Sobolev spaces $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ for $s_1 > -1/4$ and $s_2 \geq 0$. In the case of three space dimensions we prove local well-posedness in $\widetilde{H}^s(\mathbf{R}^3)$ for $s > 3/2$.

Résumé.

En utilisant la méthode de Bourgain on démontre que le problème de Cauchy pour l'équation de Kadomtsev-Petviashvili-II est localement bien posé dans les espaces de Sobolev non-isotropes $\widetilde{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, quand $s_1 > -1/4$ et $s_2 \geq 0$. Dans le cas de la dimension trois, on obtient que le problème de Cauchy pour l'équation de KP-II est localement bien posé dans $\widetilde{H}^s(\mathbf{R}^3)$, quand $s > 3/2$.

1. Introduction

We will study the initial value problem for the Kadomtsev-Petviashvili-II equation (KP-II). The Cauchy problem for KP-II equation in $\mathbf{R}^2$ has the form

$$(1) \qquad \begin{cases} (u_t + u_{xxx} + uu_x)_x + u_{yy} = 0 \\ u(0,x,y) = \phi \end{cases}$$

The Kadomtsev-Petviashvili equations are "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one-directional, with weak transverse effects. The equation (1) has the structure of an infinite dimensional Hamiltonian system and applying the inverse scattering method one can obtain global solutions of (1) in Sobolev spaces $W^{s,2} \bigcap W^{s,1}$, $s \geq 10$ (cf. [**61**]). Using a method based on an analysis of multiple Fourier series first applied for the Schrödinger equation (cf. [**12**]) and KdV equation (cf. [**13**]), Bourgain (cf. [**14**]) proved global well-posedness of (1) for initial data in $H^s(\mathbf{T}^2), s \geq 0$, $\mathbf{T}^2 = \mathbf{R}^2|\mathbf{Z}^2$. The proof is also adapted for data in $\mathbf{R}^2$. Bourgain [14] uses mainly a suitable dyadic decomposition associated to the structure of the equation (1) and an algebraic relation for the symbol of the linearized operator. No form of the classical Strichartz inequality is used for obtaining the result of [14]. In [**32**] using a method proposed by Kenig, Ponce, Vega (cf. [**42**], [**43**]) a part of the Bourgain's result is recovered. More precisely local well-posedness of the Cauchy problem (1) in $H^s(\mathbf{R}^2), s > 0$ is proved. The nonlinear estimate of [**32**] is obtained by a change of variables similar to that of [**42**] for KdV equation or [**43**] for the nonlinear Schrödinger equation. No form of the classical Strichartz estimate is used in [**32**] either.

We write (1) in the following form

$$\left\{ \begin{array}{c} i\partial_t u = p(D_1, D_2)u - iuu_x \\ u(0,x,y) = \phi \end{array} \right. \tag{2}$$

where $D_1 = \frac{1}{i}\partial_x$, $D_2 = \frac{1}{i}\partial_y$ and the operator $p(D_1, D_2)$ is defined through the Fourier transform

$$\mathcal{F}(p(D_1, D_2)u)(\zeta) = p(\zeta)\hat{u}(\zeta).$$

Here $\zeta = (\xi, \eta)$ stays for the Fourier variables corresponding to (x, y) and

$$p(\zeta) = -\xi^3 + \frac{\eta^2}{\xi}$$

With $H^{b,s_1,s_2}(\mathbf{R}^3)$ we denote the classical Sobolev spaces equipped with the norm

$$\|u; H^{b,s_1,s_2}(\mathbf{R}^3)\| = \|\langle\tau\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{u}(\tau,\xi,\eta); L^2_{(\tau,\zeta)}\|,$$

where $\langle . \rangle = 1 + |.|$. Similarly we define $H^{s_1,s_2}(\mathbf{R}^2)$ and the homogeneous spaces $\dot{H}^{s_1,s_2}(\mathbf{R}^2)$. As in [**48**] we define modified Sobolev spaces $\widetilde{H}^{b,s_1,s_2}(\mathbf{R}^3)$

$$\widetilde{H}^{b,s_1,s_2}(\mathbf{R}^3) = \{u \in H^{b,s_1,s_2}(\mathbf{R}^3) : \mathcal{F}^{-1}(|\xi|^{-1}\hat{u}(\tau,\zeta)) \in H^{b,s_1,s_2}(\mathbf{R}^3)\},$$

equipped with the norm

$$\|u; \widetilde{H}^{b,s_1,s_2}(\mathbf{R}^3)\| = \|u; H^{b,s_1,s_2}(\mathbf{R}^3)\| + \||D_1|^{-1}u; H^{b,s_1,s_2}(\mathbf{R}^3)\|.$$

Similarly we define the spaces $\widetilde{H}^{s_1,s_2}(\mathbf{R}^2)$. With $\widetilde{L}^2$ we denote the space $\widetilde{H}^0$. Note that for any $u \in \widetilde{H}^{b,s_1,s_2}$ we have

$$\int_{-\infty}^{\infty} u(t,x,y)dx = 0.$$

Let $U(t) = \exp(-itp(D_1, D_2))$ be the unitary group which defines the free evolution of (2). In both papers [**14**] and [**32**] the solutions of (1) belong to the space $\bar{B}^{b,b_1,s}$ equipped with the norm

$$\|u; \bar{B}^{b,b_1,s}\| = \|U(-t)u; \bar{X}^{b,b_1,s}\|, \tag{3}$$

where $\bar{X}^{b,b_1,s}$ is equipped with the norm

$$\|u; \bar{X}^{b,b_1,s}\| = \left\| \langle\tau\rangle^b \langle\frac{\tau}{1+|\xi|^3}\rangle^{b_1} \langle\zeta\rangle^s \hat{u}(\tau,\zeta); L^2_{(\tau,\zeta)} \right\|. \tag{4}$$

In [**14**] a small modification of (4), according to the size of the initial data is performed. The parameter s measures the Sobolev regularity of the solutions and $b \sim 1/2$, $b_1 \sim 1/4$. In the present paper we shall not make use of the term $\langle\frac{\tau}{1+|\xi|^3}\rangle^{b_1}$ in the description of the functional space where the solutions are expected to belong. More precisely we shall solve (2) in the Bourgain type spaces B^{b,s_1,s_2} equipped with the norm

$$\|u; B^{b,s_1,s_2}\| = \|U(-t)u; \widetilde{H}^{b,s_1,s_2}\|.$$

Since $\mathcal{F}(U(-t)u)(\tau,\zeta) = \hat{u}(\tau - p(\zeta),\zeta)$ we obtain that in terms of the Fourier transform variables the norm of B^{b,s_1,s_2} can be expressed as

$$\|u; B^{b,s_1,s_2}\| = \|\langle\tau + p(\zeta)\rangle^b\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{u}(\tau,\zeta)\|_{L^2_{\tau,\zeta}} + \|\langle\tau + p(\zeta)\rangle^b|\xi|^{-1}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{u}(\tau,\zeta)\|_{L^2_{\tau,\zeta}}$$

Note that if $u(t,x,y)$ is a solution of (1) then so is

$$u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$$

We have that

$$\|u_\lambda(t,.,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+2s_2+1/2}\|u(\lambda^3 t,.,.)\|_{\dot{H}^{s_1,s_2}}$$

Hence one may expect that for $s_1 + 2s_2 + 1/2 > 0$ the Cauchy problem (1) is well-posed in H^{s_1,s_2}-type spaces. Our main goal in this Chapter is to prove the following Theorem.

THEOREM 1. *Let $s_1 > -1/4$ and $s_2 \geq 0$. Then there exists $b > 1/2$ such that for any $\phi \in \widetilde{H}^{s_1,s_2}$ there exist positive $T = T(\|\phi\|_{\widetilde{H}^{s_1,s_2}})$ and a unique solution $u(t,x,y)$ of* (1) *in the time interval $[-T,T]$ satisfying*

$$u \in C([-T,T]; \widetilde{H}^{s_1,s_2}(\mathbf{R}^2)) \cap B^{b,s_1,s_2}.$$

We shall obtain solutions of (1) belonging to $B^{1/2+\epsilon,s_1,s_2}$, $s_1 > -1/4, s_2 \geq 0$ and ϵ sufficiently small. One dimensional Sobolev embedding injects the solutions in the framework of the continuous in time functions. The linear estimates are the same as in **[12]**, **[13]**, **[14]**. In the estimate of the nonlinear term uu_x we consider three main cases. The difficulty in the first one is a singularity which appears near to the origin in the integral representation of the nonlinear estimate. In the second case we apply the method of Kenig, Ponce, Vega similarly to **[32]**. In the third case we make an essential use of the Strichartz-type inequalities for KP equation (cf. **[23]**, **[48]**).

The rest of the Chapter is organized as follows. In Section 2 we recall the framework of Bourgain's method and state the nonlinear estimate in two space dimensions. In Section 3 we prove Theorem 1. Section 4 is devoted to the case of three space dimensions. In an appendix we give the proof of the classical Strichartz inequality for KP in three space dimensions.

2. Preliminaries

Note that the equation (2) is equivalent to the following integral equation

$$u(t) = U(t)u(0) - \frac{1}{2}\int_0^t U(t-t')\partial_x(u^2(t'))dt'. \tag{5}$$

Let ψ be a cut-off function such that

$$\psi \in C_0^\infty(\mathbf{R}),\ \text{supp}\ \psi \subset [-2,2], \psi = 1 \text{ over the interval } [-1,1].$$

We consider a cut-off version of (5)

$$u(t) = \psi(t)U(t)u(0) - \frac{1}{2}\psi(t/T)\int_0^t U(t-t')\partial_x(u^2(t'))dt'. \tag{6}$$

We shall solve globally in time (6). To the solutions of (6) will correspond local solutions of (5) in time interval $[-T,T]$. We have the following estimate for the first term of the right-hand side of (6)

$$\|\psi(t)U(t)\phi; B^{b,s_1,s_2}\| \leq c\|\phi; \widetilde{H}^{s_1,s_2}\|$$

For estimating the second term in the right-hand side of the integral equation (6) we use the next Proposition.

PROPOSITION 1. *(cf. [23], Lemma 3.2). Let $-1/2 < b' \leq 0 \leq b \leq b'+1$. Then*

$$\left\|\psi(t/T)\int_0^t U(t-t')\partial_x(u^2(t'))dt'; B^{b,s_1,s_2}\right\| \leq cT^{1-b+b'}\|uu_x; B^{b',s_1,s_2}\| \tag{7}$$

Remark. The estimate of Proposition 1 is in fact one dimensional (with respect to t). More precisely (7) can be reduced to the following estimate

$$\|Kg; H_t^b\| \leq cT^{1-b+b'}\|g; H_t^{b'}\|,$$

where the operator K is defined by

$$(Kg)(t) = \psi(t/T)\int_0^t g(t')dt'.$$

We refer to [23], Lemma 3.2 for details.

The Strichartz inequalities are injected into the framework of Bourgain spaces. More precisely we shall use the following estimate.

PROPOSITION 2. *Let*

$$0 < \theta \leq 1, \quad \epsilon_1 > 0, \quad 0 \leq b \leq 1/2 + \epsilon_1.$$

Then for any $u \in L^2(\mathbf{R}^3)$ the following inequality holds

$$\left\|\mathcal{F}^{-1}(\langle\tau + p(\zeta)\rangle^{-b}|\hat{u}(\tau,\zeta)|); L_t^q(L_{(x,y)}^r)\right\| \leq c\|u\|_{L^2}, \tag{8}$$

provided q and r satisfy the next relations

$$\frac{2}{q} = 1 - \frac{\theta b}{1/2+\epsilon_1}, \quad \delta(r) := 2(\frac{1}{2} - \frac{1}{r}) = \frac{(1-\theta)b}{1/2+\epsilon_1}.$$

Proof. For any $\phi \in L^2(\mathbf{R}^2)$ the classical version of the Strichartz inequality for KP equation (cf. [48], Proposition 2.3) yields

$$\|U(t)\phi; L_t^q(L_{(x,y)}^r)\| \leq c\|\phi\|_{L^2}, \tag{9}$$

provided

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 < q \leq \infty.$$

Once we have (9) Lemma 3.3 of [23] gives for any $u \in B^{1/2+\epsilon_1,0,0}$

$$\|u; L_t^q(L_{(x,y)}^r)\| \leq c\|u; B^{1/2+\epsilon_1,0,0}\|, \tag{10}$$

provided

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 < q \leq \infty.$$

Interpolating between (10) and

$$\|u; L_t^2(L_{(x,y)}^2)\| \le \|u; B^{0,0,0}\|,$$

we obtain

$$\|u; L_t^q(L_{(x,y)}^r)\| \le c\|u; B^{b,0,0}\|, \tag{11}$$

provided

$$\frac{2}{q} = 1 - \frac{\theta b}{1/2+\epsilon_1}, \quad \delta(r) = \frac{(1-\theta)b}{1/2+\epsilon_1}, \quad 0 < \theta \le 1.$$

But (11) is equivalent to (8) which completes the proof of Proposition 2.

Remark. Note that (9) is true for any $\phi \in L^2(\mathbf{R}^2)$, but $U(t)\phi$ is the solution of the linearized KP equation under an additional assumption on the initial data. It is easy to be seen that for any $\phi \in \widetilde{L}^2(\mathbf{R}^2)$, $U(t)\phi$ is the solution of the linearized KP equation.

We shall solve (6) in $B^{1/2+2\epsilon,s_1,s_2}$ for $s_1 > -1/4$, $s_2 \ge 0$ and $\epsilon > 0$ sufficiently small. Due to Proposition 1 in order to apply a fixed point argument it is sufficient to estimate $B^{-1/2+3\epsilon,s_1,s_2}$ norm of the nonlinear term. Then a small factor T^ϵ appears in the right-hand side of (7) which ensures the fixed point argument. Our main tool is the following Proposition.

PROPOSITION 3. *Let $s_1 > -1/4$ and $s_2 \ge 0$. Then for sufficiently small ϵ we have*

$$\|uu_x; B^{-1/2+3\epsilon,s_1,s_2}\| \le c\|u; B^{1/2+2\epsilon,s_1,s_2}\|^2. \tag{12}$$

3. Proof of Proposition 3

We set

$$\hat{w}(\tau,\zeta) = \langle \tau + p(\zeta)\rangle^{1/2+2\epsilon}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\hat{u}(\tau,\zeta).$$

Then (12) is equivalent to

$$\left\|\langle\tau+p(\zeta)\rangle^{-1/2+3\epsilon}\langle\xi\rangle^{1+s_1}\langle\eta\rangle^{s_2}\int K(\tau,\zeta,\tau_1,\zeta_1)\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)d\tau_1 d\zeta_1\right\|_{L^2_{\tau,\zeta}} \le$$

$$c\{\|w\|_{L^2}^2 + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|_{L^2}^2\}, \tag{13}$$

where

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}\langle\eta_1\rangle^{-s_2}\langle\eta-\eta_1\rangle^{-s_2}}{\langle\tau_1+p(\zeta_1)\rangle^{1/2+2\epsilon}\langle\tau-\tau_1+p(\zeta-\zeta_1)\rangle^{1/2+2\epsilon}}$$

By duality we obtain that (13) is equivalent to

$$\left|\int\int K_1(\tau,\zeta,\tau_1,\zeta_1)\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)\hat{v}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta\right| \le$$

$$c\{\|w\|_{L^2}^2 + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|_{L^2}^2\}\|v\|_{L^2}, \tag{14}$$

where

$$K_1(\tau,\zeta,\tau_1,\zeta_1) = \frac{\langle\xi\rangle^{1+s_1}\langle\eta\rangle^{s_2}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}\langle\eta_1\rangle^{-s_2}\langle\eta-\eta_1\rangle^{-s_2}}{\langle\tau_1+p(\zeta_1)\rangle^{1/2+2\epsilon}\langle\tau-\tau_1+p(\zeta-\zeta_1)\rangle^{1/2+2\epsilon}\langle\tau+p(\zeta)\rangle^{1/2-3\epsilon}}$$

Without loss of generality we can assume that $\hat{w} \geq 0$ and $\hat{v} \geq 0$. Further we set

$$\sigma := \sigma(\tau, \zeta) = \tau - \xi^3 + \frac{\eta^2}{\xi}, \quad \sigma_1 := \sigma(\tau_1, \zeta_1), \quad \sigma_2 := \sigma(\tau - \tau_1, \zeta - \zeta_1).$$

As in [14] the following relation plays an essential role

$$\sigma_1 + \sigma_2 - \sigma = 3\xi_1\xi(\xi - \xi_1) + \frac{(\xi_1\eta - \xi\eta_1)^2}{\xi_1\xi(\xi - \xi_1)} \tag{15}$$

Hence

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi_1\xi(\xi - \xi_1)|. \tag{16}$$

We shall also use the next elementary inequalities

$$\langle\eta_1\rangle^{-2s_2}\langle\eta - \eta_1\rangle^{-2s_2} \leq c\langle\eta\rangle^{-2s_2}, \tag{17}$$

$$\langle\xi_1\rangle^{-2s_1}\langle\xi - \xi_1\rangle^{-2s_1} \leq c\langle\xi\rangle^{-2s_1}, \quad \text{if } s_1 > 0. \tag{18}$$

By symmetry arguments we can assume that

$$|\sigma_1| \geq |\sigma_2|.$$

By J we note the left-hand side of (14). We shall divide the domain of integration of J into three main regions. In the first region $|\xi|$ is small and in the second and third region we take into account which term dominates in the left-hand side of (16).

Case 1. $|\xi| \leq 2$. We denote by J_1 the restriction of J on this region.

Case 1.1. $|\xi| \leq 2, \quad |\xi_1| \leq 1$. We denote by J_{11} the restriction of J_1 on this region.

Using (17) we obtain

$$J_{11} \leq \int\int \frac{\langle\xi\rangle^{1+s_1}\hat{w}(\tau_1, \zeta_1)\hat{w}(\tau - \tau_1, \zeta - \zeta_1)\hat{v}(\tau, \zeta)d\tau d\zeta d\tau_1 d\zeta_1}{\langle\xi_1\rangle^{s_1}\langle\xi - \xi_1\rangle^{s_1}\langle\sigma\rangle^{1/2-3\epsilon}\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}}$$

We set

$$g(\tau, \xi, \eta, \xi_1) = \frac{\langle\xi\rangle^{1+s_1}}{\langle\sigma\rangle^{1/2-3\epsilon}\langle\xi_1\rangle^{s_1}\langle\xi - \xi_1\rangle^{s_1}}$$

Now using the boundedness of $|\xi|$ and $|\xi_1|$ we obtain that g is bounded and using Hölder inequality and Proposition 2 with $b = 1/2 + 2\epsilon, \theta = 1/2$ and $\epsilon_1 = 2\epsilon$ we obtain

$$\begin{aligned} J_{11} &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-1/2-2\epsilon}\hat{w}(\tau_1, \zeta_1)); L_t^4(L^4_{(x,y)})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2-2\epsilon}\hat{w}(\tau - \tau_1, \zeta - \zeta_1)); L_t^4(L^4_{(x,y)})\|\|v\|_{L^2} \\ &\leq c\|w\|_{L^2}^2\|v\|_{L^2}. \end{aligned}$$

Remark. Note that g remains bounded for $s_1 \geq 0$ without smallness assumptions for $|\xi_1|$. But we are not able to apply the above arguments without smallness assumptions for $|\xi_1|$ when $s_1 < 0$. Let us consider a sequence $(\tau^{(n)}, \xi^{(n)}, \eta^{(n)}, \xi_1^{(n)})$ such that

$$\xi^{(n)} \to 0, \quad \xi_1^{(n)} \to \infty, \quad \tau^{(n)} = -\frac{(\eta^{(n)})^2}{\xi^{(n)}}.$$

Then it is easy to see that

$$g(\tau^{(n)}, \xi^{(n)}, \eta^{(n)}, \xi_1^{(n)}) \to \infty.$$

Therefore when $s_1 < 0$, $|\xi_1| \geq 1$ and $|\xi| \leq 2$ we have to perform a different argument in order to estimate J.

Case 1.2. $|\xi| \leq 2$, $|\xi_1| \geq 1$. We denote by J_{12} the restriction of J_1 on this region.

Using Cauchy-Schwarz inequality we obtain

$$J_{12} \leq \int I_1(\tau_1, \zeta_1) \left\{ \int_{|\xi| \leq 2} |\hat{v}(\tau, \zeta)\hat{w}(\tau - \tau_1, \zeta - \zeta_1)|^2 d\tau d\zeta \right\}^{1/2} \hat{w}(\tau_1, \zeta_1) d\tau_1 d\zeta_1,$$

where

$$I_1(\tau_1, \zeta_1) = \frac{\langle \xi_1 \rangle^{-s_1} \langle \eta_1 \rangle^{-s_2}}{\langle \sigma_1 \rangle^{1/2+2\epsilon}} \left(\int_{|\xi| \leq 2} \frac{\langle \xi \rangle^{1+s_1} \langle \xi - \xi_1 \rangle^{-2s_1} \langle \eta \rangle^{2s_2} d\tau d\zeta}{\langle \eta - \eta_1 \rangle^{2s_2} \langle \sigma \rangle^{1-6\epsilon} \langle \sigma_2 \rangle^{1+4\epsilon}} \right)^{1/2}$$

We have the following Lemma.

LEMMA 1.

$$I_1(\tau_1, \zeta_1) \leq const, \qquad if \quad |\xi_1| \geq 1.$$

Proof. We assume that $-1/4 < s_1 \leq 0$. If $s_1 \geq 0$ then we can apply the arguments of Case 1.1. in order to estimate J_{12}. Using (17) and the boundedness of $|\xi|$ we obtain

$$I_1(\tau_1, \zeta_1) \leq \frac{c\langle \xi_1 \rangle^{-2s_1}}{\langle \sigma_1 \rangle^{1/2+2\epsilon}} \left(\int_{|\xi| \leq 2} \frac{d\tau d\zeta}{\langle \sigma \rangle^{1-6\epsilon} \langle \sigma_2 \rangle^{1+4\epsilon}} \right)^{1/2}. \tag{19}$$

We perform a change of variables

$$\alpha = \sigma - \sigma_2, \qquad \beta = 3\xi\xi_1(\xi_1 - \xi).$$

Note that $|\beta| \leq 18|\xi_1|^2$. Moreover we can assume that $\xi_1 \geq 0$. If $\xi_1 \leq 0$ then the arguments are the same. Further we have

$$d\zeta = \frac{c|\beta|^{1/2} d\alpha d\beta}{|\xi_1|^{3/2}(\frac{3}{4}\xi_1^3 - \beta)^{1/2} |\sigma_1 + \beta - \alpha|^{1/2}}$$

Moreover

$$I_1(\tau_1, \zeta_1) \leq \frac{c\langle \xi_1 \rangle^{-2s_1}}{\langle \sigma_1 \rangle^{1/2+2\epsilon}} \left(\int_{|\xi| \leq 2} \frac{d\zeta}{\langle \sigma - \sigma_2 \rangle^{1-6\epsilon}} \right)^{1/2}, \tag{20}$$

where we used the following elementary inequality

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t \rangle^{1+4\epsilon} \langle t - \theta \rangle^{1-6\epsilon}} \leq \frac{c}{\langle \theta \rangle^{1-6\epsilon}}.$$

Therefore we have

$$I_1(\tau_1,\zeta_1) \le \frac{c\langle\xi_1\rangle^{-2s_1}}{\langle\sigma_1\rangle^{1/2+2\epsilon}|\xi_1|^{3/4}} \left\{\int_{-18|\xi_1|^2}^{18|\xi_1|^2}\int_{-\infty}^{\infty}\frac{|\beta|^{1/2}d\alpha d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}|\alpha-\sigma_1-\beta|^{1/2}\langle\alpha\rangle^{1-6\epsilon}}\right\}^{1/2}$$

We set

$$\int_{-\infty}^{\infty}\frac{d\alpha}{|\alpha-\sigma_1-\beta|^{1/2}\langle\alpha\rangle^{1-6\epsilon}} = \underbrace{\int_{|\alpha|\ge 2|\sigma_1+\beta|+1}\cdots}_{(1)} + \underbrace{\int_{|\alpha|\le 2|\sigma_1+\beta|+1}\cdots}_{(2)}$$

If $|\alpha|\ge 2|\sigma_1+\beta|+1$ then $|\alpha-\sigma_1-\beta|^{-1}\le 2\langle\alpha\rangle^{-1}$. Hence

$$(1)\le\int_{|\alpha|\ge 2|\sigma_1-\beta|+1}\frac{d\alpha}{|\alpha-\sigma_1-\beta|^{1/2-7\epsilon}\langle\alpha\rangle^{1+\epsilon}}\le\frac{c}{\langle\sigma_1+\beta\rangle^{1/2-7\epsilon}}$$

$$(2)\le\frac{c(2|\sigma_1+\beta|+1)^{7\epsilon}}{\langle\sigma_1+\beta\rangle^{1/2}}\le\frac{c}{\langle\sigma_1+\beta\rangle^{1/2-7\epsilon}}$$

where we used the following two elementary inequalities

$$\int_{-\infty}^{\infty}\frac{dt}{\langle t\rangle^{1+\epsilon}|t-\theta|^{1/2-7\epsilon}}\le\frac{c}{\langle\theta\rangle^{1/2-7\epsilon}},$$

$$\int_{-|b|}^{|b|}\frac{dt}{\langle t\rangle^{1-6\epsilon}|t-\theta|^{1/2}}\le\frac{c|b|^{7\epsilon}}{\langle\theta\rangle^{1/2}}.$$

Therefore we have

$$\begin{aligned} I_1(\tau_1,\zeta_1) &\le \frac{c}{\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\xi_1\rangle^{3/4+2s_1}}\left\{\int_{-18|\xi_1|^2}^{18|\xi_1|^2}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2-7\epsilon}}\right\}^{1/2} \\ &\le \frac{c}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{3/4+2s_1-7\epsilon}}\left\{\int_{-18|\xi_1|^2}^{18|\xi_1|^2}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\right\}^{1/2} \end{aligned}$$

Now we consider two cases for (τ_1,ζ_1).

- $|\xi_1|^2\le|\sigma_1|$

We have that

$$I(\tau_1,\zeta_1) \leq \frac{c}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{3/4+2s_1-7\epsilon}}\{\int_{-18\xi_1^2}^{0}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}+$$

$$\int_0^{18\xi_1^2}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\}^{1/2}$$

$$\leq \frac{c}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{3/4+2s_1-7\epsilon}}\times$$

$$\times\{\int_{-18\xi_1^2}^{0}\frac{d\beta}{\langle\sigma_1+\beta\rangle^{1/2}}+\xi_1\int_0^{18\xi^2}\frac{d\beta}{|\frac{3}{4}\xi_1^3-\beta|^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\}^{1/2}$$

$$\leq \frac{c}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{3/4+2s_1-7\epsilon}}\{\langle\sigma_1\rangle^{1/2}+\xi_1(\int_0^{18\xi_1^2-1}\frac{d\beta}{|\frac{3}{4}\xi_1^3-\beta|}+$$

$$\int_0^{18\xi_1^2-1}\frac{d\beta}{\langle\sigma_1+\beta\rangle}+\int_{18\xi_1^2-1}^{18\xi_1^2}\frac{d\beta}{|\frac{3}{4}\xi_1^3-\beta|^{1/2}\langle\sigma_1+\beta\rangle^{1/2}})\}^{1/2}$$

$$\leq \frac{c}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{3/4+2s_1-7\epsilon}}(\langle\sigma_1\rangle^{1/2}+\xi_1\ln(1+\langle\sigma_1\rangle))^{1/2}$$

$$\leq \frac{c}{\langle\sigma_1\rangle^{1/4-3/2\epsilon}\langle\xi_1\rangle^{3/4+2s_1-7\epsilon}}+\frac{c\ln^{1/2}(1+\langle\sigma_1\rangle)}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{1/4+2s_1-7\epsilon}}$$

$$\leq c+\frac{c}{\langle\xi_1\rangle^{5/4+2s_1-7\epsilon}}\leq \text{const},$$

provided $s_1 > -5/4$ and ϵ be sufficiently small.

- $|\sigma_1| \leq |\xi_1|^2$

If $|\xi_1| \leq 48$ then we can easily verify that $I_1(\tau_1,\zeta_1) \leq$ const. If $|\xi_1| \geq 48$ then

$$\left(\frac{3}{4}|\xi_1|^3-\beta\right)^{1/2} \geq c|\xi_1|^{3/2}$$

and hence

$$I_1(\tau_1,\zeta_1) \le \frac{c\langle\xi_1\rangle^{1/2}}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{3/2+2s_1-7\epsilon}}\left\{\int_{-18|\xi_1|^2}^{18|\xi_1|^2}\frac{d\beta}{\langle\sigma_1+\beta\rangle^{1/2}}\right\}^{1/2}$$

$$\le \frac{c}{\langle\sigma_1\rangle^{1/2-3/2\epsilon}\langle\xi_1\rangle^{1/2+2s_1-7\epsilon}} \le \text{const},$$

provided $s_1 > -1/4$ and ϵ be sufficiently small. This completes the proof of Lemma 1. Therefore using Lemma 1 and Cauchy-Schwarz inequality we obtain

$$J_{12} \le c\|w\|_{L^2}^2\|v\|_{L^2},$$

which completes the proof in Case 1.

Case 2. $|\sigma| \ge |\sigma_1|$, $|\xi| \ge 2$. We denote by J_2 the restriction of J on this region.

Case 2.1. $|\sigma| \ge |\sigma_1|$, $|\xi| \ge 2$, $|\xi_1| \ge 1$, $|\xi-\xi_1| \ge 1$. We denote by J_{21} the restriction of J_2 on this region. In this case we perform arguments of [**42**] similarly to [**32**]. Using Cauchy-Schwarz inequality we obtain

$$J_{21} \le \int_{\mathbf{R}^3} I(\tau,\zeta)\left\{\int_{|\sigma|\ge|\sigma_1|}|\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1\right\}^{1/2}\hat{v}(\tau,\zeta)d\tau d\zeta,$$

where

$$I(\tau,\zeta) = \frac{\langle\xi\rangle^{1+s_1}\langle\eta\rangle^{s_2}}{\langle\sigma\rangle^{1/2-3\epsilon}}\left(\int_{|\sigma|\ge|\sigma_1|}\frac{\langle\xi_1\rangle^{-2s_1}\langle\xi-\xi_1\rangle^{-2s_1}d\tau_1 d\zeta_1}{\langle\eta_1\rangle^{2s_2}\langle\eta-\eta_1\rangle^{2s_2}\langle\sigma_1\rangle^{1+4\epsilon}\langle\sigma_2\rangle^{1+4\epsilon}}\right)^{1/2}$$

If we prove that $I(\tau,\zeta)$ is bounded then an applying of Cauchy-Schwarz inequality will prove (14) when $|\sigma| \ge |\sigma_1|$. We have the following Lemma.

LEMMA 2. *(cf.* [**32**], *Lemma 2.2.)*

$$I(\tau,\zeta) \le \textit{const}, \quad \textit{if} \quad |\xi| \ge 2.$$

Proof. Lemma 2 is essentially proved in [**32**] but we shall give the proof for completeness. Using (16),(17), (18) we obtain

$$I(\tau,\zeta) \le \frac{\langle\xi\rangle^{1+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}}\left(\int_{|\sigma|\ge|\sigma_1|}\frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{1+4\epsilon}\langle\sigma_2\rangle^{1+4\epsilon}}\right)^{1/2}, \quad \text{if } s_1 \le 0. \tag{21}$$

$$I(\tau,\zeta) \le \frac{\langle\xi\rangle}{\langle\sigma\rangle^{1/2-3\epsilon}}\left(\int_{|\sigma|\ge|\sigma_1|}\frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{1+4\epsilon}\langle\sigma_2\rangle^{1+4\epsilon}}\right)^{1/2}, \quad \text{if } s_1 \ge 0. \tag{22}$$

We shall only prove boundedness of the right hand-side of (21), i.e. we assume that $0 \ge s_1 > -1/4$. The arguments in the case $s_1 > 0$ are similar (simpler). We perform a change of variables

$$\alpha = \sigma_1 + \sigma_2, \qquad \beta = 3\xi\xi_1(\xi-\xi_1).$$

Note that $\xi \in [-3|\sigma|, \min\{3/4\xi^3, 3|\sigma|\}]$, when $\xi \geq 0$ and $\xi \in [\max\{3/4\xi^3, -3|\sigma|\}, 3|\sigma|\}]$, when $\xi \leq 0$. We assume that $\xi \geq 0$. If $\xi \leq 0$ then the arguments are the same. We have that

$$d\zeta_1 = \frac{c|\beta|^{1/2}d\alpha d\beta}{|\xi|^{3/2}(\frac{3}{4}\xi^3 - \beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}}$$

Therefore we obtain

$$I(\tau,\zeta) \leq \frac{c\langle\xi\rangle}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}|\xi|^{3/4}}$$

$$\left\{\int_{-3|\sigma|}^{\min\{3/4\xi^3,3|\sigma|\}}\int_{-\infty}^{\infty}\frac{|\beta|^{1/2}d\alpha d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}\langle\alpha\rangle^{1+4\epsilon}}\right\}^{1/2} \leq$$

$$\frac{c\langle\xi\rangle^{1+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}|\xi|^{3/4}}\left\{\int_{-3|\sigma|}^{\min\{3/4\xi^3,3|\sigma|\}}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2},$$

where we used the following two elementary inequalities

$$\int_{-\infty}^{\infty}\frac{dt}{\langle t\rangle^{1+4\epsilon}\langle t-\theta\rangle^{1+4\epsilon}} \leq \frac{c}{\langle\theta\rangle^{1+4\epsilon}}, \quad \int_{-\infty}^{\infty}\frac{dt}{\langle t\rangle^{1+4\epsilon}|t-\theta|^{1/2}} \leq \frac{c}{\langle\theta\rangle^{1/2}}.$$

Now we consider two cases for (τ,ζ).

- $\frac{3}{4}|\xi|^3 \leq 4|\sigma|$

  We have that

$$\begin{aligned}
I(\tau,\zeta) &\leq \frac{c\langle\xi\rangle^{1+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}|\xi|^{3/4}}\{\int_{-3|\sigma|}^{0}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}} \\
&+ \int_0^{\frac{3}{4}\xi^3}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\}^{1/2} \\
&\leq \frac{c\langle\xi\rangle^{1/4+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}}\left\{\int_{-3|\sigma|}^{0}\frac{d\beta}{\langle\sigma+\beta\rangle^{1/2}} + \xi^{3/2}\int_0^{\frac{3}{4}\xi^3}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2} \\
&\leq \frac{c\langle\xi\rangle^{1/4+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}}\{\langle\sigma\rangle^{1/2} + \xi^{3/2}(\int_0^{\frac{3}{4}\xi^3-1}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|} \\
&+ \int_0^{\frac{3}{4}\xi^3-1}\frac{d\beta}{\langle\sigma+\beta\rangle} + \int_{\frac{3}{4}\xi^3-1}^{\frac{3}{4}\xi^3}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|^{1/2}\langle\sigma+\beta\rangle^{1/2}})\}^{1/2}
\end{aligned}$$

$$\leq \frac{c\langle\xi\rangle^{1/4+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}}(\langle\sigma\rangle^{1/2}+\xi^{3/2}\ln(1+\langle\sigma\rangle))^{1/2}$$

$$\leq \frac{c\langle\xi\rangle^{1/4+2s_1}}{\langle\sigma\rangle^{1/4+s_1-3\epsilon}}+\frac{c\langle\xi\rangle^{1+2s_1}(\ln(1+\langle\sigma\rangle))^{1/2}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}}$$

$$\leq \frac{c\langle\xi\rangle^{1/4+2s_1}}{\langle\xi\rangle^{3/4+3s_1-9\epsilon}}+\frac{c\langle\xi\rangle^{1+2s_1}}{\langle\xi\rangle^{3/2+3s_1-12\epsilon}}\leq c\langle\xi\rangle^{-1/2-s_1+12\epsilon}.$$

Hence

$$I(\tau,\zeta)\leq \text{const},$$

provided $s_1 > -1/4$ and ϵ be sufficiently small.

- $4|\sigma| \leq \frac{3}{4}|\xi|^3$

In this case we have

$$I(\tau,\zeta)\leq \frac{c\langle\xi\rangle^{1+2s_1}\langle\sigma\rangle^{1/4}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}|\xi|^{3/4}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}\leq$$

$$\frac{c\langle\xi\rangle^{1/4+2s_1}}{\langle\sigma\rangle^{1/4+s_1-3\epsilon}|\xi|^{3/4}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{d\beta}{\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}\leq$$

$$\frac{c\langle\xi\rangle^{-1/2+2s_1}\langle\sigma\rangle^{1/4}}{\langle\sigma\rangle^{1/4+s_1-3\epsilon}}\leq c\langle\xi\rangle^{-1/2-s_1+9\epsilon}\leq \text{const},$$

This completes the proof of Lemma 2.
Therefore using Lemma 2 and Cauchy-Schwarz inequality we obtain

$$J_{21}\leq c\|w\|_{L^2}^2\|v\|_{L^2},$$

which completes the proof in Case 2.1.

Case 2.2. $|\sigma|\geq|\sigma_1|,\quad |\xi|\geq 2,\quad |\xi_1|\leq 1$. We denote by J_{22} the restriction of J_2 on this region. Using Cauchy-Schwarz inequality, (16), (17) and (18) we obtain

$$J_{22}\leq\int_{\mathbf{R}^3}I(\tau,\zeta)\left\{\int_{|\sigma|\geq|\sigma_1|}||\xi_1|^{-1}\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)|^2d\tau_1d\zeta_1\right\}^{1/2}\hat{v}(\tau,\zeta)d\tau d\zeta,$$

where

$$I(\tau,\zeta)\leq\frac{\langle\xi\rangle^{1+2s_1}}{\langle\sigma\rangle^{1/2+s_1-3\epsilon}}\Big(\int_{|\sigma|\geq|\sigma_1|}\frac{d\tau_1d\zeta_1}{\langle\sigma_1\rangle^{1+4\epsilon}\langle\sigma_2\rangle^{1+4\epsilon}}\Big)^{1/2},\quad \text{if } s_1\leq 0.$$

$$I(\tau,\zeta) \leq \frac{\langle\xi\rangle}{\langle\sigma\rangle^{1/2-3\epsilon}} \left(\int_{|\sigma|\geq|\sigma_1|} \frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{1+4\epsilon}\langle\sigma_2\rangle^{1+4\epsilon}} \right)^{1/2}, \quad \text{if } s_1 \geq 0.$$

Now we can perform the arguments of Case 2.1

Case 2.3. $|\sigma| \geq |\sigma_1|, \quad |\xi| \geq 2, \quad |\xi - \xi_1| \leq 1.$

This case can be treated in the same way as Case 2.2.

Case 3. $|\sigma_1| \geq |\sigma|, \quad |\xi| \geq 2$. We denote by J_3 the restriction of J on this region.

In this case we shall use Strichartz inequality in order to estimate J. Using (17) we obtain

$$J_3 \leq \int\int \frac{\langle\xi\rangle^{1+s_1}\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)\hat{v}(\tau,\zeta)d\tau d\zeta d\tau_1 d\zeta_1}{\langle\xi_1\rangle^{s_1}\langle\xi-\xi_1\rangle^{s_1}\langle\sigma\rangle^{1/2-3\epsilon}\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}}$$

Case 3.1. $|\xi_1| \leq 1$. We denote by J_{31} the restriction of J_3 on this region.

We have that

$$\frac{1}{\langle\sigma_1\rangle^{1/2}} \leq \frac{1}{|\xi|^{1/2}|\xi_1|^{1/2}|\xi-\xi_1|^{1/2}} \leq \frac{c}{|\xi||\xi_1|^{1/2}}$$

Hence

$$\langle\xi\rangle^{1+s_1}\langle\sigma_1\rangle^{-1/2}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1} \leq c|\xi_1|^{-1} \tag{23}$$

Using (23), Hölder inequality and Proposition 2 with $\epsilon_1 = \epsilon/2$ we obtain

$$\begin{aligned} J_{31} &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2+3\epsilon}\hat{v}(\tau,\zeta)); L_t^{q_1}(L_{(x,y)}^{r_1})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-2\epsilon}|\xi_1|^{-1}\hat{w}(\tau_1,\zeta_1)); L_t^{q_2}(L_{(x,y)}^{r_2})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2-2\epsilon}\hat{w}(\tau-\tau_1,\zeta-\zeta_1)); L_t^{q_3}(L_{(x,y)}^{r_3})\| \\ &\leq c\|w\|_{L^2}\||D_x|^{-1}w\|_{L^2}\|v\|_{L^2}, \end{aligned}$$

provided

$$\frac{2}{q_1} = 1 - \frac{\theta(1/2-3\epsilon)}{1/2+\epsilon/2}, \quad \delta(r_1) = \frac{(1-\theta)(1/2-3\epsilon)}{1/2+\epsilon/2},$$

$$\frac{2}{q_2} = 1 - \frac{2\theta\epsilon}{1/2+\epsilon/2}, \quad \delta(r_2) = \frac{2(1-\theta)\epsilon}{1/2+\epsilon/2},$$

$$\frac{2}{q_3} = 1 - \frac{\theta(1/2+2\epsilon)}{1/2+\epsilon/2}, \quad \delta(r_3) = \frac{(1-\theta)(1/2+2\epsilon)}{1/2+\epsilon/2},$$

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad \delta(r_1) + \delta(r_2) + \delta(r_3) = 1.$$

Now it is sufficient to take $\theta = 1/2$ to ensure the restrictions of the Hölder inequality.

Case 3.2. $|\xi - \xi_1| \leq 1.$

This case can be treated in the same way as Case 3.1.

Case 3.3. $|\xi-\xi_1|\geq 1, \quad |\xi_1|\geq 1$. We denote by J_{33} the restriction of J_3 on this region.

We have that

$$\langle\xi\rangle^{1+s_1}\langle\sigma_1\rangle^{-1/2}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}\leq|\xi|^{1/2+s_1}|\xi_1|^{-1/2-s_1}|\xi-\xi_1|^{-1/2-s_1}\leq \text{const.}$$

Now as in Case 3.1. we obtain

$$\begin{aligned}J_{33} &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2+3\epsilon}\hat{v}(\tau,\zeta)); L_t^{q_1}(L_{(x,y)}^{r_1})\|\\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-2\epsilon}\hat{w}(\tau_1,\zeta_1)); L_t^{q_2}(L_{(x,y)}^{r_2})\|\\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2-2\epsilon}\hat{w}(\tau-\tau_1,\zeta-\zeta_1)); L_t^{q_3}(L_{(x,y)}^{r_3})\|\\ &\leq c\|w\|_{L^2}^2\|v\|_{L^2},\end{aligned}$$

provided

$$\frac{2}{q_1}=1-\frac{\theta(1/2-3\epsilon)}{1/2+\epsilon/2} \quad , \quad \delta(r_1)=\frac{(1-\theta)(1/2-3\epsilon)}{1/2+\epsilon/2},$$

$$\frac{2}{q_2}=1-\frac{2\theta\epsilon}{1/2+\epsilon/2} \quad , \quad \delta(r_2)=\frac{2(1-\theta)\epsilon}{1/2+\epsilon/2},$$

$$\frac{2}{q_3}=1-\frac{\theta(1/2+2\epsilon)}{1/2+\epsilon/2} \quad , \quad \delta(r_3)=\frac{(1-\theta)(1/2+2\epsilon)}{1/2+\epsilon/2},$$

$$\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1 \quad , \quad \delta(r_1)+\delta(r_2)+\delta(r_3)=1.$$

As in Case 3.1. we take $\theta=1/2$ to ensure the restrictions of the Hölder inequality. This completes the proof of Proposition 3.

It remains to use standard fixed point arguments to complete the proof of Theorem 1. We define an operator L

$$Lu:=\psi(t)U(t)\phi-\frac{1}{2}\psi(t/T)\int_0^t U(t-t')\partial_x(u^2(t'))dt'.$$

Using Proposition 1 and Proposition 3 we obtain

$$\|Lu;B^{1/2+2\epsilon,s_1,s_2}\|\leq c\|\phi;\widetilde{H}^{s_1,s_2}\|+cT^{\epsilon}\|u;B^{1/2+2\epsilon,s_1,s_2}\|^2. \tag{24}$$

Similarly we can obtain

$$\|Lu-Lv;B^{1/2+2\epsilon,s_1,s_2}\|\leq \tag{25}$$

$$cT^{\epsilon}\|u-v;B^{1/2+2\epsilon,s_1,s_2}\|\|u+v;B^{1/2+2\epsilon,s_1,s_2}\|.$$

Using (24),(25), we can apply the contraction mapping principle for sufficiently small T, which completes the proof of Theorem 1.

Remark. With arguments similar to these of Case 3 of the proof of Proposition 3 one can prove the next estimate for $s_1 > -1/2$ and $s_2 > 0$

$$\left| \int_{|\xi|\geq 1} \int K_3(\tau,\zeta,\tau_1,\zeta_1)\hat{w}(\tau_1,\zeta_1)\hat{w}(\tau-\tau_1,\zeta-\zeta_1)\hat{v}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta \right| \leq$$

$$c\{\|w\|_{L^2}^2 + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|_{L^2}^2\}\|v\|_{L^2}, \tag{26}$$

where

$$K_3(\tau,\zeta,\tau_1,\zeta_1) = \frac{\langle\xi\rangle^{1+s_1}\langle\eta\rangle^{s_2}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}\langle\eta_1\rangle^{-s_2}\langle\eta-\eta_1\rangle^{-s_2}}{\langle\tau_1+p(\zeta_1)\rangle^{1/2}\langle\tau-\tau_1+p(\zeta-\zeta_1)\rangle^{1/2}\langle\tau+p(\zeta)\rangle^{1/2}}$$

If we extend (26) when $|\xi| \leq 1$ then we could prove local well-posedness of (1) for $s_1 > -1/2$, $s_2 > 0$. An additional term would appear in the linear estimate but we could estimate it as in Proposition 7 bellow. We note also that just using the arguments of Case 3 above we are able to extend (26) when $|\xi| \leq 1$ for $s_1 \geq 0$, $s_2 > 0$ and therefore to prove local well-posedness in this range for (s_1, s_2).

4. The case of three space dimensions

In this section we consider KP-II equation in $\mathbf{R}^3$

$$\begin{cases} (u_t + u_{xxx} + uu_x)_x + u_{yy} + u_{zz} = 0 \\ u(0,x,y) = \phi \end{cases} \tag{27}$$

With methods similar to these of the two dimensional case we shall prove that that (27) is locally well-posed in the modified Sobolev spaces $\widetilde{H}^s$ for $s > 3/2$. In two space dimensions the Strichartz inequalities for KP equations are similar to these for two dimensional Schrödinger equation. One of the difficulties in three space dimensions is that losses of derivatives appear in Strichartz inequalities (as for the wave equations for example). We write (27) in the form

$$\begin{cases} i\partial_t u = p(D_1,D_2,D_3)u - iuu_x \\ u(0,x,y) = \phi \end{cases} \tag{28}$$

where $D_1 = \frac{1}{i}\partial_x$, $D_2 = \frac{1}{i}\partial_y$, $D_3 = \frac{1}{i}\partial_z$, and the operator $p(D_1,D_2,D_3)$ is defined through the Fourier transform

$$\mathcal{F}(p(D_1,D_2,D_3)u)(Z) = p(Z)\hat{u}(Z).$$

Here $Z = (\xi,\eta,\zeta)$ stays for the Fourier variables corresponding to (x,y,z) and

$$p(Z) = -\xi^3 + \frac{\eta^2}{\xi} + \frac{\zeta^2}{\xi}$$

With $H^{b,s}(\mathbf{R}^4)$ we denote the classical Sobolev spaces equipped with the norm

$$\|u; H^{b,s}(\mathbf{R}^4)\| = \|\langle\tau\rangle^b\langle Z\rangle^s\hat{u}(\tau,Z); L^2_{(\tau,Z)}\|,$$

Further we define modified Sobolev spaces $\widetilde{H}^{b,s}(\mathbf{R}^4)$

$$\widetilde{H}^{b,s}(\mathbf{R}^4) = \{u \in H^{b,s}(\mathbf{R}^4) : \mathcal{F}^{-1}(|\xi|^{-1}\hat{u}(\tau,Z)) \in H^{b,s}(\mathbf{R}^4)\},$$

equipped with the norm

$$\|u; \widetilde{H}^{b,s}(\mathbf{R}^4)\| = \|u; H^{b,s}(\mathbf{R}^4)\| + \||D_1|^{-1}u; H^{b,s}(\mathbf{R}^4)\|.$$

Let $U(t) = \exp(-itp(D_1, D_2, D_3))$. We shall solve (28) in the Bourgain type spaces $B^{b,s}$ equipped with the norm

$$\|u; B^{b,s}\| = \|U(-t)u; \widetilde{H}^{b,s}\|.$$

In the terms of the Fourier transform variables the norm of $B^{b,s}$ can be expressed as

$$\|u; B^{b,s}\| = \|\langle\tau + p(Z)\rangle^b \langle Z\rangle^s \hat{u}(\tau, Z); L^2_{(\tau,Z)}\| + \|\langle\tau + p(Z)\rangle^b |\xi|^{-1} \langle Z\rangle^s \hat{u}(\tau, Z); L^2_{(\tau,Z)}\|$$

The next estimate is essential for obtaining the result in three space dimensions.

PROPOSITION 4. *Let $s > 3/2$ and*

$$supp\ u \subset \{(t, x) : |t| \leq cT\}.$$

Then there exists $\gamma > 0$ such that the following inequality holds

$$\|uu_x; B^{-1/2,s}\| \leq cT^\gamma \|u; B^{1/2,s}\|^2. \tag{29}$$

Using Proposition 4 we can prove the following Theorem.

THEOREM 2. *Let $s > 3/2$. Then for any $\phi \in \widetilde{H}^s$ there exist positive $T = T(\|\phi\|_{\widetilde{H}^s})$ and a unique solution $u(t, x, y, z)$ of (27) in the time interval $[-T, T]$ satisfying*

$$u \in C([-T, T]; \widetilde{H}^s(\mathbf{R}^3)) \cap B^{1/2,s}.$$

Remark. Similarly to the two dimensional case we can define the non-isotropic Sobolev spaces $H^{s_1,s_2,s_3}_{x,y,z}(\mathbf{R}^3)$. Note that if $u(t, x, y, z)$ is a solution of (27) then so is

$$u_\lambda(t, x, y, z) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y, \lambda^2 z).$$

We have that

$$\|u_\lambda(t, ., .)\|_{\dot{H}^{s_1,s_2,s_2}} = \lambda^{s_1+4s_2-1/2} \|u(\lambda^3 t, ., .)\|_{\dot{H}^{s_1,s_2,s_2}}$$

Hence one may expect that for $s_1 + 4s_2 - 1/2 < 0$ the Cauchy problem (27) is ill-posed in H^{s_1,s_2,s_2} (in particular in L^2).

4.1. Proof of Proposition 4. In the proof of Proposition 4 we shall need the following estimate.

PROPOSITION 5. *(*Strichartz inequality*) Let*

$$0 \leq \gamma \leq 1, \quad 0 < \theta \leq 1, \quad \epsilon_1 > 0, \quad 1/2 \leq b \leq 1/2 + \epsilon_1, \quad supp\ u \subset \{(t, x) : |t| \leq cT\}$$

Then for any $u \in L^2(\mathbf{R}^4)$ the following inequality holds

$$\|\mathcal{F}^{-1}(|\xi|^{-\frac{(1-\gamma)b\delta(r)}{8(1/2+\epsilon_1)}} \langle\tau + p(Z)\rangle^{-b} |\hat{u}(\tau, Z)|); L^q_t(L^r_{(x,y,z)})\| \leq cT^{\gamma^*/2} \|u\|_{L^2}, \tag{30}$$

where $\gamma^* = \gamma$ if $b > 1/2$, $\gamma^* = \gamma - 0$ if $b = 1/2$ and provided q and r satisfy the next relations

$$\frac{2}{q} = 1 - \frac{\theta(1-\gamma)b}{1/2 + \epsilon_1}, \quad \delta(r) := 3(\frac{1}{2} - \frac{1}{r}) = \frac{(1-\theta)(1-\gamma)b}{1/2 + \epsilon_1}.$$

Proof. For any $\phi \in L^2(\mathbf{R}^3)$ Theorem 4.2 of **[5]** (cf. also the Appendix) yields

$$\| |D_x|^{-\frac{\delta(r)}{6}} U(t)\phi; L_t^q(L^r_{(x,y,z)})\| \leq c\|\phi\|_{L^2}, \tag{31}$$

provided

$$\frac{2}{q} = \delta(r), \quad 2 < q \leq \infty.$$

Like in the proof of Proposition 2 we obtain

$$\|\mathcal{F}^{-1}(|\xi|^{-\frac{(1-\gamma)b'\delta(r)}{6(1/2+\epsilon_1)}} \langle \tau + p(Z)\rangle^{-b'}|\hat{u}(\tau, Z)|); L_t^q(L^r_{(x,y,z)})\| \leq c\|u\|_{L^2}, \tag{32}$$

provided

$$\frac{2}{q} = 1 - \frac{\theta b'}{1/2+\epsilon_1}, \quad \delta(r) = \frac{(1-\theta)b'}{1/2+\epsilon_1}.$$

Now using (32) with $b' = (1-\gamma)b$ we obtain

$$\|\mathcal{F}^{-1}(|\xi|^{-\frac{(1-\gamma)b\delta(r)}{6(1/2+\epsilon_1)}} \langle \tau + p(Z)\rangle^{-b}|\hat{u}(\tau, Z)|); L_t^q(L^r_{(x,y,z)})\| \leq c\|\langle \tau + p(Z)\rangle^{-\gamma b}\hat{u}(\tau, Z); L^2\|,$$

provided

$$\frac{2}{q} = 1 - \frac{\theta(1-\gamma)b}{1/2+\epsilon_1}, \quad \delta(r) = \frac{(1-\theta)(1-\gamma)b}{1/2+\epsilon_1}.$$

The argument of **[24]**, Lemma 3.1. yields

$$\|\langle \tau + p(Z)\rangle^{-\gamma b}\hat{u}(\tau, Z)\|_{L^2} \leq cT^{\gamma^*/2}\|u\|_{L^2},$$

which completes the proof of Proposition 5.

Further we set

$$\hat{w}(\tau, Z) = \langle \tau + p(Z)\rangle^{1/2}\langle Z\rangle^s \hat{u}(\tau, Z)$$

By duality we obtain that (29) is equivalent to

$$|\int\int K(\tau, Z, \tau_1, Z_1)\hat{w}(\tau_1, Z_1)\hat{w}(\tau - \tau_1, Z - Z_1)\hat{v}(\tau, Z)d\tau_1 dZ_1 d\tau dZ| \leq$$

$$cT^{\gamma}\{\|w\|_{L^2}^2 + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|_{L^2}^2\}\|v\|_{L^2} \tag{33}$$

where

$$K(\tau, Z, \tau_1, Z_1) = \frac{\langle\xi\rangle\langle Z\rangle^s\langle Z_1\rangle^{-s}\langle Z - Z_1\rangle^{-s}}{\langle \tau_1 + p(Z_1)\rangle^{1/2}\langle \tau - \tau_1 + p(Z - Z_1)\rangle^{1/2}\langle \tau + p(Z)\rangle^{1/2}}$$

Without loss of generality we can assume that $\hat{w} \geq 0$ and $\hat{v} \geq 0$. We set

$$\sigma := \sigma(\tau, Z) = \tau - \xi^3 + \frac{\eta^2 + \zeta^2}{\xi}, \quad \sigma_1 := \sigma(\tau_1, Z_1), \quad \sigma_2 := \sigma(\tau - \tau_1, Z - Z_1)$$

We have the relation

$$\sigma_1 + \sigma_2 - \sigma = 3\xi_1\xi(\xi - \xi_1) + \frac{(\xi_1\eta - \xi\eta_1)^2}{\xi_1\xi(\xi - \xi_1)} + \frac{(\xi_1\zeta - \xi\zeta_1)^2}{\xi_1\xi(\xi - \xi_1)}$$

Hence

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi_1\xi(\xi - \xi_1)|. \tag{34}$$

By symmetry arguments we can assume that

$$|\sigma_1| \geq |\sigma_2|.$$

By J we note the left-hand side of (33). As in the proof of Proposition 3 we shall use the Strichartz inequality with v just when $|\xi| \geq 2$. We consider three main cases.

Case 1. $|\xi| \leq 2$. We denote by J_1 the restriction of J on this region. We can assume that $|Z| \leq 2|Z_1|$. If $|Z| \geq 2|Z_1|$ then $|Z| \leq 2|Z - Z_1|$ and the arguments are similar. Using Hölder, Strichartz and Sobolev inequalities we obtain

$$\begin{aligned}
J_1 &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-1/2}\hat{w}(\tau_1, \zeta_1)); L_t^{q_1}(L^2_{(x,y,z)})\| \\
&\quad \|\mathcal{F}^{-1}(\langle Z - Z_1\rangle^{-s}\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau - \tau_1, Z - Z_1)); L_t^{q_2}(L^\infty_{(x,y,z)})\|\|\|v\|_{L^2} \\
&\leq c\|\mathcal{F}^{-1}(|\xi - \xi_1|^{-\frac{(1-\gamma)\delta(r_2)}{6(1+2\epsilon_1)}}\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau - \tau_1, Z - Z_1)); L_t^{q_2}(L^{r_2}_{(x,y,z)})\|\|\|w\|_{L^2}\|v\|_{L^2}\| \\
&\leq cT^{\gamma^\star/2}\|w\|^2_{L^2}\|v\|_{L^2},
\end{aligned}$$

provided

$$\frac{2}{q_2} = 1 - \frac{\theta(1-\gamma)}{1+2\epsilon_1} \quad , \quad \delta(r_2) = \frac{(1-\theta)(1-\gamma)}{1+2\epsilon_1},$$

$$\frac{2}{q_1} = 1 - \frac{1}{1+2\epsilon_1} \quad , \quad s > \frac{(1-\gamma)\delta(r_2)}{6(1+2\epsilon_1)} + \frac{3}{r_2}.$$

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}.$$

We take $\gamma = \epsilon$ and hence

$$\theta = \frac{2\epsilon_1}{1-\epsilon},$$

i.e. $0 < \eta \leq 1$ for ϵ and ϵ_1 sufficiently small. The Sobolev embedding restriction becomes

$$s \geq 3/2 - \frac{(5 + 12\epsilon_1 + \epsilon)(1 - \epsilon - 2\epsilon_1)}{6(1+2\epsilon_1)^2},$$

which is fulfilled for $s > 2/3$ and ϵ_1, ϵ sufficiently small.

Case 2. $|\sigma| \geq |\sigma_1|$, $|\xi| \geq 2$. We denote by J_2 the restriction of J on this region. We can assume that $|Z| \leq 2|Z_1|$. If $|Z| \geq 2|Z_1|$ then $|Z| \leq 2|Z - Z_1|$ and the arguments are similar.

Case 2.1. $|\xi_1| \leq 1$. We denote by J_{21} the restriction of J_2 on this region. Using (34) we obtain

$$|\sigma| \geq c|\xi|^2|\xi_1|. \tag{35}$$

Using (35), Hölder, Strichartz and Sobolev inequalities we obtain

$$\begin{aligned}
J_{21} &\le \|\mathcal{F}^{-1}(\langle Z-Z_1\rangle^{-s}\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1,Z-Z_1));L_t^{q_1}(L^\infty_{(x,y,z)})\| \\
&\quad \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-1/2}|\xi_1|^{-1}\hat{w}(\tau_1,Z_1));L_t^{q_2}(L^2_{(x,y,z)})\|\|v\|_{L^2} \\
&\le \|\mathcal{F}^{-1}(|\xi-\xi_1|^{-\frac{(1-\gamma)\delta(r_1)}{6(1+2\epsilon_1)}}\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1,Z-Z_1));L_t^{q_1}(L^{r_1}_{(x,y,z)})\| \\
&\quad \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-1/2}|\xi_1|^{-1}\hat{w}(\tau_1,Z_1));L_t^{q_2}(L^2_{(x,y,z)})\|\|v\|_{L^2} \\
&\le cT^{\gamma^\star/2}\|w\|_{L^2}\||D_1|^{-1}w\|_{L^2}\|v\|_{L^2},
\end{aligned}$$

provided

$$\frac{2}{q_1}=1-\frac{\theta(1-\gamma)}{1+2\epsilon_1}\quad,\quad \delta(r_1)=\frac{(1-\theta)(1-\gamma)}{1+2\epsilon_1},$$

$$\frac{2}{q_2}=1-\frac{1}{1+2\epsilon_1}\quad,\quad s>\frac{(1-\gamma)\delta(r_1)}{6(1+2\epsilon_1)}+\frac{3}{r_1}.$$

$$\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{2}.$$

We take $\gamma=\epsilon$ and hence

$$\theta=\frac{2\epsilon_1}{1-\epsilon},$$

i.e. $0<\eta\le 1$ for ϵ and ϵ_1 sufficiently small. The Sobolev embedding restriction becomes

$$s\ge 3/2-\frac{(5+12\epsilon_1+\epsilon)(1-\epsilon-2\epsilon_1)}{6(1+2\epsilon_1)^2},$$

which is fulfilled for $s>2/3$ and ϵ_1,ϵ sufficiently small.

Case 2.2. $|\xi_1|\ge 1$. Using (34) we obtain

$$|\sigma|\ge c|\xi|^2.$$

Now we can use the arguments of Case 2.1.

Case 3. $|\sigma_1|\ge|\sigma|$, $|\xi|\ge 2$. We denote by J_3 the restriction of J on this region. We have that

$$J_3\le\int\int\frac{\langle\xi\rangle\langle Z\rangle^s\hat{w}(\tau_1,Z_1)\hat{w}(\tau-\tau_1,Z-Z_1)\hat{v}(\tau,Z)d\tau dZd\tau_1dZ_1}{\langle Z-Z_1\rangle^s\langle Z_1\rangle^s\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}$$

Case 3.1. $|\xi_1|\le 1$. We denote by J_{31} the restriction of J_3 on this region. Using (34) we obtain

$$|\sigma_1|\ge\frac{1}{2}|\xi|^2|\xi_1|. \tag{36}$$

Case 3.1.1. $|\xi_1| \le 1,\quad |Z| \le 2|Z_1|$. We denote by J_{311} the restriction of J_{31} on this region. Using (36), Hölder, Strichartz and Sobolev inequalities we obtain

$$\begin{aligned} J_{311} &\le \|\mathcal{F}^{-1}(\langle Z - Z_1\rangle^{-s}\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1, Z-Z_1)); L_t^{q_1}(L^{\infty}_{(x,y,z)})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2}\hat{v}(\tau, Z)); L_t^{q_2}(L^2_{(x,y,z)})\|\||D_1|^{-1}w\|_{L^2} \\ &\le \|\mathcal{F}^{-1}(|\xi-\xi_1|^{-\frac{(1-\gamma)\delta(r_1)}{6(1+2\epsilon_1)}}\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1, Z-Z_1)); L_t^{q_1}(L^{r_1}_{(x,y,z)})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2}\hat{v}(\tau, Z)); L_t^{q_2}(L^2_{(x,y,z)})\|\||D_1|^{-1}w\|_{L^2} \\ &\le cT^{\gamma^*/2}\|w\|_{L^2}\||D_1|^{-1}w\|_{L^2}\|v\|_{L^2}, \end{aligned}$$

provided

$$\frac{2}{q_1} = 1 - \frac{\theta(1-\gamma)}{1+2\epsilon_1}\quad,\quad \delta(r_1) = \frac{(1-\theta)(1-\gamma)}{1+2\epsilon_1},$$

$$\frac{2}{q_2} = 1 - \frac{1}{1+2\epsilon_1}\quad,\quad s > \frac{(1-\gamma)\delta(r_1)}{6(1+2\epsilon_1)} + \frac{3}{r_1}.$$

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}.$$

We take $\gamma = \epsilon$ and hence

$$\theta = \frac{2\epsilon_1}{1-\epsilon},$$

i.e. $0 < \theta \le 1$ for ϵ and ϵ_1 sufficiently small. The Sobolev embedding restriction becomes

$$s \ge 3/2 - \frac{(5 + 12\epsilon_1 + \epsilon)(1-\epsilon-2\epsilon_1)}{6(1+2\epsilon_1)^2},$$

which is fulfilled for $s > 2/3$ and ϵ_1, ϵ sufficiently small.

Case 3.1.2. $|\xi_1| \le 1,\quad |Z| \ge 2|Z_1|$. We denote by J_{312} the restriction of J_{31} on this region. In this case we have that $|Z| \le 2|Z - Z_1|$. Using Hölder, Strichartz and Sobolev inequalities we obtain

$$\begin{aligned} J_{312} &\le \|\mathcal{F}^{-1}(\langle Z_1\rangle^{-s}|\xi_1|^{-1}\hat{w}(\tau_1, Z_1)); L_t^2(L^{\infty}_{(x,y,z)})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-1/2}\hat{w}(\tau-\tau_1, Z-Z_1)); L_t^{q_2}(L^2_{(x,y,z)})\| \\ &\quad \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2}\hat{v}(\tau, Z)); L_t^{q_3}(L^2_{(x,y,z)})\| \\ &\le cT^{\gamma^*}\|w\|_{L^2}\||D_1|^{-1}w\|_{L^2}\|v\|_{L^2}. \end{aligned}$$

provided

$$\frac{2}{q_2} = 1 - \frac{(1-\gamma)}{1+2\epsilon_1}\quad,\quad s > \frac{3}{2},$$

$$\frac{2}{q_3} = 1 - \frac{(1-\gamma)}{1+2\epsilon_1}\quad,\quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}.$$

Now it is sufficient to take $\gamma = \frac{1-2\epsilon_1}{2}$ (i.e. $q_2 = q_3 = 4$) to ensure the restriction of the Hölder inequality.

Case 3.2. $|\xi - \xi_1| \leq 1$. This case can be treated in the same way as Case 3.1.

Case 3.3. $|\xi - \xi_1| \geq 1, \quad |\xi_1| \geq 1$. In this case we have that

$$|\sigma_1| \geq |\xi||\xi_1||\xi - \xi_1| \geq c|\xi|^2$$

Now we can estimate J like in Case 3.1. This completes the proof of Proposition 4.

4.2. Proof of Theorem 2. As in the two dimensional case we shall solve globally in time the truncated integral equation

$$u(t) = \psi(t)U(t)u(0) - \frac{1}{2}\psi(t/T)\int_0^t U(t-t')\partial_x(u^2(t'))dt'. \tag{37}$$

Note that (37) is equivalent to

$$u(t) = \psi(t)U(t)u(0) - \frac{1}{2}\psi(t/T)\int_0^t U(t-t')\partial_x(\psi^2(t'/2T)u^2(t'))dt'. \tag{38}$$

Hence we can assume the nonlinearity to be truncated too. We shall solve (38) in $B^{1/2,s}$. In this case an additional term appears in the linear estimate. For that purpose we introduce the auxiliary spaces Y^s

$$Y^s = \{u : \|u; Y^s\| < \infty\},$$

where

$$\|u;Y^s\| = \|\langle\tau\rangle^{-1}\langle Z\rangle^s\mathcal{F}(U(-t)u); L^2_Z(L^1_\tau)\| + \|\langle\tau\rangle^{-1}\langle Z\rangle^s|\xi|^{-1}\mathcal{F}(U(-t)u); L^2_Z(L^1_\tau)\|$$

The linear estimate has the form

PROPOSITION 6. *(cf.* **[24]**, *Lemma 2.1).*
The following estimate holds

$$\left\|\psi(t/T)\int_0^t U(t-t')\partial_x(u^2(t'))dt'; B^{1/2,s}\right\| \leq c\{\|uu_x; B^{-1/2,s}\| + \|uu_x; Y^s\|\}.$$

Hence in order to apply a fixed point argument we need the following Proposition.

PROPOSITION 7. *Let $s > 3/2$ and*

$$supp\ u \subset \{(t,x) : |t| \leq cT\}.$$

Then there exists $\gamma > 0$ such that the next inequality holds

$$\|uu_x; Y^s\| \leq cT^\gamma\|u; B^{1/2,s}\|^2. \tag{39}$$

Proof. Using a duality argument we obtain that (39) is equivalent to

$$|\int\int K(\tau, Z, \tau_1, Z_1)\hat{w}(\tau_1, Z_1)\hat{w}(\tau-\tau_1, Z-Z_1)\hat{v}(Z)d\tau_1 dZ_1 d\tau dZ| \leq$$

$$cT^\gamma\{\|w\|^2_{L^2} + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|^2_{L^2}\}\|v\|_{L^2}, \tag{40}$$

where

$$K(\tau, Z, \tau_1, Z_1) = \frac{\langle\xi\rangle\langle Z\rangle^s\langle Z_1\rangle^{-s}\langle Z - Z_1\rangle^{-s}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\sigma\rangle}$$

When $|\xi| \leq 2$ we can perform the same arguments as in Case 1 of the proof of Proposition 4 replacing $\hat{v}(\tau, Z)$ with $\langle\sigma\rangle^{-1}\hat{v}(Z) \in L^2_{(\tau,Z)}$. When $|\sigma_1| \geq |\sigma|$ we can use the same arguments as in the proof of Proposition 4 replacing $\hat{v}(\tau, Z)$ with $\langle\sigma\rangle^{-1/2-\epsilon}\hat{v}(Z) \in L^2_{(\tau,Z)}$ for sufficiently small ϵ. When $|\sigma| \geq |\sigma_1|$ and $|\xi| \geq 2$ we have that

$$|\sigma| \geq c|\xi|^2|\xi_1|, \quad \text{when} \quad |\xi_1| \leq 1$$

and

$$|\sigma| \geq c|\xi|^2, \quad \text{when} \quad |\xi_1| \geq 1.$$

Hence

$$\frac{\langle\xi\rangle\hat{v}(Z)}{\langle\sigma\rangle} \leq \frac{c\langle\xi\rangle\hat{v}(Z)}{\langle\sigma\rangle + |\xi_1||\xi|^2} \leq \frac{c\langle\xi\rangle\hat{v}(Z)}{|\xi_1|(\langle\sigma\rangle + |\xi|^2)}, \quad \text{when} \quad |\xi_1| \leq 1$$

and

$$\frac{\langle\xi\rangle\hat{v}(Z)}{\langle\sigma\rangle} \leq \frac{c\langle\xi\rangle\hat{v}(Z)}{\langle\sigma\rangle + |\xi|^2}, \quad \text{when} \quad |\xi_1| \geq 1.$$

It remains to note that

$$\frac{\langle\xi\rangle\hat{v}(Z)}{\langle\sigma\rangle + |\xi|^2} \in L^2_{(\tau,Z)}, \quad \text{with} \quad \left\|\frac{\langle\xi\rangle\hat{v}(Z)}{\langle\sigma\rangle + |\xi|^2}; L^2_{(\tau,Z)}\right\| \leq c\|v\|_{L^2}$$

and to apply the arguments of Proposition 4. This completes the proof of Proposition 7.

We shall solve (38) by the contraction mapping principle for sufficiently small T. Using Proposition 6, Proposition 4 and Proposition 7 we obtain

$$\|\psi(t/T)\int_0^t U(t-t')\partial_x(\psi^2(t'/2T)u^2(t'))dt'; B^{1/2,s}\| \leq$$

$$c(\|\partial_x(\psi^2(t/2T)u^2(t,Z)); B^{-1/2,s}\| + \|\partial_x(\psi^2(t/2T)u^2(t,Z)); Y^s\|) \leq$$

$$cT^\gamma\|\psi(t/2T)u(t,Z); B^{1/2,s}\|^2.$$

But we can easily prove that for any $\epsilon > 0$ there exists $c > 0$ such that

$$\|\psi(t/2T)u(t,Z); B^{1/2,s}\| \leq cT^{-\epsilon}\|u; B^{1/2,s}\|. \tag{41}$$

To prove (41) we use the next inequality due to Coifman and Meyer (cf. [18])

$$\|ab\|_{H^{1/2}(\mathbf{R})} \leq c(\|a\|_{H^{1/2}(\mathbf{R})}\|b\|_{L^\infty(\mathbf{R})} + \|a\|_{L^{\frac{2(2+\epsilon)}{\epsilon}}(\mathbf{R})}\|J^{1/2}b\|_{L^{2+\epsilon}(\mathbf{R})}), \tag{42}$$

where $J = (1 - \frac{d^2}{dx^2})^{1/2}$. Using (42) and Sobolev embedding we obtain

$$\|\psi(t/2T)u(t,Z)); B^{1/2,s}\| \leq$$

$$c(\|\psi\|_{L^\infty}\|\|u; B^{1/2,s}\| + \|J^{1/2}\psi(t/2T)\|_{L^{2+\epsilon}}\|\|u; B^{1/2,s}\|) =$$

$$c(\|\psi\|_{L^\infty} + T^{-\frac{\epsilon}{2(2+\epsilon)}}\|J^{1/2}\psi\|_{L^{2+\epsilon}})\|u; B^{1/2,s}\|,$$

which proves (41). Hence

$$\left\| \psi(t/T) \int_0^t U(t-t')\partial_x(\psi^2(t'/2T)u^2(t'))dt'; B^{1/2,s} \right\| \leq cT^{(\gamma-\epsilon)} \|u; B^{1/2,s}\|^2.$$

As in the proof of Theorem 1 a fixed point argument completes the proof of Theorem 2.

5. Appendix. Strichartz inequality for 3D KP equation

In this appendix we give the proof of (31). This estimate is essentially established in [5] but we give the proof again for completeness. In the particular case of KP equations the arguments are simpler then these [5] where a large class of estimates of type (31) for some generalized KP equations are derived. In particular a smoothing effect is established if we replace the term u_{xxx} with u_{xxxxx} in KP equation.

PROPOSITION 8. *For any $\phi \in L^2(\mathbf{R}^3)$ the following inequality holds*

$$\| |D_x|^{-\frac{\delta(r)}{6}} U(t)\phi \|_{L^q_t(L^r_{(x,y,z)})} \leq c\|\phi\|_{L^2}, \tag{43}$$

where

$$\frac{2}{q} = \delta(r).$$

First we shall prove a $L^p - L^q$ decay estimate for the solution of 3D KP equation.

LEMMA 3. *The next inequality holds*

$$\| |D_x|^{-\frac{\delta(r)}{3}} U(t)\phi; L^r_{(x,y,z)} \| \leq ct^{-\delta(r)} \|\phi\|_{L^{r'}}, \tag{44}$$

where

$$2 \leq r \leq \infty, \quad \frac{1}{r} + \frac{1}{r'} = 1$$

Proof. Using Fourier transform we obtain

$$U(t)\phi = \phi \star S(t,.,.,.)$$

where

$$S(t,x,y,z) = \int_{\mathbf{R}^3} \exp(i(x\xi + y\eta + z\zeta + tp(Z))d\xi d\eta d\zeta. \tag{45}$$

We change the variables as in [48]

$$\bar{\eta} = |t/\xi|^{1/2}\eta, \quad \bar{\zeta} = |t/\xi|^{1/2}\zeta.$$

Hence we have that

$$S(t,x,y,z) = \frac{c}{t} \int_{-\infty}^{\infty} |\xi| \exp(i(y^2/t + z^2/t + x)\xi + it\xi^3)d\xi$$

Further we have

$$U(t)\phi = \frac{c}{t} \int_{\mathbf{R}^3} \int_{-\infty}^{\infty} |\xi| \exp(i(y_1^2/t + z_1^2/t + x_1)\xi + it\xi^3)d\xi dx_1 dy_1 dz_1 \leq$$

$$\frac{c}{t} sup_{x \in [-\infty,\infty]} \left| \int_{-\infty}^{\infty} |\xi|^{\frac{1}{2}} \exp(ix\xi + it\xi^3) d\xi \right| \| |D_x|^{1/2}\phi\|_{L^1(\mathbf{R}^3)}.$$

Now using Lemma 2.1. of **[39]** we obtain

$$\|U(t)\phi\|_{L^\infty} \le \frac{c}{t^{3/2}} \| |D_x|^{1/2} \phi \|_{L^1(\mathbf{R}^3)}. \tag{46}$$

The proof of (44) follows by interpolation between (46) and the conservation of the L^2 norm for KP equations. This completes the proof of the Lemma.

Proof of Proposition 8.

By duality argument (43) is equivalent to

$$\left| \int_{\mathbf{R}^4} (|D_x|^{-\frac{\delta(r)}{6}} (S(t,.) \star \phi))\psi \right| \le c\|\phi\|_{L^2} \|\psi\|_{L_t^{q'}(L^{r'}_{(x,y,z)})}, \tag{47}$$

where

$$\frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1.$$

Using Plancherel equality we obtain that (47) is equivalent to

$$\left| \int_{\mathbf{R}^4} |\xi_1|^{-\frac{\delta(r)}{6}} \tilde{S}(\tau, Z) \hat{\phi}(Z) \hat{\psi}(\tau, Z) d\tau dZ \right| \le c\|\phi\|_{L^2} \|\psi\|_{L_t^{q'}(L^{r'}_{(x,y,z)})}. \tag{48}$$

Here $\tilde{S}(\tau, Z)$ stays for the space-time Fourier transform of S. We have that

$$\tilde{S}(\tau, Z) = \bar{\delta}(\tau - p(Z)),$$

where $\bar{\delta}$ is the Dirac delta function. Hence one can estimate the left hand-side of (48) as follows

$$\left| \int_{\mathbf{R}^4} |\xi|^{-\frac{\delta(r)}{6}} \tilde{S}(\tau, Z) \hat{\phi}(Z) \hat{\psi}(\tau, Z) d\tau dZ \right| \le c\|\phi\|_{L^2} \| |\xi|^{-\frac{\delta(r)}{6}} \hat{\psi}(p(Z), Z) : L^2_Z \|.$$

Therefore Proposition 8 will follow from the next Lemma.

LEMMA 4. *(restriction lemma) The following Fourier transform restriction inequality holds*

$$\| |\xi|^{-\frac{\delta(r)}{6}} \hat{\psi}(p(Z), Z) : L^2_Z \| \le c\|\psi\|_{L_t^{q'}(L^{r'}_{(x,y,z)})}.$$

Proof. We have that

$$\left| \int_{\mathbf{R}^3} |\xi|^{-\frac{\delta(r)}{3}} |\hat{\psi}(p(Z), Z)|^2 dZ \right| = \left| \int_{\mathbf{R}^4} \tilde{S}_1(\tau, Z) |\hat{\psi}(\tau, Z)|^2 d\tau d\tau dZ \right| =$$

$$\left| \int_{\mathbf{R}^4} (S_1 \star \psi) \bar{\psi} dt dx dy dz \right| \le c\|S_1 \star \psi\|_{L_t^q(L^r_{x,y,z})} \|\psi\|_{L_t^{q'}(L^{r'}_{x,y,z})}.$$

Here $S_1(t, x, y, z)$ is defined through its Fourier transform

$$\tilde{S}_1(\tau, Z) = |\xi|^{-\frac{\delta(r)}{3}} \bar{\delta}(\tau - p(Z))$$

Using Lemma 4 and weak Young inequality we obtain

$$\|S_1 \star \psi\|_{L_t^q(L^r_{x,y,z})} \leq$$

$$c\|\int_{\infty}^{\infty} \|\mathcal{F}^{-1}_{Z\to(x,y,z)}(|\xi|^{\frac{\delta(r)}{3}} \exp(-i(t-s)p(Z))\hat{\psi}(s,Z)); L^r_{(x,y,z)}\|ds\|_{L_t^q} \leq$$

$$c\|\int_{\infty}^{\infty} \|\psi(s,.)); L^{r'}_{(x,y,z)}\|(t-s)^{-\delta(r)}ds\|_{L_t^q} \leq c\|\psi\|_{L_t^{q'}(L^{r'}_{(x,y,z)})}.$$

The restriction of Young inequality is

$$\frac{1}{q} + 1 = \frac{1}{q'} + \delta(r), \quad \text{i.e.} \quad \frac{2}{q} = \delta(r).$$

This completes the proof of the Lemma and Proposition 8.

CHAPTER 4

Global low regularity solutions for Kadomtsev-Petviashvili equation

This Chapter essentially contains the paper [**58**] *(Global low regularity solutions for Kadomtsev-Petviashvili equation, to appear in Differential Integral Equations).*

Abstract.

We study the initial value problem for KP-II equation. We prove the existence of solutions to the integral equation corresponding to KP-II for any data in L^2, removing the additional condition imposed in [**57**]. Following a method recently developed by J. Bourgain we obtain global solutions to KP-II with data below L^2.

AMS Subject Classification: 35Q53, 35Q51, 35A07.

1. Introduction

In this Chapter we continue the study of the initial value problem for Kadomtsev-Petviashvili-II equation (KP-II). The Kadomtsev-Petviashvili equations are two dimensional extensions of the Korteweg-de Vries (KdV) equation. They are "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one-directional, with weak transverse effects. The Cauchy problem for KP-II equation in $\mathbf{R}^2$ has the form

$$(1) \qquad \begin{cases} (u_t + u_{xxx} + uu_x)_x + u_{yy} = 0 \\ u(0,x,y) = \phi(x,y) \end{cases}$$

In order to solve the initial value problem (1) different methods could be applied. For instance (cf. [**61**]) the inverse scattering technique provides global existence results for (1) under some decay assumptions on the initial data. Using energy type estimates for (1) local well-posedness in H^s for $s > 2$ could be obtained (cf. [**31**]). The condition for s is in order to control the L^∞ norm of the gradient of the solution. The equation (1) possesses an infinite number of conserved quantities. Unfortunately the only useful one to provide global a priori bound seems to be the L^2 norm.

In [**12**], [**13**] J. Bourgain developed a new method to study the local regularity of some important nonlinear evolution equations. In the favorable cases the local well-posedness results together with the conservation laws provide global well-posedness. The method was first applied for the nonlinear Schrödinger equation and KdV equation. One of the main points of the method is the choice of the functional space where the solutions are expected to belong. These are Sobolev type spaces based on L^2 which take into account the specific structure of the respective equation. The solutions are obtained by a fixed point argument for the equivalent integral equation. The linear estimates are similar for all equations. They are in fact one dimensional with respect to time (cf. [**23**]). The main difficulty concerns the nonlinear estimates. There are at least two different methods for deriving the nonlinear estimates. The first of them makes an essential use of the Strichartz-type inequalities injected into the framework of Bourgain's spaces. The second method is based on a direct estimate for the kernel in the integral representation of the nonlinear estimate.

Returning to KP-II equation, some results (using Bourgain's method) concerning local well-posedness of (1) have recently appeared. The method of Bourgain is applied to the evolution equation derived by (1)

$$\left\{\begin{array}{l} i\partial_t u = p(D_1, D_2)u - iuu_x \\ u(0,x,y) = \phi, \end{array}\right. \tag{2}$$

where $D_1 = \frac{1}{i}\partial_x$, $D_2 = \frac{1}{i}\partial_y$ and the operator $p(D_1, D_2)$ is defined through the Fourier transform

$$\mathcal{F}(p(D_1, D_2)u)(\zeta) = p(\zeta)\widehat{u}(\zeta).$$

Here $\zeta = (\xi, \eta)$ stays for the Fourier variables corresponding to (x, y) and

$$p(\zeta) = -\xi^3 + \frac{\eta^2}{\xi}$$

The method consists of applying a Picard fixed point argument in a suitable functional space to the integral equation corresponding to (2)

$$u(t) = U(t)\phi - 1/2\int_0^t U(t-t')\partial_x(u^2(t'))dt', \tag{3}$$

where $U(t) = \exp(-itp(D_1, D_2))$ is the unitary group which defines the free evolution of (2).

In [**14**] local solutions of (3) in L^2 are obtained. The nonlinear estimate uses a dyadic decomposition related to the symbol of the linearized operator. A short proof of local well-posedness in H^s, $s > 0$, which uses Strichartz inequalities, is done in [**56**]. The same result is obtained in [**32**], where the nonlinear estimates are obtained by the second method explained above. In [**57**] local well-posedness of (1) in Sobolev spaces with negative indices with respect to x is obtained. However, an additional condition on initial data is imposed. More precisely, it is assumed that $|\xi|^{-1}\widehat{\phi}(\xi, \eta) \in L^2$, which seems to be quite restrictive. Our first goal in that Chapter is to prove local existence and uniqueness of (3) for any data $\phi \in L^2$, removing the additional condition on data imposed in [**57**] (cf. Theorem 2 below). The solutions are in spaces similar to these proposed by J. Bourgain [**14**]. Therefore in some sense we give an alternative proof of the result of [**14**] in the case of data on $\mathbf{R}^2$ (in [**14**] the periodic case is also considered).

In [**15**] J. Bourgain developed a method to obtain global well-posedness in Sobolev spaces with fractional indices between these where global well-posedness is provided by the conservation laws and the local existence. The second goal of this Chapter is to show that this approach provides global solutions below L^2 (cf. Theorem 5 below). Actually we use a similar argument to this performed in [**19**] in the context of the KdV equation. In [**21**] the approach is applied for the Modified KdV equation. More precisely global well-posedness for the Modified KdV equation for data in $H^s(\mathbf{R})$, $s > 3/5$ is obtained. The proof in [**21**] does not use Bourgain's type Fourier transform restriction norms. It relies on some sharp estimates for the unitary group associated to the linearized KdV equation and in particular on the dual version of the sharp Kato's smoothing effect.

With $X^{b,b_1}(\mathbf{R}^3)$ we denote the spaces equipped with the norm

$$\|u; X^{b,b_1}(\mathbf{R}^3)\| = \left\| \langle\tau\rangle^b \left(1 + \frac{\langle\tau\rangle^{b_1}}{\langle\xi\rangle^{1/4}}\right) \widehat{u}(\tau,\xi,\eta); L^2_{(\tau,\zeta)} \right\|,$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. Now we define the Bourgain type spaces $B^{b,b_1}(\mathbf{R}^3)$ equipped with the norm

$$\|u; B^{b,b_1}\| = \|U(-t)u; X^{b,b_1}\|.$$

Since $\mathcal{F}(U(-t)u)(\tau,\zeta) = \widehat{u}(\tau - p(\zeta),\zeta)$ we obtain that in terms of the Fourier transform variables the norm of B^{b,b_1} can be expressed as

$$\|u; B^{b,b_1}\| = \left\| \langle\tau + p(\zeta)\rangle^b \left(1 + \frac{\langle\tau + p(\zeta)\rangle^{b_1}}{\langle\xi\rangle^{1/4}}\right) \widehat{u}(\tau,\zeta); L^2_{(\tau,\zeta)} \right\|.$$

By H^s_x we denote the Sobolev space which measures the regularity with respect to the x variable equipped with the norm

$$\|u\|_{H^s_x} = \|\langle\xi\rangle^s \widehat{u}(\xi,.)\|_{L^2_{(\xi,.)}}.$$

Further we define the space B^{b,b_1}_s equipped with the norm

$$\|u; B^{b,b_1}_s\| = \left\| \langle\tau + p(\zeta)\rangle^b \langle\xi\rangle^s \left(1 + \frac{\langle\tau + p(\zeta)\rangle^{b_1}}{\langle\xi\rangle^{1/4}}\right) \widehat{u}(\tau,\zeta); L^2_{(\tau,\zeta)} \right\|.$$

Clearly $B^{b,b_1}_0 = B^{b,b_1}$. Let $I \subset \mathbf{R}$ be an interval. Then we define the space $B^{b,b_1}_s(I)$ equipped with the norm

$$\|u\|_{B^{b,b_1}_s(I)} = \inf_{w \in B^{b,b_1}_s} \{\|w\|_{B^{b,b_1}_s}, w(t) = u(t) \text{ on } I\}.$$

Note that for $b > 1/2$ and $b_1 \geq 0$ one dimensional Sobolev embedding injects B^{b,b_1}_s into $C(\mathbf{R}, H^s_x(\mathbf{R}^2))$. Now we state a bilinear estimate which will be the main tool in the proof of the L^2-Theorem concerning KP-II (Theorem 2 below).

THEOREM 1. *The following inequality holds*

$$\|uu_x; B^{-b',b_1}\| \leq c\|u; B^{b,b_1}\|^2, \tag{4}$$

where

$$b > 1/2, \quad 1/2 > b' > 7/16, \quad b_1 > 11/48, \quad b' - 1/4 > b_1 > 5/8 - b'.$$

In the proof of Theorem 1 we make use of both Strichartz type inequalities and changes of variables similar to those performed by C. Kenig, G. Ponce and L. Vega in the context of KdV equation. We shall use Theorem 1 for $b \sim 1/2$, $b' \sim 1/2$ and $b_1 \sim 1/4$. Under the same conditions for b, b', b_1 the arguments of the proof of Theorem 1 yield

$$\|\partial_x(uv); B^{-b',b_1}\| \leq c\|u; B^{b,b_1}\|\|v; B^{b,b_1}\|.$$

Using Theorem 1 and the arguments of the linear estimates in Bourgain's framework (cf. Section 3) we can obtain the following Theorem.

THEOREM 2. *There exist $b > 1/2$ and $b_1 < 1/4$ such that for any $\phi \in L^2$ there exist a positive $T = T(\|\phi\|_{L^2})$ and a unique solution $u(t,x,y)$ of (3) in the time interval $[-T,T]$ satisfying*

$$u \in C([-T,T]; L^2(\mathbf{R}^2)) \cap B^{b,b_1}([-T,T]).$$

In addition the length T of the existence interval satisfies $T \geq c\|\phi\|_{L^2}^{-6}$.

Let $u(t,x,y)$ be a solution of (3). Then u is a solution of (1) only if an additional restriction on the initial data is imposed. This is because of the singularity of the symbol $p(\zeta)$ near to $\xi = 0$. Actually the operator $U(t)$ is defined for any $\phi \in L^2$ but $U(t)\phi$ has a time derivative only if we suppose that ϕ has a "zero x mean value". More precisely $U(t)\phi$ has a well-defined time derivative provided $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'$. This condition means that formally $\int \phi(x,y)dx = 0$. Hence we have the following Theorem.

THEOREM 3. *There exist $b > 1/2$ and $b_1 < 1/4$ such that for any $\phi \in L^2$ satisfying $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'$ there exist a positive $T = T(\|\phi\|_{L^2})$ and a unique solution $u(t,x,y)$ of (1) in the time interval $[-T,T]$ satisfying*

$$u \in C([-T,T]; L^2(\mathbf{R}^2)) \cap B^{b,b_1}([-T,T]).$$

As was mentioned previously, if $u(t,x,y)$ is a solution of (1) then $\|u(t,.,.)\|_{L^2}$ does not depend on t. The time interval for the solutions obtained in Theorem 3 depends only on $\|\phi\|_{L^2}$ and therefore iterating the process of deriving local solutions we arrive at the following Theorem.

THEOREM 4. *(*J. Bourgain*) For any $\phi \in L^2(\mathbf{R}^2)$ such that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'$ there exists a unique global solution of the Cauchy problem* (1).

As it was mentioned we shall generalize Theorem 4 for data rougher then L^2. Following the idea of **[15]** we decompose the initial data in low and high Fourier modes. Then we use a local existence Theorem in Sobolev spaces with negative indices (cf. Theorem 4.3 below) as a consequence of a bilinear estimate proved in Section 4.1. Then we make use of the bilinear estimate from Section 4.2 to complete the argument. Thus we are going to prove the following Theorem.

THEOREM 5. *Let $(b,b_1) = (\frac{1}{2}+, \frac{7}{32})$ and $s \in (-1/310, 0]$. Then for any $\phi \in H_x^s$ such that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'$ there exists a unique global solution of the Cauchy problem* (1) *satisfying*

$$u \in C(\mathbf{R}; H_x^s(\mathbf{R}^2)) \cap B_s^{b,b_1}, \quad u(t) - U(t)\phi \in L^2(\mathbf{R}^2).$$

The number $-1/310$ is of technical nature and is probably not optimal. Our aim here is to show that Bourgain's argument could be applied for KP-II. The essential step of the proof of Theorem 5 is the bilinear estimate proved in Section 4.2. The result of Theorem 5 could be easily generalized to the spaces $H_{x,y}^{s_1,s_2}$, provides $s_1 > -1/310$ and $s_2 \geq 0$. The point here is that for $s_2 \geq 0$ we have

$$\langle \eta \rangle^{s_2} \langle \eta_1 \rangle^{-s_2} \langle \eta - \eta_1 \rangle^{-s_2} \leq c.$$

For details we refer to **[52]**, **[57]**.

The rest of this Chapter is organized as follows. Section 2 is devoted to the proof of the nonlinear estimate (Theorem 1). First we state the Strichartz inequality injected into the framework of Bourgain spaces and some elementary calculus inequalities needed for the proof. The rest of the section consists of the proof of the integral representation of the nonlinear estimate (cf. (13) below). Section 3 is devoted to the proof of Theorem 2. First we prove the linear estimate (Proposition 3.1). Then a fixed point argument and a scaling one provide the existence. Finally we prove the uniqueness by the aid of Proposition 3.4. Section 4 is devoted to the proof of Theorem 5. First we prove two bilinear estimates needed for the proof. Then in Section 4.3 we apply Bourgain's argument to complete the proof.

We shall use the following notations. By or $\mathcal{F}$ we denote the Fourier transform, while by $\mathcal{F}^{-1}$ the inverse transform. $||.||_{L^p}$ denotes the norm in the Lebesgue space L^p. $A \sim B$ means that there exists a constant $c \geq 1$ such that $\frac{1}{c}|A| \leq |B| \leq c|A|$. The notation $a\pm$ means $a \pm \epsilon$ for arbitrary small $\epsilon > 0$. Constants are denoted by c and may change from line to line.

Remark. Some new results of H. Takaoka for KP-II equation have recently appeared. The result of Theorem 2 (cf. also Corollary 2 below) has been further developed in **[52]** where a sharper local existence without the restriction on the data is obtained in Sobolev spaces of negative indices with respect to x. (In the next Chapter we shall further improve that result). In **[53]** global well-posedness below L^2 is shown. However in **[53]** an additional condition on the initial data is imposed. More precisely the initial data is supposed to belong to a homogeneous Sobolev space of negative index with respect to x which is a restriction on it. On the other hand the regularity of the spaces considered in **[53]** is lower then this in the present Chapter. Thus the results of **[53]** do not overlap with ours.

2. Proof of Theorem 1

2.1. Preparation of the proof. The linearized KP equation has dispersive properties (cf. **[48]**) and hence there are Strichartz type inequalities for KP (cf. **[51]**). On the other hand these inequalities are naturely injected into Bourgain's framework. In this paper we shall make an intensive use of the following estimate.

PROPOSITION 2.1. *Let $\epsilon_1 > 0$ and $2 \leq q \leq 4$. Then for any $u \in L^2(\mathbf{R}^3)$ the following inequality holds*

$$||\mathcal{F}^{-1}(\langle \tau + p(\zeta)\rangle^{-b}|\hat{u}(\tau,\zeta)|)||_{L^q} \leq c||u||_{L^2}, \tag{5}$$

where $b = 2(1-\frac{2}{q})(1/2+\epsilon_1)$.

Proof. For any $\phi \in L^2(\mathbf{R}^2)$ the classical version of the Strichartz inequality for KP equation (cf. **[48]**, Proposition 2.3) yields

$$||U(t)\phi||_{L^4} \leq c||\phi||_{L^2}. \tag{6}$$

Once we have (6), Lemma 3.3 of **[23]** gives for any $u \in B^{1/2+\epsilon_1,0}$

$$||u||_{L^4} \leq c||u; B^{1/2+\epsilon_1,0}||. \tag{7}$$

Interpolating between (7) and the identity

$$||u||_{L^2} = ||u; B^{0,0}||,$$

we obtain

$$\|u\|_{L^q} \le c\|u; B^{b,0}\|, \tag{8}$$

where $b = 2(1 - \frac{2}{q})(1/2 + \epsilon_1)$. But (8) is equivalent to (5) which completes the proof of Proposition 2.1.

We shall also make use of the following calculus inequalities.

PROPOSITION 2.2. *Let $\bar{\gamma} < 1 < \gamma$. Then for any $a \in \mathbf{R}$ the following inequalities hold*

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^{\gamma}\langle t-a\rangle^{\gamma}} \le \frac{c}{\langle a\rangle^{\gamma}}, \tag{9}$$

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^{\bar{\gamma}}\langle t-a\rangle^{\gamma}} \le \frac{c}{\langle a\rangle^{\bar{\gamma}}}, \tag{10}$$

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^{\gamma}|t-a|^{1/2}} \le \frac{c}{\langle a\rangle^{1/2}}. \tag{11}$$

Now we set

$$\sigma := \sigma(\tau,\zeta) = \tau - \xi^3 + \frac{\eta^2}{\xi}, \quad \sigma_1 := \sigma(\tau_1,\zeta_1), \quad \sigma_2 := \sigma(\tau-\tau_1,\zeta-\zeta_1),$$
$$\theta := \theta(\tau,\zeta) = \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{1/4}}, \quad \theta_1 = \theta(\tau_1,\zeta_1), \quad \theta_2 = \theta(\tau-\tau_1,\zeta-\zeta_1).$$

Setting $\widehat{w}(\tau,\zeta) = \langle\sigma\rangle^b\langle\theta\rangle\widehat{u}(\tau,\zeta)$ we obtain that (4) is equivalent to

$$\left\|\frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\int\int K(\tau,\zeta,\tau_1,\zeta_1)\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)d\tau_1 d\zeta_1; L^2_{(\tau,\zeta)}\right\| \le c\|w\|^2_{L^2}. \tag{12}$$

where

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\theta_1\rangle\langle\theta_2\rangle}.$$

By duality we obtain that (12) is equivalent to

$$\left|\int\int K_1(\tau,\zeta,\tau_1,\zeta_1)\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\widehat{v}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta\right| \le c\|w\|^2_{L^2}\|v\|_{L^2}, \tag{13}$$

where

$$K_1(\tau,\zeta,\tau_1,\zeta_1) = \frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\theta_1\rangle\langle\theta_2\rangle}$$

Without loss of generality we can assume that $\widehat{w} \ge 0$ and $\widehat{v} \ge 0$. By symmetry arguments we can assume that $|\sigma_1| \ge |\sigma_2|$. To gain the loss of a derivative in the nonlinear term we shall use the relation (cf. [**14**])

$$\sigma_1 + \sigma_2 - \sigma = 3\xi_1\xi(\xi-\xi_1) + \frac{(\xi_1\eta - \xi\eta_1)^2}{\xi_1\xi(\xi-\xi_1)} \tag{14}$$

and hence

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \ge |\xi_1\xi(\xi-\xi_1)|. \tag{15}$$

By J we denote the left-hand side of (13). As it was mentioned in the introduction, there are different methods to estimate J. One of them is based on using the Strichartz inequalities. Note that J can be written in the form

$$J = \int u_1(\tau,\zeta)(u_2 \star u_3)(\tau,\zeta)d\tau d\zeta.$$

Using Plancherel identity and Hölder inequality we obtain

$$J \le \prod_{j=1}^{3} \| \mathcal{F}^{-1}(u_j)\|_{L^{q_j}},$$

where $\sum_{j=1}^{3} \frac{1}{q_j} = 1$. In the favorable cases u_j have the form

$$u_1 = \langle\sigma\rangle^{-\alpha}\widehat{v}, \quad u_2 = \langle\sigma_1\rangle^{-\alpha_1}\widehat{w}, \quad u_3 = \langle\sigma_2\rangle^{-\alpha_2}\widehat{w}$$

and an application of Proposition 2.1 (Strichartz inequality) completes the proof.

Another method reduces the evaluation of J to a direct estimate for K_1. Namely Cauchy-Schwarz inequality yields

$$\begin{aligned} J &\le \int \left(\int K_1^2 d\tau_1 d\zeta_1\right)^{1/2} \left(\int |\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1\right)^{1/2} \widehat{v}(\tau,\zeta)d\tau d\zeta \\ &\le \|K_1; L^\infty_{(\tau,\zeta)}(L^2_{(\tau_1,\zeta_1)})\| \|w\|^2_{L^2}\|v\|_{L^2}. \end{aligned}$$

Hence the difficulty is to prove

$$\|K_1; L^\infty_{(\tau,\zeta)}(L^2_{(\tau_1,\zeta_1)})\| < \infty.$$

Now we give an idea for the estimation of J proposed in [14]. We localize J in the following regions

$$\langle\sigma\rangle \sim K, \quad \langle\sigma_1\rangle \sim K_1, \quad \langle\sigma_2\rangle \sim K_2, \quad \langle\xi\rangle \sim M, \quad \langle\xi_1\rangle \sim M_1, \quad \langle\xi-\xi_1\rangle \sim M_2,$$

where K, K_1, K_2 take values $2^j, j = 0,1,2,\dots$ and M, M_1, M_2 take values $2^j, j = 0, \pm1, \pm2, \dots$ Therefore (15) yields

$$\max\{K, K_1, K_2\} \ge MM_1M_2$$

and we have

$$J \le \sum \frac{MC(K,K_1,K_2,M,M_1,M_2)}{K^{b'}K_1^bK_2^b} < v_{KM}, w_{K_1M_1} \star w_{K_2M_2} >, \tag{16}$$

where $< .,. >$ stays for the L^2 scalar product, the sum is taken over K, K_1, K_2, M, M_1, M_2 and v_{KM}, $w_{K_1M_1}$, $w_{K_2M_2}$ are localized v and w. The difficulty is to prove an inequality of type

$$\|w_{K_1M_1} \star w_{K_2M_2}\|_{L^2} \le C(K_1, K_2, M_1, M_2)\|w_{K_1M_1}\|_{L^2}\|w_{K_2M_2}\|_{L^2} \tag{17}$$

In the favorable cases use of (16) and (17) provides the estimation of J by a total constant.

2.2. Proof of the integral representation of the bilinear estimate (4). We shall estimate J considering five cases for $(\tau, \zeta, \tau_1, \zeta_1)$. We use both the Strichartz inequalities and direct estimates for the kernel $K_1(\tau, \zeta, \tau_1, \zeta_1)$.

Case 1. $|\xi| \leq 24$. We denote by J_1 the restriction of J on this region. Using that $\langle\theta\rangle \leq c\langle\sigma\rangle^{b'}$ we obtain

$$K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{c}{\langle\sigma_1\rangle^b \langle\sigma_2\rangle^b}.$$

Since $b > 1/2$ the use of the Strichartz inequality yields

$$\begin{aligned} J_1 &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-b}\widehat{w}(\tau_1,\zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-b}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^4}\|v\|_{L^2} \\ &\leq c\|w\|_{L^2}^2\|v\|_{L^2}. \end{aligned}$$

Case 2. $|\sigma| \geq |\sigma_1|$, $|\xi| \geq 24$. We denote by J_2 the restriction of J on this region. Using Cauchy-Schwarz inequality we obtain

$$J_2 \leq \int_{\mathbf{R}^3} I(\tau,\zeta)\left\{\int_{|\sigma|\geq|\sigma_1|} |\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1\right\}^{1/2} \widehat{v}(\tau,\zeta)d\tau d\zeta,$$

where

$$I(\tau,\zeta) = \frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left(\int_{|\sigma|\geq|\sigma_1|}\frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}\langle\theta_1\rangle^2\langle\theta_2\rangle^2}\right)^{1/2}.$$

We have the following Lemma.

LEMMA 2.1.

$$I(\tau,\zeta) \leq const, \quad \text{if} \quad |\xi| \geq 24.$$

Proof. Using (9) we obtain

$$I(\tau,\zeta) \leq \frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left(\int_{|\sigma|\geq|\sigma_1|}\frac{d\zeta_1}{\langle\sigma_1+\sigma_2\rangle^{2b}}\right)^{1/2}.$$

We perform a change of variables similarly to [**32**]

$$\alpha = \sigma_1 + \sigma_2, \qquad \beta = 3\xi\xi_1(\xi - \xi_1).$$

Note that $\xi \in [-3|\sigma|, \min\{3/4\xi^3, 3|\sigma|\}]$, when $\xi \geq 0$ and $\xi \in [\max\{3/4\xi^3, -3|\sigma|\}, 3|\sigma|]$, when $\xi \leq 0$. We assume that $\xi \geq 0$. If $\xi \leq 0$ then the arguments are the same. We have that

$$d\zeta_1 = \frac{c|\beta|^{1/2}d\alpha d\beta}{|\xi|^{3/2}(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}}$$

Therefore we obtain using (11)

$$I(\tau,\zeta) \leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left\{\int_{-3|\sigma|}^{\min\{3/4\xi^3,3|\sigma|\}}\int_{-\infty}^{\infty}\frac{|\beta|^{1/2}d\alpha d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}\langle\alpha\rangle^{2b}}\right\}^{1/2} \leq$$

$$\frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left\{\int_{-3|\sigma|}^{\min\{3/4\xi^3,3|\sigma|\}}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}.$$

Now we consider two cases for (τ,ζ).

- $\frac{3}{4}|\xi|^3 \leq 4|\sigma|$

We have that

$$\begin{aligned}
I(\tau,\zeta) &\leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\{\int_{-3|\sigma|}^{0}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}} \\
&\quad +\int_0^{\frac{3}{4}\xi^3}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\}^{1/2} \\
&\leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\{\int_{-3|\sigma|}^{0}\frac{d\beta}{\langle\sigma+\beta\rangle^{1/2}} \\
&\quad +\xi^{3/2}\int_0^{\frac{3}{4}\xi^3}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|^{1/2}\langle\sigma+\beta\rangle^{1/2}}\}^{1/2} \\
&\leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\{\langle\sigma\rangle^{1/2}+\xi^{3/2}(\int_0^{\frac{3}{4}\xi^3-1}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|} \\
&\quad +\int_0^{\frac{3}{4}\xi^3-1}\frac{d\beta}{\langle\sigma+\beta\rangle}+\int_{\frac{3}{4}\xi^3-1}^{\frac{3}{4}\xi^3}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|^{1/2}\langle\sigma+\beta\rangle^{1/2}})\}^{1/2} \\
&\leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}(\langle\sigma\rangle^{1/2}+\xi^{3/2}\ln(1+\langle\sigma\rangle))^{1/2} \\
&\leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'-1/4}}+\frac{c\langle\xi\rangle\langle\theta\rangle}{\langle\sigma\rangle^{b'-}}.
\end{aligned}$$

Let $|\sigma|^{b_1} \leq |\xi|^{1/4}$. Then we have

$$I(\tau,\zeta) \leq \frac{\langle\xi\rangle}{\langle\xi\rangle^{3b'}}+\frac{\langle\xi\rangle}{\langle\xi\rangle^{3b'-}} \leq \text{ const },$$

provided $b' > 1/3$. Let $|\sigma|^{b_1} \geq |\xi|^{1/4}$. Then we have

$$I(\tau,\zeta) \leq \frac{1}{\langle\sigma\rangle^{b'-b_1-1/4}}+\frac{\langle\xi\rangle^{3/4}}{\langle\sigma\rangle^{b'-b_1-}} \leq \text{ const },$$

provided $b'-b_1 > 1/4$.

- $4|\sigma| \leq \frac{3}{4}|\xi|^3$

In this case we ha

$$I(\tau,\zeta) \leq \frac{c\langle\xi\rangle^{1/4}\langle\theta\rangle}{\langle\sigma\rangle^{b'-1/4}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}$$

$$\leq \frac{c\langle\theta\rangle}{\langle\sigma\rangle^{b'-1/4}\langle\xi\rangle^{1/2}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{d\beta}{\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}$$

$$\leq \frac{c\langle\theta\rangle\langle\sigma\rangle^{1/2-b'}}{\langle\xi\rangle^{1/2}}.$$

Let $|\sigma|^{b_1} \leq |\xi|^{1/4}$. Then we have

$$I(\tau,\zeta) \leq \frac{\langle\xi\rangle^{3/2-3b'}}{\langle\xi\rangle^{1/2}} \leq \text{ const },$$

provided $b' > 1/3$.
Let $|\sigma|^{b_1} \geq |\xi|^{1/4}$. Then we have

$$I(\tau,\zeta) \leq \frac{\langle\sigma\rangle^{1/2-b'+b_1}}{\langle\xi\rangle^{3/4}} \leq \frac{\langle\xi\rangle^{3/2-3b'+3b_1}}{\langle\xi\rangle^{3/4}} \leq \text{ const },$$

provided $b' - b_1 > 1/4$. This completes the proof of Lemma 2.1. Therefore using Lemma 2.1 and Cauchy-Schwarz inequality we obtain

$$J_2 \leq c\|w\|_{L^2}^2\|v\|_{L^2}.$$

Case 3. $|\sigma_1| \geq |\sigma|, \quad |\xi| \geq 24, \quad |\xi_1| \leq 1$. We denote by J_3 the restriction of J on this region. We shall estimate J_3 by a localization with respect to ξ_1 and $\langle\sigma_1\rangle$. We set

$$J_3^{KM} = \int\int_{A^{KM}} K_1(\tau,\zeta,\tau_1,\zeta_1)\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\widehat{v}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta,$$

where

$$A^{KM} = \{(\tau_1,\zeta_1) : |\xi_1| \sim M, \langle\sigma_1\rangle \sim K\}.$$

We have

$$J_3 \leq \sum_{K,M} J_3^{KM},$$

where the sum is taken over $K = 2^k, k = 0,1,2,\ldots$ and $M = 2^m, m = 0,-1,-2,\ldots$. Now we shall estimate J_3^{KM} in two ways. First by a direct estimate for K_1 we bound J_3^{KM}. Then we estimate J_3^{KM} by the aid of the Strichartz inequality. A suitable interpolation provides the needed inequality.

Cauchy-Schwarz inequality yields

$$J_3^{KM} \leq \int_{\mathbf{R}^3} I^{KM}(\tau,\zeta)\left\{\int_{A^{KM}}|\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1\right\}^{1/2}\widehat{v}(\tau,\zeta)d\tau d\zeta,$$

where

$$I^{KM}(\tau,\zeta)=\frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left(\int_{A^{KM}}\frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}\langle\theta_1\rangle^2\langle\theta_2\rangle^2}\right)^{1/2}.$$

We change the variables

$$\alpha=\sigma_1+\sigma_2,\qquad \beta=3\xi\xi_1(\xi-\xi_1).$$

As in the proof of Lemma 2.1 we suppose that $\xi\geq 0$. We have

$$|\beta|\leq 3|\xi|(|\xi|+1)2M\leq 9|\xi|^2M.$$

Hence using that $\langle\theta\rangle\leq c\langle\sigma\rangle^{b_1}$ we obtain

$$I^{KM}(\tau,\zeta)\leq\frac{c|\xi|^{1/4}}{\langle\sigma\rangle^{b'-b_1}}$$

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{|\beta|^{1/2}d\tau_1 d\alpha d\beta}{\langle\sigma_1\rangle^{2b}\langle\alpha-\sigma_1\rangle^{2b}(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}\langle\theta_1\rangle^2}\right\}^{1/2}.$$

Since $|\beta|\leq|\sigma_1|$ we obtain

$$\frac{|\beta|^{1/2}}{\langle\theta_1\rangle^2}\leq c\langle\sigma_1\rangle^{1/2-2b_1}\leq cK^{1/2-2b_1}.$$

Hence

$$\begin{aligned}I^{KM}(\tau,\zeta)&\leq\frac{c|\xi|^{1/4}K^{1/4-b_1}}{\langle\sigma\rangle^{b'-b_1}}\left\{\int_{-\infty}^{\infty}\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\alpha d\beta}{\langle\alpha\rangle^{2b}(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}}\right\}^{1/2}\\&\leq\frac{c|\xi|^{1/4}K^{1/4-b_1}}{\langle\sigma\rangle^{b'-b_1}}\left\{\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta|^{1/2}}\right\}^{1/2}\end{aligned}$$

Since $|\xi|\geq 24$ we have that $|\frac{3}{4}\xi^3-\beta|\geq\frac{3}{8}\xi^3$ and therefore

$$\begin{aligned}I^{KM}(\tau,\zeta)&\leq\frac{cK^{1/4-b_1}}{|\xi|^{1/2}\langle\sigma\rangle^{b'-b_1}}\left\{\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\beta}{|\sigma+\beta|^{1/2}}\right\}^{1/2}\\&\leq\frac{cK^{1/4-b_1}}{|\xi|^{1/2}\langle\sigma\rangle^{b'-b_1}}M^{1/4}|\xi|^{1/2}\\&\leq cK^{1/4-b_1}M^{1/4},\end{aligned}$$

where we used the elementary inequality

$$\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\beta}{|\sigma+\beta|^{1/2}}\leq cM^{1/2}|\xi|.$$

Hence

$$I^{KM}(\tau,\zeta)\leq cK^{1/4-b_1}M^{1/4}.$$

Cauchy-Schwarz inequality yields

$$J_3^{KM} \le cK^{1/4-b_1}M^{1/4}\|w\|_{L^2}^2\|v\|_{L^2}. \tag{18}$$

Now we shall estimate $J_3^{KM}(\tau,\zeta)$ by the aid of Proposition 2.1. We have

$$K_1(\tau,\zeta,\tau_1,\zeta_1) \le \frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}}.$$

Let $|\sigma|^{b_1} \le |\xi|^{1/4}$. We denote by J_{31}^{KM} the restriction of J_3^{KM} on this region. We have

$$\begin{aligned} K_1(\tau,\zeta,\tau_1,\zeta_1) &\le \frac{c|\xi|}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \\ &\le \frac{cM^{-1/2}K^{-1/8}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-5/8}\langle\sigma_2\rangle^{b}}. \end{aligned}$$

Using Proposition 2.1 and Hölder inequality we obtain

$$\begin{aligned} J_{31}^{KM} &\le \frac{c}{M^{1/2}K^{1/8}}\|\langle\sigma\rangle^{-b'}\widehat{v}(\tau,\zeta)\|_{L^{q_1}} \\ &\quad \|\langle\sigma_1\rangle^{-b-b_1+5/8}\widehat{w}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ &\quad \|\langle\sigma_2\rangle^{-b}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ &\le \frac{c}{M^{1/2}K^{1/8}}\|w\|_{L^2}^2\|v\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b' &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b+b_1-5/8 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Now we take $\epsilon_1 = b+b_1/2+b'/2-13/16$ to ensure the restriction of Hölder inequality. Note that if $b'+b_1 > 5/8$ then ϵ_1 is positive.

Let $|\sigma|^{b_1} \ge |\xi|^{1/4}$. We denote by J_{32}^{KM} the restriction of J_3^{KM} on this region. Since $\langle\sigma_1\rangle^{3/8} \ge c|\xi|^{3/4}M^{3/8}$ we obtain

$$\begin{aligned} K_1(\tau,\zeta,\tau_1,\zeta_1) &\le \frac{c|\xi|^{3/4}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \\ &\le \frac{cM^{-3/8}K^{-1/16}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1-7/16}\langle\sigma_2\rangle^{b}}. \end{aligned}$$

Using Proposition 2.1 and Hölder inequality we obtain

$$\begin{aligned} J_{32}^{KM} &\le \frac{c}{M^{3/8}K^{1/16}}\|\langle\sigma\rangle^{-b'+b_1}\widehat{v}(\tau,\zeta)\|_{L^{q_1}} \\ &\quad \|\langle\sigma_1\rangle^{-b-b_1+7/16}\widehat{w}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ &\quad \|\langle\sigma_2\rangle^{-b}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ &\le \frac{c}{M^{3/8}K^{1/16}}\|w\|_{L^2}^2\|v\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b'-b_1 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b+b_1-7/16 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Now we take $\epsilon_1 = b+b'/2-23/32$ to ensure the restriction of Hölder inequality. Note that $\epsilon_1>0$, provided $b'>7/16$. Further we have

$$\begin{aligned} J_3^{KM} &\le J_{31}^{KM}+J_{32}^{KM} \\ &\le c\left(\frac{1}{M^{1/2}K^{1/8}}+\frac{1}{M^{3/8}K^{1/16}}\right)\|w\|_{L^2}^2\|v\|_{L^2} \qquad (19)\\ &\le \frac{c}{M^{1/2}K^{1/16}}\|w\|_{L^2}^2\|v\|_{L^2}. \end{aligned}$$

Interpolation between (18) and (19) (with weights 3/4 and 1/4 respectively) yields

$$J_3^{KM} \le \frac{cM^{1/16}}{K^{\frac{3}{4}b_1-\frac{11}{64}}}\|w\|_{L^2}^2\|v\|_{L^2}.$$

Since $b_1>11/48$, summing over K and M yields

$$J_3 \le c\|w\|_{L^2}^2\|v\|_{L^2}.$$

Case 4. $|\sigma_1|\ge|\sigma|,\quad |\xi|\ge 24,\quad |\xi|\le 2|\xi_1|$. We denote by J_4 the restriction of J on this region. In this case we have

$$\frac{\langle\theta\rangle}{\langle\theta_1\rangle}\le c\langle\xi_1\rangle^{1/4}.$$

Hence

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\le\frac{c|\xi_1|^{5/4}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}}.$$

Using Cauchy-Schwarz inequality we obtain

$$J_4\le\int I_1(\tau_1,\zeta_1)\left\{\int_{|\sigma_1|\ge|\sigma|}|\widehat{v}(\tau,\zeta)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)|^2d\tau d\zeta\right\}^{1/2}\widehat{w}(\tau_1,\zeta_1)d\tau_1d\zeta_1,$$

where

$$I_1(\tau_1,\zeta_1) = \frac{|\xi_1|^{5/4}}{\langle\sigma_1\rangle^b}\left(\int_{|\sigma_1|\geq|\sigma|}\frac{d\tau d\zeta}{\langle\sigma\rangle^{2b'}\langle\sigma_2\rangle^{2b}}\right)^{1/2}$$

We have the following Lemma.

LEMMA 2.2.

$$I_1(\tau_1,\zeta_1) \leq const.$$

Proof. (10) yields

$$\begin{aligned} I_1(\tau_1,\zeta_1) &\leq \frac{c|\xi_1|^{5/4}}{\langle\sigma_1\rangle^b}\left\{\int_{|\sigma_1|\geq|\sigma|}\frac{d\zeta}{\langle\sigma-\sigma_2\rangle^{2b'}}\right\}^{1/2} \\ &\leq \frac{c|\xi_1|^{5/4}\langle\sigma_1\rangle^{b-b'}}{\langle\sigma_1\rangle^b}\left\{\int_{|\sigma_1|\geq|\sigma|}\frac{d\zeta}{\langle\sigma-\sigma_2\rangle^{2b}}\right\}^{1/2}, \end{aligned}$$

where we used that $|\sigma-\sigma_2|\leq 2|\sigma_1|$. We perform a change of variables

$$\alpha=\sigma-\sigma_2, \qquad \beta = 3\xi\xi_1(\xi_1-\xi).$$

We can assume that $\xi_1\geq 0$. If $\xi_1\leq 0$ then the arguments are the same. Further we have

$$d\zeta = \frac{c|\beta|^{1/2}d\alpha d\beta}{|\xi_1|^{3/2}(\frac{3}{4}\xi_1^3-\beta)^{1/2}|\sigma_1+\beta-\alpha|^{1/2}}.$$

Therefore we obtain using (11)

$$I_1(\tau_1,\zeta_1)\leq\frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'}}\left\{\int_{-3|\sigma_1|}^{\min\{3/4\xi_1^3,3|\sigma_1|\}}\int_{-\infty}^{\infty}\frac{|\beta|^{1/2}d\alpha d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}|\sigma_1+\beta-\alpha|^{1/2}\langle\alpha\rangle^{2b}}\right\}^{1/2}\leq$$

$$\frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'}}\left\{\int_{-3|\sigma_1|}^{\min\{3/4\xi_1^3,3|\sigma_1|\}}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\right\}^{1/2}.$$

Now we consider two cases for (τ_1,ζ_1).

- $\frac{3}{4}|\xi_1|^3\leq 4|\sigma_1|$

We have

$$
\begin{aligned}
I_1(\tau_1,\zeta_1) &\le \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'}}\{\int_{-3|\sigma_1|}^{0}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}} \\
&\quad + \int_0^{\frac{3}{4}\xi_1^3}\frac{|\beta|^{1/2}d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\}^{1/2} \\
&\le \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'}}\{\int_{-3|\sigma_1|}^{0}\frac{d\beta}{\langle\sigma_1+\beta\rangle^{1/2}} \\
&\quad + \xi_1^{3/2}\int_0^{\frac{3}{4}\xi_1^3}\frac{d\beta}{|\frac{3}{4}\xi_1^3-\beta|^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\}^{1/2} \\
&\le \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'}}\{\langle\sigma_1\rangle^{1/2}+\xi_1^{3/2}(\int_0^{\frac{3}{4}\xi_1^3-1}\frac{d\beta}{|\frac{3}{4}\xi_1^3-\beta|} \\
&\quad + \int_0^{\frac{3}{4}\xi_1^3-1}\frac{d\beta}{\langle\sigma_1+\beta\rangle}+\int_{\frac{3}{4}\xi_1^3-1}^{\frac{3}{4}\xi_1^3}\frac{d\beta}{|\frac{3}{4}\xi_1^3-\beta|^{1/2}\langle\sigma_1+\beta\rangle^{1/2}})\}^{1/2} \\
&\le \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'}}(\langle\sigma_1\rangle^{1/2}+\xi_1^{3/2}\ln(1+\langle\sigma_1\rangle))^{1/2} \\
&\le \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'-1/4}}+\frac{c|\xi_1|^{5/4}}{\langle\sigma_1\rangle^{b'-0}} \\
&\le \frac{c|\xi_1|^{1/2}}{|\xi_1|^{3b'-3/4}}+\frac{c|\xi_1|^{5/4}}{|\xi_1|^{3b'-}} \\
&\le \text{const},
\end{aligned}
$$

provided $b' > 5/12$.

- $4|\sigma_1| \le \frac{3}{4}|\xi_1|^3$

In this case we have

$$
I_1(\tau_1,\zeta_1) \le \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'-1/4}}\left\{\int_{-3|\sigma_1|}^{3|\sigma_1|}\frac{d\beta}{(\frac{3}{4}\xi_1^3-\beta)^{1/2}\langle\sigma_1+\beta\rangle^{1/2}}\right\}^{1/2}
$$

$$\leq \frac{c|\xi_1|^{1/2}}{\langle\sigma_1\rangle^{b'-1/4}|\xi_1|^{3/4}}\left\{\int_{-3|\sigma_1|}^{3|\sigma_1|}\frac{d\beta}{\langle\sigma_1+\beta\rangle^{1/2}}\right\}^{1/2}$$

$$\leq \frac{c\langle\sigma_1\rangle^{1/4}}{\langle\sigma_1\rangle^{b'-1/4}|\xi_1|^{1/4}} \leq \frac{c|\xi_1|^{3(1/2-b')}}{|\xi_1|^{1/4}} \leq \text{const},$$

provided $b' > 5/12$. This completes the proof of Lemma 2.2. Therefore using Lemma 2.2 and Cauchy-Schwarz inequality we obtain

$$J_4 \leq c\|w\|_{L^2}^2\|v\|_{L^2}.$$

Case 5. $|\sigma_1| \geq |\sigma|, \quad |\xi| \geq 24, \quad |\xi| \geq 2|\xi_1|, \quad |\xi_1| \geq 1$. We denote by J_5 the restriction of J on this region. In this case we have that $|\xi| \leq 2|\xi - \xi_1|$.

Let $|\sigma|^{b_1} \leq |\xi|^{1/4}$. We denote by J_{51} the restriction of J_5 on this region. We have

$$\begin{aligned} K_1(\tau,\zeta,\tau_1,\zeta_1) &\leq \frac{c|\xi||\xi_1|^{1/4}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \\ &\leq \frac{c}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-5/8}\langle\sigma_2\rangle^{b}}. \end{aligned}$$

Proposition 2.1 and Hölder inequality yield

$$\begin{aligned} J_{51} &\leq \|\langle\sigma\rangle^{-b'}\widehat{v}(\tau,\zeta)\|_{L^{q_1}} \\ &\quad \|\langle\sigma_1\rangle^{-b-b_1+5/8}\widehat{w}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ &\quad \|\langle\sigma_2\rangle^{-b}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ &\leq c\|w\|_{L^2}^2\|v\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b' &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b+b_1-5/8 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Now we take $\epsilon_1 = b+b'/2+b_1/2-13/16$ to ensure the restriction of Hölder inequality. Note that if $b_1+b' > 5/8$ then ϵ_1 is positive.

Let $|\sigma|^{b_1} \geq |\xi|^{1/4}$. We denote by J_{52} the restriction of J_5 on this region. We have

$$K_1(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{c|\xi|^{3/4}|\xi_1|^{1/4}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}}.$$

and

$$|\xi|^{3/4}|\xi_1|^{1/4} \leq c|\xi|^{1/2}|\xi_1|^{1/4}|\xi - \xi_1|^{1/4} \leq c\langle\sigma_1\rangle^{1/4}\langle\sigma_1\rangle^{1/8} = c\langle\sigma_1\rangle^{3/8}.$$

Therefore

$$K_1(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{c}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1-3/8}\langle\sigma_2\rangle^{b}}.$$

Proposition 2.1 and Hölder inequality yield

$$\begin{aligned} J_{52} &\leq \|\langle\sigma\rangle^{-b'+b_1}\widehat{v}(\tau,\zeta)\|_{L^{q_1}} \\ &\quad \|\langle\sigma_1\rangle^{-b-b_1+3/8}\widehat{w}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ &\quad \|\langle\sigma_2\rangle^{-b}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ &\leq c\|w\|_{L^2}^2\|v\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3} = 1$ and

$$\begin{aligned} b' - b_1 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b + b_1 - 3/8 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Now we take $\epsilon_1 = b + b'/2 - 11/16$ to ensure the restriction of Hölder inequality. Note that if $b' > 3/8$ then ϵ_1 is positive.

Hence

$$J_5 \leq c\|w\|_{L^2}^2\|v\|_{L^2}.$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. First we prove the linear estimate following the lines of the Bourgain's method (cf. **[12]**, **[13]**, **[14]**, **[23]**, **[32]**). Then using Theorem 1 and a scaling argument we prove the existence. Finally we prove the uniqueness using an argument of **[42]**. Recall that we shall apply a Picard fixed point Theorem to the integral equation

$$u(t) = U(t)\phi - 1/2\int_0^t U(t-t')\partial_x(u^2(t'))dt'. \tag{20}$$

3.1. The linear estimate. We define the operator T_ϕ

$$T_\phi(u) = U(t)\phi - 1/2\int_0^t U(t-t')\partial_x(u^2(t'))dt'. \tag{21}$$

Let ψ be a cut-off function such that

$$\psi \in C_0^\infty(\mathbf{R}),\ \text{supp}\ \psi \subset [-2,2], \psi = 1 \text{ over the interval } [-1,1].$$

Now we state the linear estimate.

PROPOSITION 3.1. *Let $b > 1/2$, $b' > 0$, $b + b' \leq 1$ and $b_1 \geq 0$. Then*

$$\|\psi T_\phi(u)\|_{B^{b,b_1}} \leq c(\|\phi\|_{L^2} + \|uu_x\|_{B^{-b',b_1}}). \tag{22}$$

Proof. Recall that $p(\zeta) = -\xi^3 + \frac{\eta^2}{\xi}$ and $\sigma = \tau + p(\zeta)$. Note that

$$\begin{aligned}
\psi(t)\int_0^t U(t-t')u(t')u_x(t')dt' &= \psi(t)\int e^{i(x\xi+y\eta-tp(\zeta))}\mathcal{F}(uu_x)(\tau,\zeta)\frac{e^{it\sigma}-1}{\sigma}d\tau d\zeta \\
&= \psi(t)\sum_{k\geq 1}\frac{(it)^k}{k!}\int_{|\sigma|\leq 1} e^{i(x\xi+y\eta-tp(\zeta))}\mathcal{F}(uu_x)(\tau,\zeta)\sigma^{k-1}d\tau d\zeta \\
&\quad +\psi(t)\int_{|\sigma|\geq 1} e^{i(x\xi+y\eta+t\tau)}\mathcal{F}(uu_x)(\tau,\zeta)\frac{1}{\sigma}d\tau d\zeta \\
&\quad -\psi(t)\int_{|\sigma|\geq 1} e^{i(x\xi+y\eta-tp(\zeta))}\mathcal{F}(uu_x)(\tau,\zeta)\frac{1}{\sigma}d\tau d\zeta \\
&= (1)+(2)+(3).
\end{aligned}$$

Further we have

$$\|(1)\| \leq c\|t^k\psi(t)\|_{H^{b+b_1}}\|uu_x\|_{B^{-b',b_1}},$$

for any $b \geq 0$ and $b' \geq 0$. In order to estimate (2) we need the following Lemma.

LEMMA 3.1. *For any $b \geq 0$ and $b_1 \geq 0$ the next inequality holds*

$$\|\psi u\|_{B^{b,b_1}} \leq c\|u\|_{B^{b,b_1}}. \tag{23}$$

Proof. Let $\widehat{v}(\tau,\zeta) = \langle\sigma\rangle^b\langle\theta\rangle\widehat{u}(\tau,\zeta)$. Then (23) is equivalent to

$$\int \langle\sigma\rangle^{2b}\langle\theta\rangle^2\left|\int\widehat{\psi}(\tau-\tau_1)\langle\sigma^*\rangle^{-b}\langle\theta^*\rangle^{-1}\widehat{v}(\tau_1,\zeta)d\tau_1\right|^2 d\tau d\zeta \leq \|v\|_{L^2}^2,$$

where $\sigma^* = \tau_1 + p(\zeta)$ and $\theta^* = \frac{\langle\sigma^*\rangle^{b_1}}{\langle\xi\rangle^{1/4}}$. A duality argument shows that the above inequality is equivalent to

$$\left|\int \frac{\langle\sigma\rangle^b\langle\theta\rangle}{\langle\sigma^*\rangle^b\langle\theta^*\rangle}\widehat{\psi}(\tau-\tau_1)\widehat{v}(\tau_1,\zeta)\widehat{w}(\tau,\zeta)d\tau d\tau_1 d\zeta\right| \leq c\|v\|_{L^2}\|w\|_{L^2}. \tag{24}$$

We can suppose that $\widehat{\psi} \geq 0$, $\widehat{v} \geq 0$, $\widehat{w} \geq 0$. Since $\langle\sigma\rangle \leq |\tau-\tau_1| + \langle\sigma^*\rangle$ and $\langle\theta\rangle \leq |\tau-\tau_1|^{b_1} + \langle\theta^*\rangle$ we obtain

$$\frac{\langle\sigma\rangle^b\langle\theta\rangle}{\langle\sigma^*\rangle^b\langle\theta^*\rangle} \leq 1 + |\tau-\tau_1|^{b+b_1}.$$

Now we set

$$\widehat{\chi}(\tau-\tau_1) = (1+|\tau-\tau_1|^{b+b_1})\widehat{\psi}(\tau-\tau_1).$$

We denote by I the left-hand side of (24). Cauchy-Schwarz and Young inequalities yield

$$\begin{aligned} I^2 &\leq \left(\int (\chi *_\tau v)(\tau,\zeta)\widehat{w}(\tau,\zeta)d\tau d\zeta\right)^2 \\ &\leq \int (\chi *_\tau v)^2(\tau,\zeta)d\tau d\zeta \|w\|_{L^2}^2 \\ &\leq c\|\chi\|_{L^1}^2 \|v\|_{L^2}^2 \|w\|_{L^2}^2, \end{aligned}$$

which completes the proof of Lemma 3.1.

Now due to Lemma 3.1 we have

$$\begin{aligned} \|(2)\|_{B^{b,b_1}} &\leq c\|\langle\sigma\rangle^{b-1}\langle\theta\rangle\mathcal{F}(uu_x)\|_{L^2} \\ &\leq c\|uu_x\|_{B^{-b',b_1}}. \end{aligned}$$

It remains to estimate (3)

$$\begin{aligned} \|(3)\|_{B^{b,b_1}} &\leq c\|\psi\|_{H_t^{b+b_1}} \left(\int\left(\int \frac{|\mathcal{F}(uu_x)(\tau,\zeta)|d\tau}{\langle\sigma\rangle}\right)^2 d\zeta\right)^{1/2} \\ &\leq c\|\psi\|_{H_t^{b+b_1}} \int \frac{d\tau}{\langle\sigma\rangle^{2b}} \int \frac{|\mathcal{F}(uu_x)(\tau,\zeta)|^2 d\tau d\zeta}{\langle\sigma\rangle^{2(1-b)}} \\ &\leq c\|\psi\|_{H_t^{b+b_1}} \|uu_x\|_{B^{b-1,b_1}} \\ &\leq c\|uu_x\|_{B^{-b',b_1}}. \end{aligned}$$

Now we estimate the free evolution

$$\|\psi(t)U(t)\phi\|_{B^{b,b_1}} \leq c\|\psi\|_{H_t^{b+b_1}}\|\phi\|_{L^2},$$

which completes the proof of Proposition 3.1.

3.2. Existence. Using Proposition 3.1 and Theorem 1 with $b = \frac{1}{2}+\epsilon$, $b' = \frac{1}{2}-\epsilon$ and $b_1 = \frac{1}{4} - 2\epsilon$ we obtain for sufficiently small ϵ.

PROPOSITION 3.2. *If $\epsilon > 0$ is sufficiently small then the following inequalities hold*

$$\|\psi T_\phi(u)\|_{B^{\frac{1}{2}+\epsilon,\frac{1}{4}-2\epsilon}} \le c(\|\phi\|_{L^2} + \|u\|^2_{B^{\frac{1}{2}+\epsilon,\frac{1}{4}-2\epsilon}}) \tag{25}$$

$$\|\psi(T_\phi(u) - T_\phi(v))\|_{B^{\frac{1}{2}+\epsilon,\frac{1}{4}-2\epsilon}} \le c\|u-v\|_{B^{\frac{1}{2}+\epsilon,\frac{1}{4}-2\epsilon}}\|u+v\|_{B^{\frac{1}{2}+\epsilon,\frac{1}{4}-2\epsilon}} \tag{26}$$

Now a standard fixed point argument yields

PROPOSITION 3.3. *There exists $c_0 > 0$ such that if $\|\phi\|_{L^2} \le c_0$ then the map ψT_ϕ has a fixed point u which is the unique solution of $u = \psi T_\phi(u)$, i.e. local solution of (3) in time interval* $[-1, 1]$.

To prove local existence for arbitrary data in L^2 we shall perform a scaling argument. If $u(t,x,y)$ is a function then we set $u_T(t,x,y) = T^{2/3}u(Tt, T^{1/3}x, T^{2/3}y)$. Similarly if $\phi(x,y)$ is a function then we set $\phi_T(x,y) = T^{2/3}\phi(T^{1/3}x, T^{2/3}y)$. Finally we set $\psi_T(t) = \psi(t/T)$. We have that $\|\phi_T\|_{L^2} = T^{1/6}\|\phi\|_{L^2}$. Hence due to Proposition 3.3 if T is sufficiently small then the equation $u = \psi T_{\phi_T}(u)$ has a solution. Note that if $u(t,x,y)$ is a solution of KP equation with data $\phi(x,y)$ then so is $u_T(t,x,y)$ with data $\phi_T(x,y)$. This observation and a direct computation shows that we have the next relation

$$[\psi T_{\phi_T}(u)]_{T^{-1}} = \psi_T T_\phi(u_{T^{-1}}). \tag{27}$$

Hence $u_{T^{-1}}$ is a solution of $\psi_T T_\phi(u_{T^{-1}}) = u_{T^{-1}}$. Therefore, we proved the existence of a solution to (3) for T sufficiently small in time interval $[-T, T]$.

3.3. Uniqueness.

To prove the uniqueness we need the next Proposition.

PROPOSITION 3.4. *Let $1/2 > b' > b'' > b_1 > 0$. Then there exists $\theta > 0$ such that for $\delta \in (0,1)$ the following estimate holds*

$$\|\psi_\delta u\|_{B^{-b',b_1}} \le c\delta^\theta \|u\|_{B^{-b'',b_1}}.$$

Proof. Let $Y^{b,s}$ be the space equipped with the norm

$$\|u\|_{Y^{b,s}} = \|\langle\tau - \xi^3 + \frac{\eta^2}{\xi}\rangle^b \langle\xi\rangle^s \widehat{u}(\tau,\zeta)\|_{L^2_{(\tau,\zeta)}}.$$

Write the norm in B^{b,b_1} as

$$\|u\|_{B^{b,b_1}} \sim \|u\|_{Y^{b,0}} + \|u\|_{Y^{b+b_1,-1/4}}.$$

If $0 < b < a < 1/2$ and $\delta \in (0,1)$ then there exists $\theta > 0$ such that

$$\|\psi_\delta u\|_{Y^{-a,s}} \le c\delta^\theta \|u\|_{Y^{-b,s}}. \tag{28}$$

The estimate (28) is proved in **[42]** in the context of KdV equation (cf. inequality (3.29) in **[42]**). For KP equation the proof is essentially the same. Now using (28) we arrive at

$$\begin{aligned}\|\psi_\delta u\|_{B^{-b',b_1}} &\sim \|\psi_\delta u\|_{Y^{-b',0}} + \|\psi_\delta u\|_{Y^{-b'+b_1,-1/4}} \\ &\le c\delta^{\theta_1}\|u\|_{Y^{-b'',0}} + c\delta^{\theta_2}\|u\|_{Y^{-b''+b_1,-1/4}} \\ &\le c\delta^\theta \|u\|_{B^{-b'',b_1}}\end{aligned}$$

This completes the proof of Proposition 3.4.

Let $u_1, u_2 \in B^{1/2+\epsilon,1/4-2\epsilon}([-T,T])$ be two solutions of (2) in the time interval $[-T,T]$ and $\delta \in (0,T)$ to be specified later. Clearly

$$\psi_\delta(u_1-u_2) = \frac{\psi_\delta}{2}\int_0^t U(t-t')\partial_x(u_1^2-u_2^2)(t')dt'.$$

Let $I=[-\delta,\delta]$. Then Proposition 3.1, Proposition 3.4 and Theorem 1 yield for sufficiently small $\epsilon > 0$

$$\begin{aligned}
\|u_1-u_2\|_{B^{\frac12+\epsilon,\frac14-2\epsilon}(I)} &\le c\|\psi_\delta^2(u_1-u_2)\|_{B^{\frac12+\epsilon,\frac14-2\epsilon}} \\
&\le c\|\psi_\delta^2\partial_x(u_1^2-u_2^2)\|_{B^{-\frac12+\epsilon,\frac14-2\epsilon}} \\
&\le c\delta^\theta\|\psi_\delta\partial_x(u_1^2-u_2^2)\|_{B^{-\frac12+\frac32\epsilon,\frac14-2\epsilon}} \\
&\le c\delta^\theta\|\psi_\delta(u_1-u_2)\|_{B^{\frac12+\epsilon,\frac14-2\epsilon}}\|u_1+u_2\|_{B^{\frac12+\epsilon,\frac14-2\epsilon}} \\
&\le c\delta^\theta\left(\|u_1-u_2\|_{B^{\frac12+\epsilon,\frac14-2\epsilon}(I)}+\right).
\end{aligned}$$

Here the constant c depends only on $\|u_1+u_2\|_{B^{\frac12+\epsilon,\frac14-2\epsilon}}$. We chose δ such that $c\delta^\theta < 1/2$ to conclude that $u_1=u_2$ on I. Now we iterate the last argument to prove the uniqueness. This completes the proof of Theorem 2.

4. Proof of Theorem 5

4.1. A bilinear estimate. In this section we prove a bilinear estimate following the lines of the proof of Theorem 1.

THEOREM 4.1. *Let (b,b',b_1,s) be such that*

$$b>1/2,\quad b'>s+1/8,\quad b'>b_1+s,\quad b'-b_1>1/4+s,\quad b'+3b_1>1,$$

$$b'+2b_1>7/8,\quad b'>5/12+s/3,\quad b'+b_1>1/2+s/2,\quad b'>3/8+s/2.$$

Then the following inequality holds

$$\|\partial_x(uv)\|_{B_{-s}^{-b',b_1}} \le c\|u\|_{B_{-s}^{b,b_1}}\|v\|_{B_{-s}^{b,b_1}} \tag{29}$$

Proof. Recall that

$$\sigma := \sigma(\tau,\zeta) = \tau-\xi^3+\frac{\eta^2}{\xi},\quad \sigma_1 := \sigma(\tau_1,\zeta_1),\quad \sigma_2 := \sigma(\tau-\tau_1,\zeta-\zeta_1),$$

$$\theta := \theta(\tau,\zeta) = \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{1/4}},\quad \theta_1 = \theta(\tau_1,\zeta_1),\quad \theta_2 = \theta(\tau-\tau_1,\zeta-\zeta_1).$$

We have that (29) is equivalent to

$$\left|\int\int K(\tau,\zeta,\tau_1,\zeta_1)\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta\right| \leq$$

$$c\|u\|_{L^2}\|v\|_{L^2},\|w\|_{L^2}, \tag{30}$$

where

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{|\xi|\langle\theta\rangle\langle\xi\rangle^{-s}\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\theta_1\rangle\langle\theta_2\rangle}$$

Without loss of generality we can assume that $\widehat{u}\geq 0$, $\widehat{v}\geq 0$ and $\widehat{w}\geq 0$. By symmetry arguments we can assume that $|\sigma_1|\geq|\sigma_2|$. We denote by J the left-hand side of (30). Consider several cases for $(\tau,\zeta,\tau_1,\zeta_1)$.

Case 1. $|\xi|\leq 24$, $|\xi_1|\leq 48$. We denote by J_1 the restriction of J on this region. Using that $\langle\theta\rangle\leq c\langle\sigma\rangle^{b'}$ we obtain

$$K(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{c}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}.$$

Since $b>1/2$ the use of the Strichartz inequality yields

$$\begin{aligned} J_1 &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-b}\widehat{u}(\tau_1,\zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-b}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^4}\|w\|_{L^2}\\ &\leq c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.\end{aligned}$$

Case 2. $|\xi|\leq 24$, $|\xi_1|\geq 48$, $|\sigma_1|\geq|\sigma|$. We denote by J_2 the restriction of J on this region. We have that $\langle\theta\rangle\leq c\langle\theta_1\rangle\langle\xi_1\rangle^{1/4}$ and hence

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\leq \frac{c|\xi|\langle\xi_1\rangle^{1/4+s}\langle\xi-\xi_1\rangle^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\\ &\leq \frac{c|\xi|\langle\xi_1\rangle^{1/4+2s}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}.\end{aligned}$$

Since $|\xi|^{1/8+s}\langle\xi_1\rangle^{1/4+2s}\leq c\langle\sigma_1\rangle^{1/8+s}$ we obtain

$$K(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{c|\xi|^{7/8-s}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b-s-1/8}\langle\sigma_2\rangle^b}.$$

By Strichartz inequality we have

$$\begin{aligned} J_2 &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-b+s+1/8}\widehat{u}(\tau_1,\zeta_1))\|_{L^{q_1}}\\ &\quad \|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-b}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^{q_2}}\\ &\quad \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-b'}\widehat{w}(\tau,\zeta))\|_{L^{q_3}}\\ &\leq c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2},\end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b-s-1/8 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b' &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Note that if $b'>s+1/8$ then $\epsilon_1>0$.

Case 3. $|\xi|\le 24,\quad |\xi_1|\ge 48,\quad |\sigma|\ge|\sigma_1|$. We denote by J_3 the restriction of J on this region. We have that $\langle\theta\rangle\le c\langle\sigma\rangle^{b_1}$ and hence

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\le \frac{c|\xi|\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} \\ &\le \frac{c|\xi|^{1-s}}{\langle\sigma\rangle^{b'-b_1-s}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} \\ &\le \frac{c}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}, \end{aligned}$$

provided $b'>b_1+s$. Now a use of Strichartz inequality yields

$$\begin{aligned} J_3 &\le \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-b}\widehat{u}(\tau_1,\zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-b}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^4}\|w\|_{L^2} \\ &\le c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \end{aligned}$$

Case 4. $|\xi|\ge 24,\quad |\sigma|\ge|\sigma_1|,\quad \min\{|\xi_1|,|\xi-\xi_1|\}\le 1$. We denote by J_4 the restriction of J on this region. Note that in this case we have

$$\langle\xi\rangle^{-s}\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s\le c.$$

Therefore we are in position to perform the arguments of the proof of Theorem 1, Case 2. Under the assumptions $b'>1/3$ and $b'-b_1>1/4$ we obtain

$$J_4\le c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 5. $|\xi|\ge 24,\quad |\sigma|\ge|\sigma_1|,\quad \min\{|\xi_1|,|\xi-\xi_1|\}\ge 1$. We denote by J_5 the restriction of J on this region. Note that in this case we have

$$\langle\xi\rangle^{-s}\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s\le c\langle\xi\rangle^{-2s}\langle\sigma\rangle^s.$$

Using Cauchy-Schwarz inequality we obtain

$$J_5\le\int_{\mathbf{R}^3}I(\tau,\zeta)\left\{\int_{|\sigma|\ge|\sigma_1|}|\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)|^2d\tau_1d\zeta_1\right\}^{1/2}\widehat{w}(\tau,\zeta)d\tau d\zeta,$$

where

$$I(\tau,\zeta)=\frac{|\xi|^{1-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}}\left(\int_{|\sigma|\ge|\sigma_1|}\frac{d\tau_1d\zeta_1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}\langle\theta_1\rangle^2\langle\theta_2\rangle^2}\right)^{1/2}.$$

We have the following Lemma.

LEMMA 4.1.

$$I(\tau,\zeta) \le const, \quad if \quad |\xi| \ge 24.$$

Proof. Using (9) we obtain

$$I(\tau,\zeta) \le \frac{|\xi|^{1-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}} \left(\int_{|\sigma|\ge|\sigma_1|} \frac{d\zeta_1}{\langle\sigma_1+\sigma_2\rangle^{2b}}\right)^{1/2}.$$

Perform a change of variables $\alpha = \sigma_1+\sigma_2$, $\beta = 3\xi\xi_1(\xi-\xi_1)$. Note that $\xi \in [-3|\sigma|, \min\{3/4\xi^3, 3|\sigma|\}]$, when $\xi \ge 0$ and $\xi \in [\max\{3/4\xi^3, -3|\sigma|\}, 3|\sigma|\}]$, when $\xi \le 0$. We assume that $\xi \ge 0$. If $\xi \le 0$ then the arguments are the same. We have

$$d\zeta_1 = \frac{c|\beta|^{1/2} d\alpha d\beta}{|\xi|^{3/2}(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}}$$

Therefore we obtain using (11)

$$I(\tau,\zeta) \le \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}} \left\{\int_{-3|\sigma|}^{\min\{3/4\xi^3,3|\sigma|\}} \int_{-\infty}^{\infty} \frac{|\beta|^{1/2} d\alpha d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}|\sigma+\beta-\alpha|^{1/2}\langle\alpha\rangle^{2b}}\right\}^{1/2} \le$$

$$\frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}} \left\{\int_{-3|\sigma|}^{\min\{3/4\xi^3,3|\sigma|\}} \frac{|\beta|^{1/2} d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}.$$

Now we consider two cases for (τ,ζ).

- $\frac{3}{4}|\xi|^3 \le 4|\sigma|$

We have that

$$\begin{aligned}
I(\tau,\zeta) &\le \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}}\{\int_{-3|\sigma|}^{0} \frac{|\beta|^{1/2} d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}} \\
&+ \int_0^{\frac{3}{4}\xi^3} \frac{|\beta|^{1/2} d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\}^{1/2} \\
&\le \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}}\{\int_{-3|\sigma|}^{0} \frac{d\beta}{\langle\sigma+\beta\rangle^{1/2}} \\
&+\xi^{3/2} \int_0^{\frac{3}{4}\xi^3} \frac{d\beta}{|\frac{3}{4}\xi^3-\beta|^{1/2}\langle\sigma+\beta\rangle^{1/2}}\}^{1/2} \\
&\le \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}}\{\langle\sigma\rangle^{1/2} + \xi^{3/2}(\int_0^{\frac{3}{4}\xi^3-1} \frac{d\beta}{|\frac{3}{4}\xi^3-\beta|}
\end{aligned}$$

$$+\int_0^{\frac{3}{4}\xi^3-1}\frac{d\beta}{\langle\sigma+\beta\rangle}+\int_{\frac{3}{4}\xi^3-1}^{\frac{3}{4}\xi^3}\frac{d\beta}{|\frac{3}{4}\xi^3-\beta|^{1/2}\langle\sigma+\beta\rangle^{1/2}})\}^{1/2}$$

$$\leq \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s}}(\langle\sigma\rangle^{1/2}+\xi^{3/2}\ln(1+\langle\sigma\rangle))^{1/2}$$

$$\leq \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s-1/4}}+\frac{c\langle\xi\rangle^{1-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s-}}.$$

Let $|\sigma|^{b_1}\leq|\xi|^{1/4}$. Then we have

$$I(\tau,\zeta)\leq\frac{\langle\xi\rangle^{1-2s}}{\langle\xi\rangle^{3b'-3s}}+\frac{\langle\xi\rangle^{1-2s}}{\langle\xi\rangle^{3b'-3s-}}\leq \text{ const },$$

provided $b'>1/3+s/3$.
Let $|\sigma|^{b_1}\geq|\xi|^{1/4}$. Then we have

$$I(\tau,\zeta) \leq \frac{\langle\xi\rangle^{-2s}}{\langle\sigma\rangle^{b'-b_1-s-1/4}}+\frac{\langle\xi\rangle^{3/4-2s}}{\langle\sigma\rangle^{b'-b_1-s-}}$$

$$\leq \frac{\langle\xi\rangle^{-2s}}{\langle\xi\rangle^{3b'-3b_1-3s-3/4}}+\frac{\langle\xi\rangle^{3/4-2s}}{\langle\sigma\rangle^{3b'-3b_1-3s-}}$$

$$\leq \text{ const },$$

provided $b'-b_1>1/4+s$.

- $4|\sigma|\leq\frac{3}{4}|\xi|^3$

In this case we have

$$I(\tau,\zeta) \leq \frac{c\langle\xi\rangle^{1/4-2s}\langle\theta\rangle}{\langle\sigma\rangle^{b'-s-1/4}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{d\beta}{(\frac{3}{4}\xi^3-\beta)^{1/2}\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}$$

$$\leq \frac{c\langle\theta\rangle}{\langle\sigma\rangle^{b'-s-1/4}\langle\xi\rangle^{1/2+2s}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{d\beta}{\langle\sigma+\beta\rangle^{1/2}}\right\}^{1/2}$$

$$\leq \frac{c\langle\theta\rangle\langle\sigma\rangle^{1/2-b'+s}}{\langle\xi\rangle^{1/2+2s}}.$$

Let $|\sigma|^{b_1}\leq|\xi|^{1/4}$. Then we have

$$I(\tau,\zeta)\leq\frac{\langle\xi\rangle^{3/2-3b'+3s}}{\langle\xi\rangle^{1/2+2s}}\leq \text{ const },$$

provided $b' > 1/3 + s/3$.
Let $|\sigma|^{b_1} \geq |\xi|^{1/4}$. Then we have

$$\begin{aligned} I(\tau,\zeta) &\leq \frac{\langle\sigma\rangle^{1/2-b'+b_1+s}}{\langle\xi\rangle^{3/4+2s}} \\ &\leq \frac{\langle\xi\rangle^{3/2-3b'+3b_1+3s}}{\langle\xi\rangle^{3/4+2s}} \\ &\leq \text{const}\ , \end{aligned}$$

provided $b' - b_1 > 1/4 + s/3$. This completes the proof of the Lemma. Therefore using Cauchy-Schwarz inequality we obtain

$$J_5 \leq c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 6. $|\xi| \geq 24 \quad |\sigma_1| \geq |\sigma|, \quad |\xi_1| \leq 1$. We denote by J_6 the restriction of J on this region. In this case we have that

$$\langle\xi\rangle^{-s}\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s \leq c.$$

Hence we are in the situation of Case 3 of the proof of Theorem 1. Now we shall apply a slightly more precise argument. We shall again localize J_6 with respect to ξ_1 and $\langle\sigma_1\rangle$. Set

$$J_6^{KM} = \int\int_{A^{KM}} K(\tau,\zeta,\tau_1,\zeta_1)\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta,$$

where

$$A^{KM} = \{(\tau_1,\zeta_1)\ :\ |\xi_1| \sim M,\ \langle\sigma_1\rangle \sim K\}.$$

We have

$$J_6 \leq \sum_{K,M} J_6^{KM},$$

where the sum is taken over $K = 2^k, k = 0,1,2,\dots$ and $M = 2^m, m = 0,-1,-2,\dots$. As in the proof of Theorem 1, Case 3 we can obtain

$$J_6^{KM} \leq cK^{1/4-b_1}M^{1/4}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{31}$$

Further we have

$$K(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\theta_1\rangle\langle\theta_2\rangle}.$$

Let $|\sigma|^{b_1} \leq |\xi|^{1/4}$. We denote by J_{61}^{KM} the restriction of J_6^{KM} on this region. We have

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\leq \frac{c|\xi|}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^b} \\ &\leq \frac{cM^{-1/2}K^{-\delta}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-\delta-1/2}\langle\sigma_2\rangle^b}. \end{aligned}$$

Here $\delta > 0$ is to be specified later. Using Proposition 2.1 and Hölder inequality we obtain

$$\begin{aligned} J_{61}^{KM} &\le \frac{c}{M^{1/2}K^{\delta}} \|\langle\sigma\rangle^{-b'}\widehat{w}(\tau,\zeta)\|_{L^{q_1}} \\ &\quad \|\langle\sigma_1\rangle^{-b-b_1+\delta+1/2}\widehat{u}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ &\quad \|\langle\sigma_2\rangle^{-b}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ &\le \frac{c}{M^{1/2}K^{\delta}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b' &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b+b_1-\delta-1/2 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Note that if $\delta < b'+b_1-1/2$ then ϵ_1 is positive.

Let $|\sigma|^{b_1} \ge |\xi|^{1/4}$. We denote by J_{62}^{KM} the restriction of J_6^{KM} on this region. We have

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\le \frac{c|\xi|^{3/4}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \\ &\le \frac{cM^{-3/8}K^{-\delta}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1-\delta-3/8}\langle\sigma_2\rangle^{b}}. \end{aligned}$$

Using Proposition 2.1 and Hölder inequality we obtain

$$\begin{aligned} J_{62}^{KM} &\le \frac{c}{M^{3/8}K^{\delta}} \|\langle\sigma\rangle^{-b'+b_1}\widehat{w}(\tau,\zeta)\|_{L^{q_1}} \\ &\quad \|\langle\sigma_1\rangle^{-b-b_1+\delta+3/8}\widehat{v}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ &\quad \|\langle\sigma_2\rangle^{-b}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ &\le \frac{c}{M^{3/8}K^{\delta}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b'-b_1 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}),\\ b+b_1-\delta-3/8 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}),\\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Note that if $\delta < b'-3/8$ then $\epsilon_1>0$. Hence

$$J_6^{KM} \le \frac{c}{M^{1/2}K^{\delta}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{32}$$

Interpolation between (31) and (32) with weights $\frac{2}{3}+$ and $\frac{1}{3}-$ respectively yields

$$J_6^{KM} \le \frac{cM^{\delta_1}}{K^{\delta_2}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2},$$

with $\delta_1>0$ and $\delta_2>0$ provided

$$(1/4-b_1)(\frac{2}{3}+)-\delta(\frac{1}{3}-)<0,$$

i.e. $\delta > 1/2-2b_1$. But δ should be such that

$$\delta \le \min(b'+b_1-1/2, b'-3/8).$$

Hence we are able to chose proper δ provided b and b_1 satisfy

$$b'+2b_1>7/8, \quad b'+3b_1>1.$$

Summing over K and M yields

$$J_6 \le c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 7. $|\xi|\ge 24,\ |\sigma_1|\ge|\sigma|,\ |\xi|\le 2|\xi_1|$. We denote by J_7 the restriction of J on this region. In this case we have

$$\frac{\langle\theta\rangle}{\langle\theta_1\rangle}\le c\langle\xi_1\rangle^{1/4}, \quad \frac{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}{\langle\xi\rangle^s}\le c\frac{\langle\xi_1\rangle^{2s}}{\langle\xi\rangle^s},$$

Hence

$$K(\tau,\zeta,\tau_1,\zeta_1)\le\frac{c|\xi_1|^{5/4+s}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}}.$$

Now the arguments of of the proof of Theorem 1, Case 4 provides a bound for J_7 provided $b'>5/12+s/3$.

Case 8. $|\xi|\ge 24,\ |\sigma_1|\ge|\sigma|,\ |\xi|\ge 2|\xi_1|,\ |\xi_1|\ge 1$. We denote by J_8 the restriction of J on this region. In this case we have that $|\xi|\le 2|\xi-\xi_1|$ and

$$\frac{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}{\langle\xi\rangle^s}\le c\langle\xi-\xi_1\rangle^s.$$

Hence

$$K(\tau,\zeta,\tau_1,\zeta_1) \le \frac{c|\xi||\xi-\xi_1|^s\langle\theta\rangle}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\theta_1\rangle}.$$

Let $|\sigma|^{b_1} \leq |\xi|^{1/4}$. We denote by J_{81} the restriction of J_8 on this region. We have

$$K(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{c|\xi||\xi_1|^{1/4}|\xi-\xi_1|^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^b}$$

$$\leq \frac{c}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-s/2-1/2}\langle\sigma_2\rangle^b}.$$

Proposition 2.1 and Hölder inequality yield

$$\begin{aligned} J_{81} \leq\ & \|\langle\sigma\rangle^{-b'}\widehat{w}(\tau,\zeta)\|_{L^{q_1}} \\ & \|\langle\sigma_1\rangle^{-b-b_1+s/2+1/2}\widehat{u}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ & \|\langle\sigma_2\rangle^{-b}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ \leq\ & c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=1$ and

$$\begin{aligned} b' &= 2(1/2+\epsilon_1)(1-\frac{2}{q_1}), \\ b+b_1-s/2-1/2 &= 2(1/2+\epsilon_1)(1-\frac{2}{q_2}), \\ b &= 2(1/2+\epsilon_1)(1-\frac{2}{q_3}). \end{aligned}$$

Note that if $b_1+b'>1/2+s/2$ then ϵ_1 is positive.

Let $|\sigma|^{b_1} \geq |\xi|^{1/4}$. We denote by J_{82} the restriction of J_8 on this region. We have

$$K(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{c|\xi|^{3/4}|\xi_1|^{1/4}|\xi-\xi_1|^s}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^b}$$

$$\leq \frac{c}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1-3/8-s/2}\langle\sigma_2\rangle^b}.$$

Proposition 2.1 and Hölder inequality yield

$$\begin{aligned} J_{82} \leq\ & \|\langle\sigma\rangle^{-b'+b_1}\widehat{w}(\tau,\zeta)\|_{L^{q_1}} \\ & \|\langle\sigma_1\rangle^{-b-b_1+3/8+s/2}\widehat{u}(\tau_1,\zeta_1)\|_{L^{q_2}} \\ & \|\langle\sigma_2\rangle^{-b}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\|_{L^{q_3}} \\ \leq\ & c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \end{aligned}$$

provided $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$ and

$$\begin{aligned} b' - b_1 &= 2(1/2 + \epsilon_1)(1 - \frac{2}{q_1}), \\ b + b_1 - s/2 - 3/8 &= 2(1/2 + \epsilon_1)(1 - \frac{2}{q_2}), \\ b &= 2(1/2 + \epsilon_1)(1 - \frac{2}{q_3}). \end{aligned}$$

Note that if $b' > 3/8 + s/2$ then ϵ_1 is positive. Hence

$$J_8 \le c\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

This completes the proof of the Theorem .

Now we state a corollary which will be used in the proof of Theorem 5.

COROLLARY 1. *Let* $(b, b', b_1) = (\frac{1}{2}+, \frac{1}{2}-, \frac{7}{32})$. *Then for* $s \in [0, 1/32)$ *the following inequality holds*

$$\|\partial_x(uv)\|_{B_{-s}^{-b',b_1}} \le c\|u\|_{B_{-s}^{b,b_1}}\|v\|_{B_{-s}^{b,b_1}}. \tag{33}$$

Using Corollary 1 we arrive at the following local existence result.

COROLLARY 2. *Let* $(b, b_1) = (\frac{1}{2}+, \frac{7}{32})$ *and* $s \in (-1/32, 0]$. *Then for any* $\phi \in H_x^s(\mathbf{R}^2)$ *there exist positive* $T = T(\|\phi\|_{H_x^s})$ *and a unique solution* $u(t, x, y)$ *of (3) in the time interval* $[-T, T]$ *satisfying*

$$u \in C([-T, T]; H_x^s(\mathbf{R}^2)) \cap B_s^{b,b_1}.$$

In addition the length T *of the existence interval satisfies* $T \ge c\|\phi\|_{H_x^s}^{-\frac{6}{1+2s}}$.

Proof. Once we have Corollary 1 the proof of Corollary 2 follows the lines of the proof of Theorem 2. The assertion for the length of the existence interval follows from the inequality

$$\|\phi_T\|_{H_x^s} \le cT^{\frac{1+2s}{6}}\|\phi\|_{H_x^s}.$$

4.2. Another bilinear estimate. This section is devoted to the proof of the following estimate.

THEOREM 4.2. *Let* $(b, b', b_1) = (\frac{1}{2}+, \frac{1}{2}-, \frac{7}{32})$. *Then for* $s \in [0, 1/32)$ *the following inequality holds.*

$$\|\partial_x(uv)\|_{B_0^{-b',0}} \le c\|u\|_{B_{-s}^{b,b_1}}\|v\|_{B_{-s}^{b,b_1}}. \tag{34}$$

Proof. Note that (34) is equivalent to

$$\left|\int\int K(\tau, \zeta, \tau_1, \zeta_1)\widehat{u}(\tau_1, \zeta_1)\widehat{v}(\tau - \tau_1, \zeta - \zeta_1)\widehat{w}(\tau, \zeta)d\tau_1 d\zeta_1 d\tau d\zeta\right| \le$$

$$c\|u\|_{L^2}\|v\|_{L^2}, \|w\|_{L^2}, \tag{35}$$

where

$$K(\tau, \zeta, \tau_1, \zeta_1) = \frac{|\xi|\langle\xi_1\rangle^s\langle\xi - \xi_1\rangle^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\theta_1\rangle\langle\theta_2\rangle}$$

Without loss of generality we can assume that $\widehat{u} \geq 0$, $\widehat{v} \geq 0$ and $\widehat{w} \geq 0$. By symmetry arguments we can assume that $|\sigma_1| \geq |\sigma_2|$. We denote by J the left-hand side of (35). Consider several cases for $(\tau, \zeta, \tau_1, \zeta_1)$.

Case 1. $|\xi| \leq 24$, $|\xi_1| \leq 48$. We denote by J_1 the restriction of J on this region. As in the proof of Theorem 1 we bound J_1, provided $b > 1/2$.

Case 2. $|\xi| \leq 24$, $|\xi_1| \geq 48$, $|\sigma_1| \geq |\sigma|$. We denote by J_2 the restriction of J on this region. In this case we have that

$$K(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{c|\xi|\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}$$

$$\leq \frac{c|\xi|^{1-s}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b-s}\langle\sigma_2\rangle^b}.$$

Now we bound J_2 by the aid of Proposition 2.1, provided $b > s$, i.e. $s \leq 1/2$.

Case 3. $|\xi| \leq 24$, $|\xi_1| \geq 48$, $|\sigma| \geq |\sigma_1|$. We denote by J_3 the restriction of J on this region. We have

$$K(\tau,\zeta,\tau_1,\zeta_1) \leq \frac{c|\xi|\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}$$

$$\leq \frac{c|\xi|^{1-s}}{\langle\sigma\rangle^{b'-s}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^b}$$

Now we can estimate J_3 by the aid of Proposition 2.1, provided $b' > s$, i.e. $s < 1/2$.

Case 4. $|\xi| \geq 24$, $|\sigma| \geq |\sigma_1|$, $\min\{|\xi_1|, |\xi-\xi_1|\} \leq 1$. We denote by J_4 the restriction of J on this region. Note that in this case we have

$$\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s \leq c|\xi|^s.$$

Now, in order to bound J_4, we can perform the arguments of the proof of Theorem 4.1, Case 4, provided $b' > 1/3 + s/3$ and $b' - b_1 > 1/4 + s/3$, i.e. $s < 1/2$ and $s < 3/32$.

Case 5. $|\xi| \geq 24$, $|\sigma| \geq |\sigma_1|$, $\min\{|\xi_1|, |\xi-\xi_1|\} \geq 1$. We denote by J_5 the restriction of J on this region. In this case we have

$$\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s \leq c\langle\sigma\rangle^s.$$

The arguments of the proof of Theorem 1, Case 2 provides a bound for J_5 provided $b' > 1/3+s$ and $b' - b_1 > 1/4 + s$, i.e. $s < 1/6$ and $s < 1/32$.

Case 6. $|\xi| \geq 24$ $|\sigma_1| \geq |\sigma|$, $|\xi_1| \leq 1$. We denote by J_6 the restriction of J on this region. In this case we use the same localization as in Case 6 of the proof of Theorem 4.1. We have

$$\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s \leq c\langle\xi\rangle^{s/2}\langle\xi-\xi_1\rangle^{s/2} \leq \frac{cK^{s/2}}{M^{s/2}}.$$

Hence the following inequalities could be obtained similarly to (31) and (32) respectively

$$J_6^{KM} \leq cK^{1/32+s/2}M^{1/4-s/2}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \tag{36}$$

$$J_6^{KM} \leq K^{-\delta+s/2}M^{-1/2-s/2}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2},$$

where $1/2 - 2b_1 \langle \delta < \min\{b' - 3/8, b' + b_1 - 1/2\}$, i.e. $1/16 < \delta < 1/8$. We take $\delta = \frac{1}{8}-$. Hence

$$J_6^{KM} \leq K^{-1/8+s/2+}M^{-1/2-s/2}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \tag{37}$$

We shall interpolate between (36) and (37) with weights α and $1-\alpha$ respectively. Choose α such that

$$\alpha(1/4 - s/2) + (1-\alpha)(-1/2 - s/2) = 0+ \quad \text{(to have a positive power of } M\text{)},$$

i.e. $\alpha = 2/3 + 2s/3+$. Further we need

$$\alpha(1/32 + s/2) + (1-\alpha)(-1/8 + s/2+) < 0, \quad \text{(to have a negative power of } K\text{)},$$

i.e. $s < 1/29$. Hence

$$J_6^{KM} \leq \frac{cM^{\delta_1}}{K^{\delta_2}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2},$$

with $\delta_1 > 0$ and $\delta_2 > 0$. Summing over K and M completes the proof in this case.

Case 7. $|\xi| \geq 24, \quad |\sigma_1| \geq |\sigma|, \quad |\xi| \leq 2|\xi_1|$. We denote by J_7 the restriction of J on this region. In this case we have

$$\langle \xi_1 \rangle^s \langle \xi - \xi_1 \rangle^s \leq c|\xi_1|^{2s}.$$

Now the arguments of the proof of Theorem 1, Case 4 provides a bound for J_7 provided $b' > 5/12 + 2s/3$, i.e. $s < 1/4$.

Case 8. $|\xi| \geq 24, \quad |\sigma_1| \geq |\sigma|, \quad |\xi| \geq 2|\xi_1|, \quad |\xi_1| \geq 1$. We denote by J_8 the restriction of J on this region. In this case we have that $|\xi| \leq 2|\xi - \xi_1|$ and

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\leq \frac{c|\xi||\xi_1|^{1/4+s}|\xi-\xi_1|^s}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^b} \\ &\leq \frac{c}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-1/2-s/2}\langle\sigma_2\rangle^b} \end{aligned}$$

Now a use of Proposition 2.1 gives a bound for J_7, provided $b' + b_1 - 1/2 > s/2$, i.e. $s < 7/16$. This completes the proof of the Theorem .

4.3. End of the proof of Theorem 5. Decompose the initial data $\phi \in H_x^{-s}$ in two parts

$$\phi = \phi_0 + \eta_0,$$

where

$$\phi_0 = \int_{|\xi| \leq N} \widehat{\phi}(\xi,\eta)e^{i(x\xi+y\eta)}d\xi d\eta.$$

Here $N \gg 1$ is a large number to be specified later. Note that $\|\phi_0\|_{L^2} \sim N^s$ and $\|\eta_0\|_{H_x^{-\sigma}} \sim N^{s-\sigma}$. Consider the following system of integral equations

$$u(t) = U(t)\phi_0 + \int_0^t U(t-t')u(t')u_x(t')dt', \tag{38}$$

$$v(t) = U(t)\eta_0 + \int_0^t U(t-t')\partial_x(v^2+2uv)(t')dt'. \tag{39}$$

We have the following local existence Theorem for (38)-(39).

THEOREM 4.3. *Let* $(b, b_1) = (\frac{1}{2}+, \frac{7}{32})$ *and* $\sigma \in (s, 1/32)$. *Then there exists a unique solution* (u, v) *of (38)-(39) on the time interval* $I = [-\delta, \delta]$ *satisfying*

$$u \in C([-T,T]; L^2(\mathbf{R}^2)) \cap B_0^{b,b_1}(I), \quad v \in C([-T,T]; H_x^{-\sigma}(\mathbf{R}^2)) \cap B_{-\sigma}^{b,b_1}(I).$$

In addition the length δ *of the existence interval satisfies* $\delta \sim N^{-6s}$ *and the following inequalities hold*

$$\|u\|_{B_0^{b,b_1}} \le cN^{\frac{37}{16}s+}, \tag{40}$$

$$\|v\|_{B_{-\sigma}^{b,b_1}} \le cN^{\frac{37}{16}s-\sigma+2s\sigma+}. \tag{41}$$

Proof. We apply the scaling argument of the proof of Theorem 2 and Corollary 2. Recall that if $u(t,x,y)$ is a function on $\mathbf{R}^3$ then we define u_δ by $u_\delta = \delta^{2/3}u(\delta t, \delta^{1/3}x, \delta^{2/3}y)$ and similarly for given $\phi(x,y)$ we define ϕ_δ as $\phi_\delta = \delta^{2/3}\phi(\delta^{1/3}x, \delta^{2/3}y)$. Consider (38)-(39) with initial data $((\phi_0)_\delta, (\eta_0)_\delta)$. We have that

$$\|(\phi_0)_\delta\|_{L^2} = \delta^{1/6}\|\phi_0\|_{L^2} \sim \delta^{1/6}N^s$$

and

$$\|(\eta_0)_\delta\|_{H_x^{-\sigma}} \le \delta^{\frac{1-2\sigma}{6}}\|\eta_0\|_{H_x^{-\sigma}} \sim \delta^{\frac{1-2\sigma}{6}}N^{s-\sigma}$$

As in the proof of Theorem 2 we shall use a contraction argument to obtain a local solution (u, v) of (38)-(39) in $B_0^{b,b_1} \oplus B_{-\sigma}^{b,b_1}$ on the time interval $[-1, 1]$ with data $((\phi_0)_\delta, (\eta_0)_\delta)$. More precisely we define an operator T

$$T : B_0^{b,b_1} \oplus B_{-\sigma}^{b,b_1} \longmapsto B_0^{b,b_1} \oplus B_{-\sigma}^{b,b_1},$$

where $T(u,v) = (T_1(u,v), T_2(u,v))$ and

$$T_1(u,v) = \psi(t)U(t)(\phi_0)_\delta + \psi(t)\int_0^t U(t-t')u(t')u_x(t')dt',$$

$$T_2(u,v) = \psi(t)U(t)(\eta_0)_\delta + \psi(t)\int_0^t U(t-t')\partial_x(v^2+2uv)(t')dt'.$$

Using Proposition 3.1 and Corollary 4.1 we obtain

$$\|T_1(u,v)\|_{B_0^{b,b_1}} \le c\{\|(\phi_0)_\delta\|_{L^2} + \|u\|^2_{B_0^{b,b_1}}\},$$

$$\|T_2(u,v)\|_{B_{-\sigma}^{b,b_1}} \le c\{\|(\eta_0)_\delta\|_{H_x^{-\sigma}} + \|u\|_{B_0^{b,b_1}}\|v\|_{B_{-\sigma}^{b,b_1}} + \|v\|^2_{B_{-\sigma}^{b,b_1}}\}.$$

Now a use of the contraction mapping principle provides a fixed point (u,v) of T for δ such that $\|(\phi_0)_\delta\|_{L^2} = \delta^{1/6}\|\phi_0\|_{L^2} = c_0 \ll 1$. Note that with this choice of δ the norm $\|(\eta_0)_\delta\|_{H_x^{-\sigma}}$ is small. Furthermore since $((\phi_0)_\delta, (\eta_0)_\delta)$ is small in $L^2 \oplus H_x^{-\sigma}$ then we can perform a contraction argument in $B_1 \oplus B_2$, where B_1 is a ball with radius $c\|(\phi_0)_\delta\|_{L^2}$ in B_0^{b,b_1} and B_2 is a ball with radius $c\|(\eta_0)_\delta\|_{H_x^{-\sigma}}$ in $B_{-\sigma}^{b,b_1}$. Hence we obtain that (u,v) satisfies

$$\|u\|_{B_0^{b,b_1}} \leq c\|(\phi_0)_\delta\|_{L^2} \sim c, \tag{42}$$

$$\|v\|_{B_{-\sigma}^{b,b_1}} \leq c\|(\eta_0)_\delta\|_{H_x^{-\sigma}} \sim N^{-\sigma+2s\sigma}. \tag{43}$$

Since (u,v) is a solution of (38)-(39) on $[-1,1]$ with data $((\phi_0)_\delta, (\eta_0)_\delta)$ then $(u_{\delta^{-1}}, v_{\delta^{-1}})$ is a solution of (38)-(39) with data (ϕ_0, η_0). This proves the existence. The uniqueness follows from the argument performed in the proof of Proposition 3.4. In order to prove (40) and (41) we need the next Lemma.

LEMMA 4.2. *Let $\delta \leq 1$ and $s \geq 0$. Then*

$$\|u_{\delta^{-1}}\|_{B_{-s}^{b,b_1}} \leq c\delta^{1/3-b-b_1}\|u\|_{B_{-s}^{b,b_1}}.$$

Proof of the Lemma. Recall that $\sigma(\tau,\zeta) = \tau - \xi^3 + \frac{\eta^2}{\xi}$ and $\zeta = (\xi,\eta)$. We have

$$\begin{aligned}
\|u_{\delta^{-1}}\|^2_{B_{-s}^{b,b_1}} &= \delta^{-4/3}\|u(./\delta, ./\delta^{1/3}, ./\delta^{2/3})\|^2_{B_{-s}^{b,b_1}} \\
&= \delta^{-4/3}\int \langle\sigma\rangle^{2b}\left(1 + \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{1/4}}\right)^2 \langle\xi\rangle^{-s} \\
&\quad |\mathcal{F}u(./\delta, ./\delta^{1/3}, ./\delta^{2/3})(\tau,\zeta)|^2 d\tau d\zeta \\
&= \delta^{8/3-2b-2b_1}\int (\delta + \sigma(\delta\tau, \delta^{1/3}\xi, \delta^{2/3}\eta))^{2b}\langle\xi\rangle^{-s} \\
&\quad \left(\delta^{b_1} + \frac{(\delta+\sigma(\delta\tau,\delta^{1/3}\xi,\delta^{2/3}\eta))^{b_1}}{\langle\xi\rangle^{1/4}}\right)^2 |\widehat{u}(\delta\tau,\delta^{1/3}\xi,\delta^{2/3}\eta)|^2 d\tau d\zeta \\
&\leq \delta^{2/3-2b-2b_1}\|u\|^2_{B_{-s}^{b,b_1}}
\end{aligned}$$

Hence

$$\|u_{\delta^{-1}}\|_{B_{-s}^{b,b_1}} \leq c\delta^{1/3-b-b_1}\|u\|_{B_{-s}^{b,b_1}}.$$

This completes the proof of the Lemma. Using Lemma 4.2, (42) and (43) we arrive at (40), (41). This completes the proof of Theorem 4.3.

We decompose v in the following way

$$v(t) = U(t)\eta_0 + w(t).$$

Further we set

$$\phi_1 = u(\delta) + w(\delta), \quad \eta_1 = U(\delta)\eta_0.$$

It is clear that $\|\eta_0\|_{H_x^{-\sigma}} = \|\eta_1\|_{H_x^{-\sigma}}$. Now we shall estimate the L^2 norm of ϕ_1.

$$\begin{aligned} \|\phi_1\|_{L^2} &\le \|u(\delta)\|_{L^2} + \|w(\delta)\|_{L^2} \\ &= \|\phi_0\|_{L^2} + \|w(\delta)\|_{L^2} \\ &\sim N^s + \|w(\delta)\|_{L^2}. \end{aligned}$$

Now we estimate $\|w(\delta)\|_{L^2}$ by the aid of (40), (41), Proposition 3.1 and Theorem 4.2

$$\begin{aligned} \|w(\delta)\|_{L^2} &\le sup_{t\in(-\infty,\infty)}\|\psi(t)w(t)\|_{L^2} \\ &\le \|\psi w\|_{B_0^{b,0}} \\ &\le c(\|\partial_x(uv)\|_{B_0^{-b',0}} + \|\partial_x(v^2)\|_{B_0^{-b',0}}) \\ &\le c(\|u\|_{B_{-\sigma}^{b,b_1}}\|v\|_{B_{-\sigma}^{b,b_1}} + \|v\|^2_{B_{-\sigma}^{b,b_1}}) \\ &\le c(N^{\frac{37}{16}s+}N^{\frac{37}{16}s-\sigma+2s\sigma+} + N^{\frac{37}{8}s-2\sigma+4s\sigma+}) \\ &\le cN^{\frac{37}{8}s-\sigma+2s\sigma+}. \end{aligned}$$

Hence

$$\|\phi_1\|_{L^2} \le \|\phi_0\|_{L^2} + cN^{\frac{37}{8}s-\sigma+2s\sigma+}.$$

Therefore the L^2 norm increase by at most $cN^{\frac{37}{8}s-\sigma+2s\sigma+}$ passing from time 0 to time δ. We iterate the argument replacing ϕ_0 and η_0 with ϕ_1 and η_1 respectively. We would like to iterate the process $M(N)$ times. In order to keep the control on the L^2 we need that

$$M(N)N^{\frac{37}{8}s-\sigma+2s\sigma+} \sim N^s, \quad \text{i.e.} \quad M(N) \sim N^{\sigma-\frac{29}{8}s-2s\sigma-}.$$

Hence we are able to achieve an interval of size

$$M(N)\delta \sim N^{\sigma-\frac{77}{8}s-2s\sigma-}.$$

Therefore to achieve any interval we should impose the condition

$$\sigma - \frac{77}{8}s - 2s\sigma > 0. \tag{44}$$

The largest possible value for σ is 1/32. We take $\sigma = \frac{1}{32}-$. Hence (44) is equivalent to $s < 1/310$. This completes the proof of Theorem 5.

CHAPTER 5

On the local regularity of Kadomtsev-Petviashvili-II equation

This Chapter essentially contains the joint paper with Hideo Takaoka [54] *(On the local regularity of Kadomtsev-Petviashvili II equation, Preprint (1999)).*

Abstract.

We prove the local well-posedness for the KP-II equation in the anisotropic Sobolev spaces $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ for $s_1 > -1/3$, $s_2 \geq 0$. On the other hand we prove that the crucial bilinear estimate needed for the local well-posedness fails for $s_1 < -1/3$, $s_2 = 0$.

1. Introduction

In this Chapter we continue the study of the Cauchy problem for the Kadomtsev-Petviashvili-II (KP-II) equation with data in an anisotropic Sobolev space of low order. The Kadomtsev-Petviashvili equations are two dimensional extensions of the Korteweg-de Vries (KdV) equation. They are derived in [**33**] as "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one directional with weak transverse effects. Consider the initial value problem for KP-II equation

$$(1)\qquad \begin{cases} (u_t + u_{xxx} + uu_x)_x + u_{yy} = 0, & (t,x,y) \in \mathbf{R}^3, \\ u(0,x,y) = \phi(x,y), \end{cases}$$

where the initial data $\phi(x,y)$ belongs to an anisotropic Sobolev space $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ defined by

$$H^{s_1,s_2}_{x,y}(\mathbf{R}^2) = \left\{ \phi \in \mathcal{S}'(\mathbf{R}^2) : \|\phi\|_{H^{s_1,s_2}_{x,y}} < \infty \right\},$$

where

$$\|\phi\|_{H^{s_1,s_2}_{x,y}} = \|(1-\partial_x^2)^{s_1/2}(1-\partial_y^2)^{s_2/2}\phi\|_{L^2_{x,y}}.$$

The spaces $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ are a natural set for the initial data of (1) since their homogeneous versions are invariant under the scale transformations preserving the KP equations. More precisely if $u(t,x,y)$ solves (1) with data $\phi(x,y)$ then so does

$$u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y),$$

with data

$$\phi_\lambda(x,y) = \lambda^2 \phi(\lambda x, \lambda^2 y).$$

Consider the homogeneous versions $\dot{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ of $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ equipped with the norm

$$\|\phi\|_{\dot{H}^{s_1,s_2}_{x,y}} = \|(-\partial_x^2)^{s_1/2}(-\partial_y^2)^{s_2/2}\phi\|_{L^2_{x,y}}.$$

Then

$$\|u_\lambda(t,.,.)\|_{\dot{H}^{s_1,s_2}_{x,y}} = \lambda^{s_1+2s_2+1/2}\|u(\lambda^3 t,.,.)\|_{\dot{H}^{s_1,s_2}_{x,y}}.$$

Hence for $s_1 + 2s_2 + 1/2 = 0$ the space $\dot{H}^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ is invariant under the scale transformation which preserves the KP equations. Therefore the critical values for (s_1, s_2) for initial data in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ is $s_1 + 2s_2 + 1/2 = 0$. One may expect well-posedness (ill-posedness) for $s_1 + 2s_2 + 1/2 > 0$ $(s_1 + 2s_2 + 1/2 < 0)$.

Our goal is to seek for the lowest possible (s_1, s_2) such that (1) is locally well-posed in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ (more precisely, within the framework of $B^{b,b_1,b_2}_{s_1,s_2}$, which is defined in Definition 1). The notion of local well-posedness in the functional space X means the existence, the uniqueness, the persistence property (i.e. a solution describes a continuous curve in X whenever $\phi \in X$) and the continuous dependence of the solution upon the data. There are a large number of recent papers devoted to the Cauchy problem for (1) (cf. **[14, 20, 31, 32, 48, 52, 53, 57, 58, 60, 61]** and the references therein). The KP equations are infinite dimensional integrable Hamiltonian systems. Thus the inverse scattering technique could be applied to (1) under appropriate decay assumptions on the initial data ϕ (cf. **[61]**). In **[61]** a smallness assumption on the data is also assumed. Using energy estimates one can obtain local well-posedness of (1) in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ provided $s_1 > 3/2$ and $s_2 > 1/2$. Actually the proof does not rely on the specific structure of KP equations and could be performed in a rather general context. The condition for (s_1, s_2) is imposed in order to control the L^∞ norm of the x derivative of the solution by the usual Sobolev embedding. The equation (1) possesses an infinite number of conservation laws. Unfortunately the only one providing an a priori bound seems to be the L^2 norm. In **[20]** A.V. Faminski obtained weak solutions of (1) in a suitable weighted space. In **[12]**, **[13]** J. Bourgain developed a new method to study the local regularity of nonlinear dispersive equations. This method has a special advantage for the quadratic nonlinearities and has been first applied to the nonlinear Schrödinger equation and to the KdV equation. An essential element of the method is the new Fourier transform restriction spaces strongly related to the symbol of the respective equation. The method has been successfully applied for KP-II equation as well. In **[14]** J. Bourgain showed that (1) is locally well-posed for data in L^2. The proof is mainly performed for periodic boundary conditions but could be applied to the continuous case too. The arguments of **[14]** do not use the Strichartz inequality for the KP equations. One uses dyadic decompositions related to the symbol of the linearized operator, in order to prove the crucial bilinear estimate. An additional difficulty appears in the continuous case because of the appearance of small frequencies cases in the integral representation of the nonlinear estimate. In **[57]** the local well-posedness of (1) in Sobolev spaces with negative indices with respect to x variable is obtained. However, an additional condition on initial data is imposed. More precisely, it is assumed that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in L^2$, which is rather restrictive. The constraint on the data is in order to eliminate the difficulties with the small Fourier modes in the integral representation of the nonlinear estimate. This condition on the data was removed in **[52]**, where local well-posedness for the integral equation corresponding to (1) in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/4$, $s_2 \geq 0$ is obtained. In **[53]**,**[58]** the global well-posedness of (1) with data below L^2 is shown by using a new idea due to J. Bourgain of decomposing the initial data into low and high Fourier modes. In all papers **[52]**,**[53]**,**[57]**, **[58]** a modified version of the proof of

[14] is performed. Namely, the Strichartz inequality for the KP equations are injected into the framework of the Bourgain spaces associated to the KP equations and one uses the simple calculus techniques which C. E. Kenig, G. Ponce and L. Vega first performed in the context of the KdV equation.

In this Chapter we prove that (1) is locally well-posed for data in the anisotropic Sobolev spaces $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, $s_1 > -1/3$, $s_2 \geq 0$. The proof is a suitable combination of the arguments of **[52]** and **[58]**. The result seems to be the optimal one for initial data in $H^{s_1,0}_{x,y}(\mathbf{R}^2)$, due to the counter examples of the relevant bilinear estimate constructed in the second part of the Chapter. Now we define the Bourgain spaces associated to the KP-II equation.

DEFINITION 1. *For $b, b_1, b_2, s_1, s_2 \in \mathbf{R}$ we define $B^{b,b_1,b_2}_{s_1,s_2}$ as a Bourgain type space associated to the KP-II equation*

$$B^{b,b_1,b_2}_{s_1,s_2} = \left\{ u \in \mathcal{S}'(\mathbf{R}^3) : \|u\|_{B^{b,b_1,b_2}_{s_1,s_2}} < \infty \right\},$$

where

$$\|u\|_{B^{b,b_1,b_2}_{s_1,s_2}} = \left\| \left\langle \tau - \xi^3 + \frac{\eta^2}{\xi} \right\rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \left(1 + \frac{\langle \tau - \xi^3 + \frac{\eta^2}{\xi} \rangle^{b_1}}{\langle \xi \rangle^{b_2}} \right) \widehat{u}(\tau, \xi, \eta) \right\|_{L^2_{\tau,\xi,\eta}}$$

and $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. Let $I \subset \mathbf{R}$ be an interval. Then we define the space $B^{b,b_1,b_2}_{s_1,s_2}(I)$ equipped with the norm

$$\|u\|_{B^{b,b_1,b_2}_{s_1,s_2}(I)} = \inf_{w \in B^{b,b_1,b_2}_{s_1,s_2}} \left\{ \|w\|_{B^{b,b_1,b_2}_{s_1,s_2}}, \quad w(t) = u(t) \text{ on } I \right\}.$$

Now we state the crucial bilinear estimate which is essential for the proof of the local existence result concerning the KP-II equation.

THEOREM 1. *Let the real numbers $b, b', b_1, b_2, s_1, s_2$ be such that*

$$b_1 \geq 0, \quad b_2 \geq 0, \quad b > 1/2, \quad s_2 \geq 0, \quad s_1 > b_1 - b',$$

$$s_1 > 1 - 3b', \quad s_1 > 1 + 3b_1 - 3b' - b_2, \quad b' > b_1 + 1/4,$$

$$b' + 3b_1 > 1, \quad b' + 2b_1 + b_2 - 2b_1 b_2 > 1, \quad s_1 > b_2 - 1,$$

$$b' + b_1 > 1/2, \quad 2b' + b_2 > 1, \quad s_1 > 1 - 3b_1 - 3b' + b_2.$$

Then the following inequality holds

$$\|\partial_x(uv)\|_{B^{-b',b_1,b_2}_{s_1,s_2}} \lesssim \|u\|_{B^{b,b_1,b_2}_{s_1,s_2}} \|v\|_{B^{b,b_1,b_2}_{s_1,s_2}}. \tag{2}$$

Using Theorem 1 with $b = \frac{1}{2}+$, $b' = \frac{1}{2}-$, $b_1 = \frac{1}{6}+$ (depending on b'), $b_2 = \frac{1}{3}$ we arrive at the following Theorem.

THEOREM 2. *Let $s_1 > -1/3$, $s_2 \geq 0$. Then*

$$\|\partial_x(uv)\|_{B^{-\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \lesssim \|u\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \|v\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}}. \tag{3}$$

Remark 1.1. The estimate of Theorem 1 contains the bilinear estimates of Proposition 3.4 established in [**52**]. Namely taking $b = \frac{1}{2}+$, $b' = \frac{1}{2}-$, $b_1 = \frac{1}{4}$, $b_2 = s_1 + \frac{3}{4}$ we obtain for $s_1 > -1/4$

$$\|\partial_x(uv)\|_{B_{s_1,s_2}^{-\frac{1}{2}+,\frac{1}{4},s_1+\frac{3}{4}}} \lesssim \|u\|_{B_{s_1,s_2}^{\frac{1}{2}+,\frac{1}{4},s_1+\frac{3}{4}}} \|v\|_{B_{s_1,s_2}^{\frac{1}{2}+,\frac{1}{4},s_1+\frac{3}{4}}}.$$

The proof of Theorem 1 uses the Fourier transform restriction method due to J. Bourgain. As in the case of the KdV equation the difficulty stems from the derivative nonlinearity. The smoothing effects result from an arithmetic identity involving the symbol $\tau - \xi^3 + \frac{\eta^2}{\xi}$ of the linearized KP-II equation (cf. (13) below). A "bad" sign in the similar identity for the symbol of the KP-I equation is the obstruction to perform the method for the KP-I equation (which is derived by changing the sign of the term u_{yy} in (1)).

Since the KP equations are two dimensional generalizations of the KdV equation one may try to adapt the arguments of some of the existing results for the KdV equation in the context of KP equations. Unfortunately a lot of properties of the KdV equation do not have analogues in the case of the KP equations. For example the sharp version of the Kato smoothing effect for the linearized KdV equation is used in order to gain regularity for the KdV equation in [**40, 41**]. The point is that one controls the $L_x^\infty(L_t^2)$ norm of the gradient of the solution to the linearized KdV equation by the L_x^2 norm of the initial data. In order to prove this (sharp) version of Kato smoothing effect one changes the role of the time variable t and the space variable x in the oscillatory integral representing the solution of the linearized KdV equation. Then an application of Plancherel identity provides the needed bound. If we try to use the above argument for the linearized KP equations we should choose one of the space variables x or y to be changed with the time variable and hence we lose the symmetry of the estimate. On the other hand in [**48**] a local smoothing effect for the KP-II equation is obtained. The proof uses the original Kato idea (cf. [**37**]) in the context of the KdV equation. Since the Kato local smoothing effect can be used to obtain global weak solutions for the KdV equation (cf. [**37, 44**]) for data in L^2 the local smoothing effect of [**48**] strongly suggests the global existence of weak solutions for KP-II equation with data in L^2. Actually A.V Faminskii [**20**] proves such a result for data in an L^2 weighted space. Another tool used in the context of the KdV equation are estimates for the corresponding maximal function. We do not know of any satisfactory maximal function inequalities for the KP equations.

On the other hand, there are Strichartz inequalities for the KP equations (cf. [**48**]). They are in fact similar to those of the 2D Schrödinger equation and hence at this level the KP equations behave as the 2D Schrödinger equation. It is known that it is difficult to solve in the Bourgain spaces a derivative nonlinear Schrödinger equation even in 1D because of the lower order dispersion compared to the KdV equation (where the dispersion is sufficient to compensate the loss of one derivative). Therefore the fact that one is able to solve the KP-II equation (recuperate a derivative loss) in Bourgain spaces shows that the dispersion of the KP-II equation is in some sense stronger than that of the 2D Schrödinger equation. We note also that the Hessian matrix of the symbol of the linearized KP equation is a constant and hence the Strichartz inequalities do not gain some additional regularity, contrary to the KdV equation, where the second derivative of the symbol of the linearized KdV equation increase at infinity.

There are difficulties to prove Theorems 1 and 2 by the calculus arguments of [**42**] alone in the case of the KP-II equation. In the integral representation of the bilinear estimate one needs the integrability with respect to η, the dual of y in the Fourier variables. For that purpose we need the following extra factor

$$1+\frac{\langle\tau-\xi^3+\frac{\eta^2}{\xi}\rangle^{b_1}}{\langle\xi\rangle^{b_2}} \tag{4}$$

in the definition of the Fourier transform restriction spaces. We need (4) also to treat the small frequencies in the proof of the crucial bilinear estimate. At this stage we do not know of a proof of the bilinear estimate (2) which does not make use of the Strichartz inequality for the KP-II equation.

A consequence of Theorem 2 is a local existence result for (1). The method consists in applying a fixed point argument to the integral equation corresponding to (1). The linear estimates are similar to those of other dispersive models where the method of Bourgain could be applied. Then a Picard fixed point Theorem provides the existence. The proof of the uniqueness uses an additional argument first applied in the context of the KdV equation (cf. [**42**]). We have the following Theorem.

THEOREM 3. *Let* $s_1 > -1/3$ *and* $s_2 \geq 0$. *Then for any* $\phi \in H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$, *such that* $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbf{R}^2)$ *there exist a positive* $T = T(\|\phi\|_{H^{s_1,s_2}_{x,y}})$ *(* $\lim_{\rho\to 0} T(\rho) = \infty$*) and a unique solution* $u(t,x,y)$ *of the initial value problem (1) on the time interval* $I = [-T,T]$ *satisfying* $u \in C(I, H^{s_1,s_2}_{x,y}(\mathbf{R}^2)) \cap B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}(I)$.

Remark 1.2. Actually we shall solve an integral equation corresponding to (1) for any data $\phi \in H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$. The condition $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbf{R}^2)$ is imposed in order to insure that the solution of the integral equation solves (1) too in the distribution sense.

Now we state a result showing that in some cases the crucial bilinear estimate needed for the local well-posedness fails.

THEOREM 4. *The estimate*

$$\|\partial_x(uv)\|_{B^{b-1,b_1,b_2}_{s_1,0}} \lesssim \|u\|_{B^{b,b_1,b_2}_{s_1,0}}\|v\|_{B^{b,b_1,b_2}_{s_1,0}} \tag{5}$$

fails if one of the following cases hold

(a) $b + b_1 < 2/3$,

(b) $s_1 < 3(b+b_1) - b_2 - 2$,

(c) $s_1 < -1/3$.

The proof of Theorem 4 uses ideas due to C. E. Kenig, G. Ponce, L. Vega (cf. [**42**]). Combining the results of Theorem 2 and Theorem 4 we see that the Sobolev exponent $s_1 = -1/3$ is critical for the bilinear estimate (5). This observation strongly suggests that $s_1 = -1/3$ is also the critical exponent for the local well-posedness of the KP-II equation in the anisotropic Sobolev spaces $H^{s_1,0}_{x,y}(\mathbf{R}^2)$, when using a contraction argument in the Bourgain spaces related to the KP-II equation. Note that the exponent $s_1 = -1/2$ is suggested by the scaling argument. Hence similarly to the KdV equation, it seems that in the case of the KP-II equation the critical for the local well-posedness Sobolev exponent differs from the scaling one. The scaling exponent for KdV equation is $-3/2$. In [**42**] local well-posedness for data in $H^s(\mathbf{R})$, $s > -3/4$ is obtained. In fact, the flow map turns out to be real analytic. On the other hand in [**15**] it is proven that if the data-solution map is supposed to be of class C^3 from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$ then KdV equation is locally ill-posed for data in $H^s(\mathbf{R})$, $s < -3/4$. For the moment we are not able to prove a similar result for the KP-II equation.

Finally we remark that the existence of solitary wave solutions for the KP-I equation was used in [**49**] in order to prove that the KP-I equation is locally ill-posed in $H^{s_1,0}_{x,y}(\mathbf{R}^2)$, $s_1 < -1/2$. Unfortunately this approach does not work in the context of the KP-II equation. Namely, a use of Pohojaev type identities shows easily that there is no nontrivial localized solitary wave solution for the KP-II equation (cf. [**10**]).

Notations. We denote by $\hat{}$ or $\mathcal{F}$ the Fourier transform and by $\mathcal{F}^{-1}$ the inverse transform. $\|.\|_{L^p}$ denotes the norm in the Lebesgue space L^p. $A \sim B$ means that there exists a constant $c \geq 1$ such that $\frac{1}{c}|A| \leq |B| \leq c|A|$. For any positive A and B the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there exists a positive constant c such that $A \leq cB$ (resp. $A \geq cB$). The notation $a\pm$ means $a \pm \varepsilon$ for arbitrary small $\varepsilon > 0$. Constants are denoted by c and may change from line to line. By $\mathrm{mes}(A)$ we denote the measure of a set A.

The rest of the Chapter is organized as follows. Section 2 is devoted to the proof of the crucial bilinear estimate. We first state the Strichartz inequality for KP equations injected into the framework of Bourgain spaces and some simple calculus inequalities needed for the proof. The rest of the section is devoted to the proof of the integral representation of the bilinear estimate. In Section 3 we apply a Picard fixed point argument to an integral equation corresponding to (1). In Section 4 we give the examples of failure of the bilinear estimate.

2. Proof of Theorem 1

2.1. Preparation of the proof. Let $\zeta = (\xi, \eta)$, $\zeta_1 = (\xi_1, \eta_1)$ and

$$\sigma := \sigma(\tau, \zeta) = \tau - \xi^3 + \frac{\eta^2}{\xi}, \quad \sigma_1 := \sigma(\tau_1, \zeta_1), \quad \sigma_2 := \sigma(\tau - \tau_1, \zeta - \zeta_1),$$

$$\theta := \theta(\tau, \zeta) = \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{b_2}}, \quad \theta_1 := \theta(\tau_1, \zeta_1), \quad \theta_2 := \theta(\tau - \tau_1, \zeta - \zeta_1).$$

Now we state a version of the Strichartz inequality for the KP-II equation.

LEMMA 2.1. *Let $2 \leq q \leq 4$. Then for any $u \in L^2(\mathbf{R}^3)$ the following inequality holds*

$$\|\mathcal{F}^{-1}(\langle\sigma\rangle^{-b}|\hat{u}(\tau, \zeta)|)\|_{L^q} \lesssim \|u\|_{L^2}, \tag{6}$$

where $b = 2(1 - \frac{2}{q})(\frac{1}{2}+)$.

Proof. For any $\phi \in L^2(\mathbf{R}^2)$ the classical version of the Strichartz inequality for the KP equation (cf. [**48**], Proposition 2.3) yields

$$\|U(t)\phi\|_{L^4} \lesssim \|\phi\|_{L^2}, \tag{7}$$

where $U(t) = \exp(-t(\partial_x^3 + \partial_x^{-1}\partial_y^2))$. Once we have (7), Lemma 3.3 of [**23**] gives for any $u \in L^2(\mathbf{R}^3)$

$$\|\mathcal{F}^{-1}(\langle\sigma\rangle^{-\frac{1}{2}-}|\widehat{u}(\tau,\zeta)|)\|_{L^4} \lesssim \|u\|_{L^2}. \tag{8}$$

Interpolating between (8) and the Plancherel identity completes the proof of Lemma 2.1. Now we state a corollary of Lemma 2.1 which will be intensively used hereafter.

LEMMA 2.2. *Let $\alpha_1, \alpha_2, \alpha_3 \in [0, \frac{1}{2} + \varepsilon]$ and $\widehat{u}$, $\widehat{v}$, $\widehat{w}$ be positive. Then*

$$\int \frac{\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}(\tau,\zeta)}{\langle\sigma_1\rangle^{\alpha_1}\langle\sigma_2\rangle^{\alpha_2}\langle\sigma\rangle^{\alpha_3}} d\tau d\zeta d\tau_1 d\zeta_1 \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \tag{9}$$

provided $\alpha_1 + \alpha_2 + \alpha_3 \geq 1 + 2\varepsilon$.

Proof. We denote by I the left-hand side of (9). Clearly we can assume that $\alpha_1+\alpha_2+\alpha_3 = 1 + 2\varepsilon$. Then the Hölder inequality, the Plancherel identity and Lemma 2.1 yield

$$\begin{aligned} I &\leq \|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-\alpha_1}\widehat{u})\|_{L^{q_1}}\|\mathcal{F}^{-1}(\langle\sigma_2\rangle^{-\alpha_2}\widehat{v})\|_{L^{q_2}}\|\mathcal{F}^{-1}(\langle\sigma\rangle^{-\alpha}\widehat{w})\|_{L^{q_3}} \\ &\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \end{aligned}$$

provided $\alpha_j = 2(1 - \frac{2}{q_j})(\frac{1}{2} + \varepsilon)$, $j = 1,2,3$ and $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$. But the last condition is equivalent to $\alpha_1 + \alpha_2 + \alpha_3 = 1 + 2\varepsilon$ which completes the proof of Lemma 2.2. We shall also make use of the following calculus inequalities.

LEMMA 2.3. *For any $a \in \mathbf{R}$ the following inequalities hold*

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^{1\pm}\langle t-a\rangle^{1+}} \lesssim \frac{1}{\langle a\rangle^{1\pm}}, \tag{10}$$

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^{1+}|t-a|^{\frac{1}{2}}} \lesssim \frac{1}{\langle a\rangle^{\frac{1}{2}}}. \tag{11}$$

A duality argument shows that (2) is equivalent to

$$\int\int K(\tau,\zeta,\tau_1,\zeta_1)\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta \lesssim$$

$$\|u\|_{L^2}\|v\|_{L^2}, \|w\|_{L^2}, \tag{12}$$

where $\widehat{u}$, $\widehat{v}$, $\widehat{w}$ can assumed to be positive and

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{|\xi|\langle\theta\rangle\langle\xi\rangle^{s_1}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\theta_1\rangle\langle\theta_2\rangle} \frac{\langle\eta\rangle^{s_2}}{\langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2}}.$$

2.2. Proof of the integral representation of the bilinear estimate. We shall consider only the case $s_1 \in (-1/3, 0]$. Let $s = -s_1$. Moreover we assume $s_2 = 0$. The case $s_2 > 0$ can be treated in the same way, using that for $s_2 \geq 0$ one has

$$\langle\eta\rangle^{s_2} \lesssim \langle\eta_1\rangle^{s_2}\langle\eta - \eta_1\rangle^{s_2}.$$

Hence the kernel $K(\tau, \zeta, \tau_1, \zeta_1)$ in (12) becomes

$$K(\tau, \zeta, \tau_1, \zeta_1) = \frac{|\xi|\langle\theta\rangle\langle\xi\rangle^{-s}\langle\xi_1\rangle^{s}\langle\xi - \xi_1\rangle^{s}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\theta_1\rangle\langle\theta_2\rangle}.$$

To compensate the loss of a derivative in the nonlinear term we shall use the relation (cf. [14])

$$\sigma_1 + \sigma_2 - \sigma = 3\xi_1\xi(\xi - \xi_1) + \frac{(\xi_1\eta - \xi\eta_1)^2}{\xi_1\xi(\xi - \xi_1)}$$

and hence

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi_1\xi(\xi - \xi_1)|. \tag{13}$$

By symmetry arguments we can assume that $|\sigma_1| \geq |\sigma_2|$. Denote by J the left-hand side of (12). We consider several cases for $(\tau, \zeta, \tau_1, \zeta_1)$.

Case 1. $|\xi| \leq 24$, $|\xi_1| \leq 48$. Denote by J_1 the contribution of this region to J. In this case $\langle\theta\rangle \lesssim \langle\sigma\rangle^{b'}$ and hence

$$K(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{1}{\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}}.$$

Since $b > 1/2$ a use of Lemma 2.2 provides a bound for J_1.

Case 2. $|\xi| \leq 24$, $|\xi_1| \geq 48$, $|\sigma_1| \geq |\sigma|$. We denote by J_2 the contribution of this region to J. We have that $\langle\theta\rangle \lesssim \langle\sigma\rangle^{b_1}$ and hence

$$K(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{|\xi|\langle\xi_1\rangle^{2s}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}}.$$

In this case $|\sigma_1|$ dominates in (13). Therefore $|\xi|^s\langle\xi_1\rangle^{2s} \lesssim \langle\sigma_1\rangle^s$ and hence we obtain

$$K(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{|\xi|^{1-s}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b-s}\langle\sigma_2\rangle^{b}}.$$

Since $b' > b_1 + s$ a use of Lemma 2.2 provides a bound for J_2.

Case 3. $|\xi| \leq 24$, $|\xi_1| \geq 48$, $|\sigma| \geq |\sigma_1|$. We denote by J_3 the contribution of this region to J. We have that $\langle\theta\rangle \lesssim \langle\sigma\rangle^{b_1}$ and since in this case $|\sigma|$ dominates in (13), we obtain

$$K(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{|\xi|^{1-s}}{\langle\sigma\rangle^{b'-b_1-s}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}} \lesssim \frac{1}{\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}},$$

since $b' > b_1 + s$. Now Lemma 2.2 provides a bound for J_3.

Case 4. $|\xi| \geq 24$, $|\sigma| \geq |\sigma_1|$, $\min\{|\xi_1|, |\xi - \xi_1|\} \geq 1$.

Case 4.1. $|\xi| \le 100|\xi_1|$ and $|\xi| \le 100|\xi - \xi_1|$. Denote by J_{41} the contribution of this region to J. Due to (13) on the support of J_{41} we have

$$|\xi|^{1-s}|\xi_1|^s|\xi - \xi_1|^s \lesssim |\xi|^{(1+s)/3}|\xi_1|^{(1+s)/3}|\xi - \xi_1|^{(1+s)/3} \lesssim \langle\sigma\rangle^{(1+s)/3}.$$

- Let $\langle\sigma\rangle^{b_1} \le |\xi|^{b_2}$. Then

$$K(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{1}{\langle\sigma\rangle^{b'-(1+s)/3}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}.$$

Since $b' > (1+s)/3$ we can use Lemma 2.2.

- Let $\langle\sigma\rangle^{b_1} \ge |\xi|^{b_2}$. Then similarly as previously we obtain

$$K(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{|\xi|^{1-s-b_2}|\xi_1|^s|\xi - \xi_1|^s}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b} \lesssim \frac{1}{\langle\sigma\rangle^{b'-b_1-\max\{s,(1+s-b_2)/3\}}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}.$$

Since $b' - b_1 > (1 + s - b_2)/3$ and $b' - b_1 > s$ we can again use Lemma 2.2 to complete the proof of this case.

Case 4.2. $|\xi| \ge 100|\xi_1|$ or $|\xi| \ge 100|\xi - \xi_1|$. Denote by J_{42} the contribution of this region to J. On the support of J_{42} one has

$$|\xi|^{1-s}|\xi_1|^s|\xi - \xi_1|^s \lesssim |\xi|^{1+s}.$$

Using the Cauchy-Schwarz inequality, we obtain

$$J_{42} \lesssim \int I(\tau, \zeta) \left\{ \int |\widehat{u}(\tau_1, \zeta_1)\widehat{v}(\tau - \tau_1, \zeta - \zeta_1)|^2 d\tau_1 d\zeta_1 \right\}^{1/2} \widehat{w}(\tau, \zeta) d\tau d\zeta,$$

where

$$I(\tau, \zeta) = \frac{|\xi|^{1+s}\langle\theta\rangle}{\langle\sigma\rangle^{b'}} \left(\int_{|\sigma| \ge |\sigma_1|} \frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}\langle\theta_1\rangle^2\langle\theta_2\rangle^2} \right)^{1/2}.$$

We are going to show that $I(\tau, \zeta)$ is bounded for $|\xi| \ge 24$. Using (10) we obtain

$$I(\tau, \zeta) \lesssim \frac{|\xi|^{1+s}\langle\theta\rangle}{\langle\sigma\rangle^{b'}} \left(\int \frac{d\xi_1 d\eta_1}{\langle\sigma_1 + \sigma_2\rangle^{2b}} \right)^{1/2}.$$

We perform the change of variables $(\xi_1, \eta_1) \longmapsto (\nu, \mu)$

$$\nu = 3\xi\xi_1(\xi - \xi_1), \qquad \mu = \sigma_1 + \sigma_2.$$

Note that $\nu \in [-3|\sigma|, \min\{3/4\xi^3, 3|\sigma|\}]$, when $\xi \ge 0$ and $\nu \in [\max\{3/4\xi^3, -3|\sigma|\}, 3|\sigma|\}]$, when $\xi \le 0$. We assume that $\xi \ge 0$. If $\xi \le 0$, the arguments are the same. Further we have

$$d\xi_1 d\eta_1 = \frac{c|\nu|^{1/2} d\nu d\mu}{|\xi|^{3/2}(\frac{3}{4}\xi^3 - \nu)^{1/2}|\sigma + \nu - \mu|^{1/2}}.$$

Therefore we obtain using (11)

$$I(\tau,\zeta) \lesssim \frac{\langle\xi\rangle^{1/4+s}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\int_{-\infty}^{\infty}\frac{|\nu|^{1/2}d\mu d\nu}{(\frac{3}{4}\xi^3-\nu)^{1/2}|\sigma+\nu-\mu|^{1/2}\langle\mu\rangle^{2b}}\right\}^{1/2}$$

$$\lesssim \frac{\langle\xi\rangle^{1/4+s}\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{|\nu|^{1/2}d\nu}{(\frac{3}{4}\xi^3-\nu)^{1/2}\langle\sigma+\nu\rangle^{1/2}}\right\}^{1/2}.$$

• Let $|\sigma|^{b_1}\leq|\xi|^{b_2}$. Since in this case $\min\{|\xi_1|,|\xi-\xi_1|\}\leq\frac{1}{100}|\xi|$ we have that $|\nu|\leq\frac{3}{8}|\xi|^3$ and hence $|\nu|^{1/2}\leq(\frac{3}{4}\xi^3-\nu)^{1/2}\sim|\xi|^{3/2}$. Then we have

$$I(\tau,\zeta)\lesssim\frac{\langle\xi\rangle^{1/4+s}}{\langle\sigma\rangle^{1/4}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{|\nu|^{1-2b'}}{|\frac{3}{4}\xi^3-\nu|^{1/2}\langle\sigma+\nu\rangle^{1/2}}d\nu\right\}^{1/2}\lesssim\frac{\langle\xi\rangle^{1/4+s}}{\langle\xi\rangle^{3b'-3/4}}\leq \text{const},$$

since $b'>1/3+s/3$.

• Let $|\sigma|^{b_1}\geq|\xi|^{b_2}$. Then we have

$$I(\tau,\zeta)\lesssim\frac{\langle\xi\rangle^{1/4+s-b_2}}{\langle\sigma\rangle^{1/4}}\left\{\int_{-3|\sigma|}^{3|\sigma|}\frac{|\nu|^{1-2b'+2b_1}}{|\frac{3}{4}\xi^3-\nu|^{1/2}\langle\sigma+\nu\rangle^{1/2}}d\nu\right\}^{1/2}\lesssim\frac{\langle\xi\rangle^{1/4+s-b_2}}{\langle\xi\rangle^{3(b'-b_1-1/4)}}\leq \text{const},$$

since $b'-b_1>\max\{(1+s-b_2)/3,1/4\}$.

Hence $I(\tau,\zeta)$ is bounded for $|\xi|\geq 24$. Using the Cauchy-Schwarz inequality we obtain

$$J_{42}\lesssim\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 5. $|\xi|\geq 24,\quad |\sigma|\geq|\sigma_1|,\quad \min\{|\xi_1|,|\xi-\xi_1|\}\leq 1$. In this case we have

$$\langle\xi\rangle^{-s}\langle\xi_1\rangle^{s}\langle\xi-\xi_1\rangle^{s}\leq \text{const}.$$

and we can thus perform the arguments of Case 4.2 with $s=0$.

Case 6. $|\xi|\geq 24,\quad |\sigma_1|\geq|\sigma|,\quad |\xi_1|\leq 1$. Denote by J_6 the contribution of this region to J. On the support of J_6 one has

$$\langle\xi\rangle^{-s}\langle\xi_1\rangle^{s}\langle\xi-\xi_1\rangle^{s}\leq \text{const}.$$

Hence we assume that $s=0$. Consider the dyadic levels

$$A^{KM}=\{(\tau_1,\zeta_1):\quad |\xi_1|\sim M,\quad \langle\sigma_1\rangle\sim K\}.$$

Denote by J_6^{KM} the contribution of A^{KM} to J_6. Then

$$J_6\lesssim\sum_{K,M}J_6^{KM},$$

where the sum is taken over $K=2^k, k=0,1,2,\dots$ and $M=2^m, m=0,-1,-2,\dots$. We shall bound J_6^{KM} in two ways. First we shall use an argument similar to that of the Case 4.2 above. Then we shall estimate J_6^{KM} by the aid of Lemma 2.2. A suitable interpolation argument will provide the needed inequality.

The Cauchy-Schwarz inequality yields

$$J_6^{KM}\lesssim\int I^{KM}(\tau,\zeta)\left\{\int|\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)|^2d\tau_1d\zeta_1\right\}^{1/2}\widehat{w}(\tau,\zeta)d\tau d\zeta,$$

where

$$I^{KM}(\tau,\zeta)=\frac{|\xi|\langle\theta\rangle}{\langle\sigma\rangle^{b'}}\left(\int_{A^{KM}}\frac{d\tau_1 d\zeta_1}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}\langle\theta_1\rangle^2\langle\theta_2\rangle^2}\right)^{1/2}.$$

Similarly to Case 4.2 we perform the change of variables

$$\nu=3\xi\xi_1(\xi-\xi_1),\qquad \mu=\sigma_1+\sigma_2.$$

As in Case 4.2 we can suppose that $\xi\geq 0$. We have

$$|\nu|\leq 3|\xi|(|\xi|+1)2M\leq 9|\xi|^2M.$$

Hence using that $\langle\theta\rangle\lesssim\langle\sigma\rangle^{b'}$ we obtain

$$I^{KM}(\tau,\zeta)\lesssim\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{c|\xi|^{1/2}|\nu|^{1/2}d\tau_1 d\mu d\nu}{\langle\sigma_1\rangle^{2b}\langle\mu-\sigma_1\rangle^{2b}(\frac{3}{4}\xi^3-\nu)^{1/2}|\sigma+\nu-\mu|^{1/2}\langle\theta_1\rangle^2}\right\}^{1/2}.$$

Due to (13) we have that $|\nu|\leq|\sigma_1|$ and therefore

$$\frac{|\nu|^{1/2}}{\langle\theta_1\rangle^2}\lesssim\langle\sigma_1\rangle^{1/2-2b_1}\lesssim K^{1/2-2b_1}.$$

Hence

$$\begin{aligned}I^{KM}(\tau,\zeta)&\lesssim|\xi|^{1/4}K^{1/4-b_1}\left\{\int_{-\infty}^{\infty}\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\mu d\nu}{\langle\mu\rangle^{2b}(\frac{3}{4}\xi^3-\nu)^{1/2}|\sigma+\nu-\mu|^{1/2}}\right\}^{1/2}\\&\lesssim|\xi|^{1/4}K^{1/4-b_1}\left\{\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\nu}{(\frac{3}{4}\xi^3-\nu)^{1/2}|\sigma+\nu|^{1/2}}\right\}^{1/2}\end{aligned}$$

Since $|\xi|\geq 24$ we have that $|\frac{3}{4}\xi^3-\nu|\geq\frac{3}{8}\xi^3$ and therefore

$$\begin{aligned}I^{KM}(\tau,\zeta)&\lesssim\frac{K^{1/4-b_1}}{|\xi|^{1/2}}\left\{\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\nu}{|\sigma+\nu|^{1/2}}\right\}^{1/2}\\&\lesssim\frac{K^{1/4-b_1}}{|\xi|^{1/2}}M^{1/4}|\xi|^{1/2}\\&\lesssim K^{1/4-b_1}M^{1/4},\end{aligned}$$

where we used the elementary inequality

$$\int_{-9M|\xi|^2}^{9M|\xi|^2}\frac{d\nu}{|\sigma+\nu|^{1/2}}\lesssim M^{1/2}|\xi|.$$

Now we have by the Cauchy-Schwarz inequality

(14) $$J_6^{KM}\lesssim K^{1/4-b_1}M^{1/4}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Note that in (14) we gain a small factor $M^{1/4}$. When $b_1\leq 1/4$ the only use of $M^{1/4}$ provides a bound for J_6. But as it was mentioned in the introduction our goal is to lower as much as possible b_1 in order to weaken the restriction of Case 3 above. For that purpose we shall now estimate J_6^{KM} by the aid of Lemma 2.2.

• Let $|\sigma|^{b_1} \leq |\xi|^{b_2}$. We denote by J_{61}^{KM} the contribution of this region to J_6^{KM}. Since $|\xi_1| \leq 1$ and using (13) we arrive at

$$K(\tau,\zeta,\tau_1,\zeta_1) \lesssim \frac{|\xi|}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \lesssim \frac{M^{-1/2}K^{-\delta_1}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-\delta_1-1/2}\langle\sigma_2\rangle^{b}}.$$

Here $\delta_1 > 0$ is to be specified later. Due to Lemma 2.2 we obtain a bound

$$J_{61}^{KM} \lesssim \frac{1}{M^{1/2}K^{\delta_1}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \tag{15}$$

provided $2b + b' + b_1 - \delta_1 - 1/2 > 1$.

• Let $|\sigma|^{b_1} \geq |\xi|^{b_2}$. We denote by J_{62}^{KM} the restriction of J_6^{KM} on this region. We have

$$K(\tau,\zeta,\tau_1,\zeta_1) \lesssim \frac{|\xi|^{1-b_2}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \lesssim \frac{M^{-(1-b_2)/2}K^{-\delta_2}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1-\delta_2-(1-b_2)/2}\langle\sigma_2\rangle^{b}}.$$

Due to Lemma 2.2 we have a bound

$$J_{62}^{KM} \lesssim \frac{1}{M^{(1-b_2)/2}K^{\delta_2}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \tag{16}$$

provided $2b + b' - \delta_2 - (1 - b_2)/2 > 1$. The factors $K^{-\delta_i}$, $i = 1, 2$ in (15), (16) help us to lower b_1. Thus an interpolation between (14) and (15) (resp. (16)) with weights $\frac{2}{3}+$ and $\frac{1}{3}-$ (resp. $\frac{2-2b_2}{3-2b_2}+$ and $\frac{1}{3-2b_2}-$) respectively yields

$$J_{61}^{KM} \lesssim \frac{M^{\delta_1^*}}{K^{\delta_2^*}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2},$$

with $\delta_1^* > 0$ and $\delta_2^* > 0$ provided

$$(1/4 - b_1)\left(\frac{2}{3}+\right) - \delta_1\left(\frac{1}{3}-\right) < 0, \quad \text{i.e. } \delta_1 > 1/2 - 2b_1$$

$$(\text{resp. } J_{62}^{KM} \lesssim \frac{M^{\delta_1^*}}{K^{\delta_2^*}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2},$$

with $\delta_1^* > 0$ and $\delta_2^* > 0$ provided

$$(1/4 - b_1)\left(\frac{2-2b_2}{3-2b_2}+\right) - \delta_2\left(\frac{1}{3-2b_2}-\right) < 0, \quad \text{i.e. } \delta_2 > (1/2 - 2b_1)(1 - b_2)).$$

But for the applications of Lemma 2.2, δ_i should be such that

$$\delta_1 < 2b + b' + b_1 - 3/2, \quad \delta_2 < 2b + b' + b_2/2 - 3/2.$$

Since $b > 1/2$, it remains to note that we are able to choose properly δ_i provided b, b_1 and b_2 satisfy

$$b' + 3b_1 > 1, \quad b' + 2b_1 + b_2 - 2b_1b_2 > 1.$$

Summing over K and M yields

$$J_6 \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 7. $|\xi| \geq 24$, $|\sigma_1| \geq |\sigma|$, $|\xi| \leq 2|\xi_1|$. We denote by J_7 the contribution of this region to J. In this case we have

$$\frac{\langle\theta\rangle}{\langle\theta_1\rangle} \lesssim \frac{\langle\xi_1\rangle^{b_2}}{\langle\xi\rangle^{b_2}} \quad \text{and} \quad \frac{\langle\xi_1\rangle^{s}\langle\xi-\xi_1\rangle^{s}}{\langle\xi\rangle^{s}} \lesssim \frac{\langle\xi_1\rangle^{2s}}{\langle\xi\rangle^{s}}.$$

Using the above inequalities and taking into account that $|\sigma_1|$ dominate in this case we arrive at

$$K(\tau,\zeta,\tau_1,\zeta_1) \lesssim \frac{|\xi_1|^{1+s}}{\langle\sigma\rangle^{b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b}},$$

for $s \leq 1-b_2$. We use now a symmetry argument. The arguments of Case 4.2 can be used to estimate J_7. The point is that we get the same estimate just replacing (τ,ζ) with (τ_1,ζ_1).

Case 8. $|\xi| \geq 24$, $|\sigma_1| \geq |\sigma|$, $|\xi| \geq 2|\xi_1|$, $|\xi_1| \geq 1$. Denote by J_8 the contribution of this region to J. In this case we have that $|\xi| \leq 2|\xi-\xi_1|$ and

$$\frac{\langle\xi_1\rangle^{s}\langle\xi-\xi_1\rangle^{s}}{\langle\xi\rangle^{s}} \lesssim \langle\xi_1\rangle^{s}.$$

Hence

$$K(\tau,\zeta,\tau_1,\zeta_1) \lesssim \frac{|\xi||\xi_1|^{s}\langle\theta\rangle}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\theta_1\rangle}.$$

• Let $|\sigma|^{b_1} \leq |\xi|^{b_2}$. We denote by J_{81} the contribution of this region to J_8. Since $|\sigma_1|$ dominates in (13), we obtain

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\lesssim \frac{|\xi||\xi_1|^{s+b_2}}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \\ &\lesssim \frac{1}{\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{b+b_1-\max\{1/2,(s+b_2+1)/3\}}\langle\sigma_2\rangle^{b}}. \end{aligned}$$

Since $b'+b_1 > (s+b_2+1)/3$ and $b > 1/2$, we have that $2b+b'+b_1-(s+b_2+1)/3 > 1$. Since $b'+b_1 > 1/2$ and $b > 1/2$, we have that $2b+b'+b_1-1/2 > 1$. Therefore a use of Lemma 2.2 provides a bound for J_{81}.

• Let $|\sigma|^{b_1} \geq |\xi|^{b_2}$. Denote by J_{82} the contribution of this region to J_8 . Similarly to the estimate for J_{81} we have

$$\begin{aligned} K(\tau,\zeta,\tau_1,\zeta_1) &\lesssim \frac{|\xi|^{1-b_2}|\xi_1|^{s+b_2}}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1}\langle\sigma_2\rangle^{b}} \\ &\lesssim \frac{1}{\langle\sigma\rangle^{b'-b_1}\langle\sigma_1\rangle^{b+b_1-\max\{(1-b_2)/2,(s+1)/3\}}\langle\sigma_2\rangle^{b}}. \end{aligned}$$

Since $b' > (s+1)/3$ and $b > 1/2$, we have that $2b+b'-(s+1)/3 > 1$. Since $2b'+b_2 > 1$ and $b > 1/2$, we have that $2b+b'-(1-b_2)/2 > 1$. Therefore we can apply Lemma 2.2 in order to give a bound for J_{82}.

This completes the proof of Theorem 1.

Remark 2.1. One can summarize the restrictions on $s = -s_1$, b', b_1, b_2 of the proof of Theorem 1 in the following way:

$$s_1 > b_1 - b' \quad - \quad \text{when } |\xi| \text{ is small.}$$

$$s_1 > b_1 - b', \quad s_1 > 1 - 3b', \quad - \quad \text{when } |\sigma| \text{ dominates in (13)}$$
$$b' > b_1 + 1/4, \quad s_1 > 1 + 3b_1 - 3b' - b_2.$$

$$b' + 3b_1 > 1, \quad b' + 2b_1 + b_2 - 2b_1 b_2 > 1 \quad - \quad \text{when } |\sigma_1| \text{ dominates in (13) and } |\xi_1| \text{ is small.}$$

$$s_1 > b_2 - 1, \quad 2b' + b_2 > 1, \quad b' + b_1 > 1/2, \quad - \quad \text{when } |\sigma_1| \text{ dominates in (13) and } |\xi_1| \geq 1$$
$$s_1 > -3b' - 3b_1 + b_2 + 1, \quad s_1 > 1 - 3b'.$$

3. Proof of Theorem 3

We shall apply a Picard fixed point Theorem to the integral equation

$$u(t) = U(t)\phi - 1/2 \int_0^t U(t - t')\partial_x(u^2(t'))dt', \tag{17}$$

where $U(t) = \exp(-t(\partial_x^3 + \partial_x^{-1}\partial_y^2))$ is the unitary group describing the free evolution of the KP-II equation. The operator $U(t)$ could be regarded as a kernel operator

$$U(t)\phi = S_t \star \phi,$$

with a kernel $S_t(x, y)$ defined by the oscillatory integral

$$S_t(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(ix\xi + iy\eta + it(\xi^3 - \frac{\eta^2}{\xi})\right) d\xi d\eta.$$

Suppose that $u(t, x, y)$ be a solution of (17). Then $u(t, x, y)$ satisfies the KP-II only if an additional condition on the data ϕ is imposed. Actually $U(t)\phi(x, y)$ is well defined for any $\phi \in \mathcal{S}'(\mathbf{R}^2)$ but $U(t)\phi(x, y)$ has a well-defined time derivative when we can give a sense to the expression $|\xi|^{-1}\widehat{\phi}(\xi, \eta)$.

Let ψ be a cut-off function such that $\psi \in C_0^\infty(\mathbf{R})$, supp $\psi \subset [-2, 2]$, $\psi = 1$ over the interval $[-1, 1]$. Let $\psi_\delta(t) = \psi(t/\delta)$. Now we state the linear estimate.

THEOREM 3.1. *Let $b > 1/2$, $b' > 0$, $b + b' \leq 1$, $b_1 \geq 0$, $b_2 \geq 0$ and $s_1, s_2 \in \mathbf{R}$. Let the operator T_ϕ be defined by*

$$T_\phi(u) = U(t)\phi - 1/2 \int_0^t U(t - t')\partial_x(u^2(t'))dt'.$$

Then the following estimate holds

$$\|\psi T_\phi(u)\|_{B^{b,b_1,b_2}_{s_1,s_2}} \lesssim \left(\|\phi\|_{H^{s_1,s_2}_{x,y}} + \|uu_x\|_{B^{-b',b_1,b_2}_{s_1,s_2}}\right). \tag{18}$$

Proof. Let Y^{b,s_1,s_2} be the space equipped with the norm

$$\|u\|_{Y^{b,s_1,s_2}} = \left\| \left\langle \tau - \xi^3 + \frac{\eta^2}{\xi} \right\rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \widehat{u}(\tau,\xi,\eta) \right\|_{L^2_{\tau,\xi,\eta}} .$$

Then we can write the norm in $B^{b,b_1,b_2}_{s_1,s_2}$ as

$$\|u\|_{B^{b,b_1,b_2}_{s_1,s_2}} \sim \|u\|_{Y^{b,s_1,s_2}} + \|u\|_{Y^{b+b_1,s_1-b_2,s_2}} .$$

Now we state a theorem containing the linear estimates in the Bourgain space framework. These estimates are in fact one dimensional (with respect to time) and do not depend on the particular choice of the unitary group $U(t)$.

THEOREM 3.2. *Let $b + b' \leq 1$. Then for $s_1, s_2 \in \mathbf{R}$ we have*

$$\|\psi(t)U(t)\phi\|_{Y^{b,s_1,s_2}} \lesssim \|\phi\|_{H^{s_1,s_2}_{x,y}}, \tag{19}$$

$$\|\psi(t)\int_0^t U(t-t')F(t')dt'\|_{Y^{b,s_1,s_2}} \lesssim \|F\|_{Y^{-b',s_1,s_2}} . \tag{20}$$

Proof of Theorem 3.2. To prove (19) it suffices to notice that

$$\|u\|_{Y^{b,s_1,s_2}} = \|U(-t)u\|_{H^{b,s_1,s_2}_{t,x,y}},$$

where $H^{b,s_1,s_2}_{t,x,y}(\mathbf{R}^3)$ is a classical Sobolev space equipped with the norm

$$\|u\|_{H^{b,s_1,s_2}_{t,x,y}} = \|\langle \tau \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \widehat{u}(\tau,\xi,\eta)\|_{L^2_{\tau,\xi,\eta}} .$$

For the proof of (20) we refer to [**42**] (cf. also [**23**]).

We return to the proof of Theorem 3.1. Due to Theorem 3.2 we obtain

$$\begin{aligned} \|\psi T_\phi(u)\|_{B^{b,b_1,b_2}_{s_1,s_2}} &\sim \|\psi T_\phi(u)\|_{Y^{b,s_1,s_2}} + \|\psi T_\phi(u)\|_{Y^{b+b_1,s_1-b_2,s_2}} \\ &\lesssim \left(\|\phi\|_{H^{s_1,s_2}_{x,y}} + \|\phi\|_{H^{s_1-b_2,s_2}_{x,y}} + \|uu_x\|_{Y^{-b',s_1,s_2}} + \|uu_x\|_{Y^{-b'+b_1,s_1-b_2,s_2}} \right) \\ &\lesssim \left(\|\phi\|_{H^{s_1,s_2}_{x,y}} + \|uu_x\|_{B^{-b',b_1,b_2}_{s_1,s_2}} \right) . \end{aligned}$$

This completes the proof of Theorem 3.1.

Using Theorem 3.1 with $b = \frac{1}{2}+$, $b' = \frac{1}{2}-$, $b_1 = \frac{1}{6}+$, $b_2 = \frac{1}{3}$ and Theorem 2 we obtain for $s_1 > -1/3$ and $s_2 \geq 0$

$$\|\psi T_\phi(u)\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \lesssim \left(\|\phi\|_{H^{s_1,s_2}_{x,y}} + \|u\|^2_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \right) . \tag{21}$$

Similarly we can obtain

$$\|\psi(T_\phi(u) - T_\phi(v))\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \lesssim \|u-v\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} \|u+v\|_{B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}} . \tag{22}$$

Hence due to (21) and (22) there exists $c_0 > 0$ such that if $\|\phi\|_{H^{s_1,s_2}_{x,y}} \leq c_0$ then the map ψT_ϕ has a fixed point, which is a solution of the integral equation (17) in the time interval $[-1, 1]$. To prove local existence for arbitrary data in $H^{s_1,s_2}_{x,y}(\mathbf{R}^2)$ we shall use the scale change invariance of KP equations. As was noticed in the introduction if $u(t, x, y)$ is a solution of the KP-II equation with data $\phi(x, y)$ then so is

$$u_\lambda(t, x, y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y),$$

with data

$$\phi_\lambda(x, y) = \lambda^2 \phi(\lambda x, \lambda^2 y).$$

We have that

$$\|\phi_\lambda\|_{H^{s_1,s_2}_{x,y}} = O(\lambda^{\min\{0,s_1,2s_2,s_1+2s_2\}+1/2}), \text{ as } \lambda \to 0.$$

Since for $s_1 > -1/3$, $s_2 \geq 0$ we have $s_1 + 2s_2 + 1/2 > 0$, we can always assume that $\|\phi\|_{H^{s_1,s_2}_{x,y}}$ is small enough. To prove the uniqueness we need the following theorem.

THEOREM 3.3. *Let* $1/2 > b' > b'' > b_1 > 0$, $b_2 \in \mathbf{R}$. *Then there exists* $\theta > 0$ *such that for* $\delta \in (0, 1)$ *and* $s_1, s_2 \in \mathbf{R}$ *the following estimate holds*

$$\|\psi_\delta u\|_{B^{-b',b_1,b_2}_{s_1,s_2}} \lesssim \delta^\theta \|u\|_{B^{-b'',b_1,b_2}_{s_1,s_2}}.$$

Proof of Theorem 3.3. If $0 < b < a < 1/2$ and $\delta \in (0, 1)$ then there exists $\theta > 0$ such that

$$\|\psi_\delta u\|_{Y^{-a,s_1,s_2}} \lesssim \delta^\theta \|u\|_{Y^{-b,s_1,s_2}}. \tag{23}$$

The estimate (23) is proved in [42] in the context of the KdV equation (cf. inequality (3.29) in [42]). For the KP equation the proof is essentially the same. Now using (23) we arrive at

$$\begin{aligned}
\|\psi_\delta u\|_{B^{-b',b_1,b_2}_{s_1,s_2}} &\sim \|\psi_\delta u\|_{Y^{-b',s_1,s_2}} + \|\psi_\delta u\|_{Y^{-b'+b_1,s_1-1/4,s_2}} \\
&\lesssim \delta^{\theta_1} \|u\|_{Y^{-b'',s_1,s_2}} + \delta^{\theta_2} \|u\|_{Y^{-b''+b_1,s_1-1/4,s_2}} \\
&\lesssim \delta^\theta \|u\|_{B^{-b'',b_1,b_2}_{s_1,s_2}}.
\end{aligned}$$

This completes the proof of Theorem 3.3.

Let $u_1, u_2 \in B^{\frac{1}{2}+,\frac{1}{6}+,\frac{1}{3}}_{s_1,s_2}([-T, T])$ be two solutions of (17) in the time interval $I = [-T, T]$ and $\delta \in (0, T)$ to be specified later. Clearly

$$\psi_\delta(u_1 - u_2) = \frac{\psi_\delta}{2} \int_0^t U(t - t')\partial_x(u_1^2 - u_2^2)(t')dt'.$$

Let $I_1 = [-\delta, \delta]$. Then Theorem 2, Theorem 3.1 and Theorem 3.3 yield

$$\begin{aligned}
\|u_1 - u_2\|_{B^{\frac12+,\frac18+,\frac13}_{s_1,s_2}(I_1)} &\le c\|\psi_\delta^2(u_1-u_2)\|_{B^{\frac12+,\frac18+,\frac13}_{s_1,s_2}} \\
&\le c\|\psi_\delta^2\partial_x(u_1^2-u_2^2)\|_{B^{-\frac12+,\frac18+,\frac13}_{s_1,s_2}} \\
&\le c\delta^\theta\|\psi_\delta\partial_x(u_1^2-u_2^2)\|_{B^{-\frac12+,\frac18+,\frac13}_{s_1,s_2}} \\
&\le c\delta^\theta\|\psi_\delta(u_1-u_2)\|_{B^{\frac12+,\frac18+,\frac13}_{s_1,s_2}}\|u_1+u_2\|_{B^{\frac12+,\frac18+,\frac13}_{s_1,s_2}} \\
&\le c\delta^\theta\left(\|u_1-u_2\|_{B^{\frac12+,\frac18+,\frac13}_{s_1,s_2}(I_1)}+\right).
\end{aligned}$$

Here the constant c depends only on $\|u_1 + u_2\|_{B^{\frac12+,\frac18+,\frac13}_{s_1,s_2}}$. We chose δ such that $c\delta^\theta < 1/2$ to conclude that $u_1 = u_2$ on I_1. Now we iterate the last argument to prove the uniqueness. This completes the proof of Theorem 3.

4. Proof of Theorem 4

4.1. Proof of (a) and (b). We shall construct examples where the bilinear estimate which is crucial for local well-posedness fails, by considerations similar to [**42**], p.591 (cf. also [**15**], Section 6). Let χ_A be the characteristic function of the set A. The sets A and B are defined as follows

$$A = \{(\tau,\xi,\eta) : N \le \xi \le N + N^{-1/2},\ |\eta| \le N^{1/2} \text{ and } \left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \le 1\},$$

$$B = \{(\tau,\xi,\eta) : (-\tau,-\xi,-\eta) \in A\}.$$

Note that A and B are symmetric with respect to the origin, of $\mathbf{R}^3_{\tau,\xi,\eta}$. Let C be the parallelepiped centered at the origin of dimensions $cN^{3/2} \times N^{-2} \times N^{1/2}$ with the largest side pointing in the $(3N^2, 1, 0)$ direction of $\mathbf{R}^3_{\tau,\xi,\eta}$ and the shortest side pointing in the $(-\frac13 N^{-2}, 1, 0)$ direction of $\mathbf{R}^3_{\tau,\xi,\eta}$. The parallelepiped C has

$$\pm c\frac{N^{5/2} \pm 3}{2N\sqrt{9N^4+1}} \sim \pm N^{-1/2},$$

as ξ components of the vertices. Clearly

$$\|\chi_A\|_{L^2} \sim \|\chi_B\|_{L^2} \sim \|\chi_C\|_{L^2} \sim 1.$$

We remark that A includes the set

$$D = \left\{(\tau,\xi,\eta) : N \le \xi \le N + N^{-1/2},\ |\eta| \le \frac12 N^{1/2} \text{ and } |\tau - \xi^3| \le \frac12\right\}.$$

Actually if $(\tau,\xi,\eta)\in D$ then

$$\left|\tau-\xi^3+\frac{\eta^2}{\xi}\right|\le|\tau-\xi^3|+\frac{\eta^2}{\xi}\le\frac{1}{2}+\frac{1}{4}<1$$

and hence $(\tau,\xi,\eta)\in A$. On the other hand D includes the parallelepiped

$D'=\{(\tau,\xi,\eta)\ :\ |\eta|\le\frac{1}{2}N^{1/2}$, for fixed η the sections parallel to the (τ,ξ) plane are rectangles having a vertex (N^3,N,η) with dimensions $N^{-2}\times N^{3/2}$ and the longest side pointing in $(3N^2,1,0)$ direction of $\mathbf{R}^3_{\tau,\xi,\eta}\}$.

Similarly B includes the set D'' which is symmetric to D' with respect to the origin of $\mathbf{R}^3_{\tau,\xi,\eta}$. Since the volume of D' is a constant independent of N and the orientation of D, D'' and C in the space is the same, we obtain

$$(\chi_A\star\chi_B)(\tau,\xi,\eta)\ge(\chi_{D'}\star\chi_{D''})(\tau,\xi,\eta)\gtrsim\chi_C(\tau,\xi,\eta). \tag{24}$$

First we shall show the necessity of $b+b_1\ge 2/3$. Assume that $b+b_1<2/3$ and (5) holds. We shall consider only the case $b+b_1\ge 0$, since the case $b+b_1<0$ is easier. Recall that $\zeta=(\xi,\eta)$, $\zeta_1=(\xi_1,\eta_1)$ and

$$\sigma:=\sigma(\tau,\zeta)=\tau-\xi^3+\frac{\eta^2}{\xi},\quad \sigma_1:=\sigma(\tau_1,\zeta_1),\quad \sigma_2:=\sigma(\tau-\tau_1,\zeta-\zeta_1).$$

By polarization and duality (5) is equivalent to

$$\left\|\int\frac{1}{\langle\xi_1\rangle^{s_1}\langle\sigma_1\rangle^{b}\left\langle\frac{\langle\sigma_1\rangle^{b_1}}{\langle\xi_1\rangle^{b_2}}\right\rangle}\frac{|\xi|\langle\xi\rangle^{s_1}f(\tau,\xi,\eta)}{\langle\sigma\rangle^{1-b}}\left\langle\frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{b_2}}\right\rangle\frac{g(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)}{\langle\xi-\xi_1\rangle^{s_1}\langle\sigma_2\rangle^{b}\left\langle\frac{\langle\sigma_2\rangle^{b_1}}{\langle\xi-\xi_1\rangle^{b_2}}\right\rangle}d\tau d\zeta\right\|_{L^2_{\tau_1,\xi_1,\eta_1}}$$

$$\lesssim\|f\|_{L^2_{\tau,\xi,\eta}}\|g\|_{L^2_{\tau,\xi,\eta}}. \tag{25}$$

Take

$$f(\tau,\xi,\eta)=\chi_A(\tau,\xi,\eta),\quad g(\tau,\xi,\eta)=\chi_A(\tau,\xi,\eta)+\chi_B(\tau,\xi,\eta). \tag{26}$$

Note that on C one has $|\tau|\le c\left(1+N^2|\xi|\right)$. Consider the sets

$$S_1=C\cap\{(\tau,\xi,\eta)\ :\ \eta^2\ge N^2\xi^2,\ |\eta|\ge c\}$$

and

$$S_2=C\cap\{(\tau,\xi,\eta)\ :\ \eta^2<N^2\xi^2,\ |\eta|\ge c\}.$$

Clearly $S_1\cup S_2\subset C$. Further we have

$$\langle\sigma\rangle\lesssim\left(1+\frac{\eta^2}{|\xi|}\right),\quad\text{on } S_1$$

and

$$\langle\sigma\rangle\lesssim\left(1+N^2|\xi|\right),\quad\text{on } S_2.$$

Now substituting (26) into (25) and using (24) we obtain

$$\begin{aligned}
\text{const} &\gtrsim N\left(\int_{S_1} \frac{\chi_C(\tau,\xi,\eta)}{\langle\sigma\rangle^{2(b+b_1)}} d\tau d\xi d\eta\right)^{1/2} \\
&\quad + N\left(\int_{S_2} \frac{\chi_C(\tau,\xi,\eta)}{\langle\sigma\rangle^{2(b+b_1)}} d\tau d\xi d\eta\right)^{1/2} \\
&\gtrsim N\left(\int_c^{cN^{1/2}} \int_0^{c\eta/N} \left(\frac{\xi}{\eta^2}\right)^{2(b+b_1)} d\xi d\eta\right)^{1/2} \\
&\quad + N\left(\int_{cN^{-1}}^{cN^{-1/2}} \int_c^{cN\xi} (N^2\xi)^{-2(b+b_1)} d\eta d\xi\right)^{1/2} \\
&\sim N N^{-3(b+b_1)/2},
\end{aligned}$$

which proves the necessity for $b + b_1 \geq 2/3$.

Now we show that an argument similar as above provides the necessity for $s_1 \geq -1/2$. In the next section we shall obtain an improvement of this result. Note that (5) is equivalent to

$$\left\| \int \frac{|\xi|\langle\xi\rangle^{s_1}}{\langle\sigma\rangle^{1-b}} \left\langle \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{b_2}} \right\rangle \frac{f(\tau_1,\xi_1,\eta_1)}{\langle\xi_1\rangle^{s_1}\langle\sigma_1\rangle^{b}\left\langle\frac{\langle\sigma_1\rangle^{b_1}}{\langle\xi_1\rangle^{b_2}}\right\rangle} \frac{g(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)}{\langle\xi-\xi_1\rangle^{s_1}\langle\sigma_2\rangle^{b}\left\langle\frac{\langle\sigma_2\rangle^{b_1}}{\langle\xi-\xi_1\rangle^{b_2}}\right\rangle} d\tau_1 d\xi_1 d\eta_1 \right\|_{L^2_{\tau,\xi,\eta}}$$

$$\lesssim \|f\|_{L^2_{\tau,\xi,\eta}} \|g\|_{L^2_{\tau,\xi,\eta}}. \tag{27}$$

We shall take f and g as in (26). Then arguments similar as above show that substituting (26) into (27) yields

$$\begin{aligned}
\text{const} &\gtrsim N^{-2s_1}\left(\int_{S_1} \frac{|\xi|^2\chi_C(\tau,\xi,\eta)}{\langle\sigma\rangle^{2(1-b-b_1)}} d\tau d\xi d\eta\right)^{1/2} \\
&\quad + N^{-2s_1}\left(\int_{S_2} \frac{|\xi|^2\chi_C(\tau,\xi,\eta)}{\langle\sigma\rangle^{2(1-b-b_1)}} d\tau d\xi d\eta\right)^{1/2} \\
&\gtrsim N^{-2s_1}\left(\int_c^{cN^{1/2}} \int_0^{c\eta/N} \xi^2 \left(\frac{\xi}{\eta^2}\right)^{2(1-b-b_1)} d\xi d\eta\right)^{1/2} \\
&\quad + N^{-2s_1}\left(\int_{cN^{-1}}^{cN^{-1/2}} \int_c^{cN\xi} \xi^2 (N^2\xi)^{2(b+b_1-1)} d\eta d\xi\right)^{1/2} \\
&\sim N^{-2s_1} N^{3(b+b_1)/2-2},
\end{aligned}$$

which proves the necessity for $s_1 \geq 3(b+b_1)/4 - 1 \geq -1/2$.

Now we show the necessity of $s_1 \geq 3(b+b_1) - b_2 - 2$. A duality argument shows that (5) is equivalent to

$$\int \frac{|\xi|\langle\xi\rangle^{s_1} g(\tau,\xi,\eta)}{\langle\sigma\rangle^{1-b}} \left\langle \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{b_2}} \right\rangle \frac{f(\tau_1,\xi_1,\eta_1)}{\langle\xi_1\rangle^{s_1}\langle\sigma_1\rangle^{b}\left\langle\frac{\langle\sigma_1\rangle^{b_1}}{\langle\xi_1\rangle^{b_2}}\right\rangle} \frac{h(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)}{\langle\xi-\xi_1\rangle^{s_1}\langle\sigma_2\rangle^{b}\left\langle\frac{\langle\sigma_2\rangle^{b_1}}{\langle\xi-\xi_1\rangle^{b_2}}\right\rangle} d\tau d\tau_1 d\xi d\xi_1 d\eta d\eta_1$$

$$\lesssim \|g\|_{L^2_{\tau,\xi,\eta}} \|f\|_{L^2_{\tau,\xi,\eta}} \|h\|_{L^2_{\tau,\xi,\eta}}. \tag{28}$$

Let the set A' be defined as follows:

$$A' = \{(\tau + 2N^3, \xi + 2N, \eta) : (\tau,\xi) \in C,\ |\eta| \leq \sqrt{N}/2\}.$$

Clearly $\mathrm{mes}(A') \sim 1$. Set

$$g(\tau,\xi,\eta) = \chi_{A'}(\tau,\xi,\eta), \quad f(\tau,\xi,\eta) = h(\tau,\xi,\eta) = \chi_A(\tau,\xi,\eta).$$

An argument analogous to [42] p. 591 gives

$$(f \star h)(\tau,\xi,\eta) \gtrsim (\chi_A \star \chi_A)(\tau,\xi,\eta) \gtrsim \chi_{A'}(\tau,\xi,\eta). \tag{29}$$

On the set A', we have $|\xi| \sim N$ and $\left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \lesssim N^3$. Clearly

$$\|g\|_{L^2} \sim \|f\|_{L^2} \sim \|h\|_{L^2} \sim 1.$$

On the set A, we have $|\xi| \sim N$ and $\left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \leq \mathrm{const}$. Substituting (29) into (28), we obtain

$$\begin{aligned} \mathrm{const} &\gtrsim N^{-2s_1} \int_{A'} \frac{|\xi|\langle\xi\rangle^{s_1}\chi_{A'}(\tau,\xi,\eta)}{\langle\sigma\rangle^{1-b}} \left\langle \frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{b_2}} \right\rangle d\tau d\xi d\eta \\ &\gtrsim N^{-s_1+1+3(b+b_1-1)-b_2} \int_{A'} d\tau d\xi d\eta \\ &\sim N^{-s_1+3(b+b_1)-b_2-2}, \end{aligned}$$

which shows the necessity of the condition $s_1 \geq 3(b+b_1) - b_2 - 2$. This completes the proof of Theorem 4, (a) and (b).

4.2. Proof of (c). The aim of this Section is to give a proof of (c). The essential ingredient of the proof is the following Lemma.

LEMMA 4.1. *Let $N \gg 1$ be a large number. Then there exist sets A, B, W in $\mathbf{R}^3_{\tau,\xi,\eta}$ such that*

$$mes(A) \sim mes(B) \sim mes(W) \sim O(1),$$

$$(\tau,\xi,\eta) \in \ supp\ (A \cup B) \implies \left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \lesssim 1, \quad |\xi| \sim N,$$

$$(\tau,\xi,\eta) \in \ supp\ W \implies \left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \lesssim N^2, \quad |\xi| \sim O(1)$$

and

$$\chi_A \star \chi_B \gtrsim \chi_W.$$

Once we prove Lemma 4.1 the proof of Theorem 4 (c) is straightforward. Recall that (5) is equivalent to

$$\left\| \int \frac{|\xi|\langle\xi\rangle^{s_1}}{\langle\sigma\rangle^{1-b}} \left\langle\frac{\langle\sigma\rangle^{b_1}}{\langle\xi\rangle^{b_2}}\right\rangle \frac{f(\tau_1,\xi_1,\eta_1)}{\langle\xi_1\rangle^{s_1}\langle\sigma_1\rangle^{b}\left\langle\frac{\langle\sigma_1\rangle^{b_1}}{\langle\xi_1\rangle^{b_2}}\right\rangle} \frac{g(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)}{\langle\xi-\xi_1\rangle^{s_1}\langle\sigma_2\rangle^{b}\left\langle\frac{\langle\sigma_2\rangle^{b_1}}{\langle\xi-\xi_1\rangle^{b_2}}\right\rangle} d\tau_1 d\xi_1 d\eta_1 \right\|_{L^2_{\tau,\xi,\eta}}$$

$$\lesssim \|f\|_{L^2_{\tau,\xi,\eta}} \|g\|_{L^2_{\tau,\xi,\eta}}. \tag{30}$$

Now take $f = \chi_A$, $g = \chi_B$, where the sets A and B are the same as in Lemma 4.1. Then clearly

$$O(1) \gtrsim N^{-2s_1} N^{2(b+b_1-1)},$$

which shows the necessity of

$$s_1 \geq b + b_1 - 1 \geq -1/3$$

due to (a). Hence it remains to perform the proof of Lemma 4.1.

Remark 4.1. We note that the example providing the necessity of $s_1 \geq -1/2$, constructed in the previous section uses $f = \chi_A$, $g = \chi_B$ in (30), with $\chi_A \star \chi_B \gtrsim \chi_C$. The sets A, B, C in $\mathbf{R}^3_{\tau,\xi,\eta}$ have measures $O(1)$, the symbol $\tau - \xi^3 + \frac{\eta^2}{\xi}$ of the KP-II equation is bounded on A and B, $|\xi| \sim N$ on A and B, $\left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \lesssim N^{3/2}$ and $|\xi| \sim N^{-1/2}$ on C. In this section we improve this argument by considering sets A and B with the same properties as in the example providing the necessity of $s_1 \geq -1/2$, but the set C will be such that $|\xi| \sim O(1)$ and $\left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \lesssim N^2$ on C.

Proof of Lemma 4.1. Let $\gamma = \frac{1}{2\sqrt{N}}$ and the set $A_{\beta,\eta} \subset \mathbf{R}^2$ be defined as follows

$$A_{\beta,\eta} := \{(\tau,\xi) \;:\; N+\beta \leq \xi \leq N+\beta+\gamma,\; \left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \leq 2\},$$

where $0 \leq \beta \leq 1$ and $0 \leq \eta \leq N$. The set $A_{\beta,\eta}$ includes the rectangle $R^1_{\beta,\eta}$ of dimensions $cN^{-2} \times N^{3/2}$ with the longest side pointing in the

$$\left(3(N+\beta)^2 + \frac{\eta^2}{(N+\beta)^2}, 1\right)$$

direction of $\mathbf{R}^2_{\tau,\xi}$. The intersections of the planes $\{\eta = \text{const}\}$ with the set A in Lemma 4.1 will be chosen as subsets of $R^1_{\beta,\eta}$. The coordinates of the vertices of $R^1_{\beta,\eta}$ in (τ,ξ) plane are the following:

$$\left((N+\beta)^3 - \tfrac{\eta^2}{N+\beta}, N+\beta\right),$$

$$\left((N+\beta)^3 - \tfrac{\eta^2}{N+\beta} - \tau^1_\beta, N+\beta+\xi^1_\beta\right),$$

$$\left((N+\beta+\gamma)^3 - \tfrac{\eta^2}{N+\beta+\gamma} - 1, N+\beta+\gamma\right),$$

$$\left((N+\beta+\gamma)^3 - \tfrac{\eta^2}{N+\beta+\gamma} + \tau_\beta^1 - 1, N+\beta+\gamma-\xi_\beta^1\right),$$

where

$$\tau_\beta^1 = \tau(N+\beta,\eta,\gamma) := \frac{1-\gamma^3-3\gamma^2(N+\beta)+\frac{\eta^2\gamma^2}{(N+\beta)^2(N+\beta+\gamma)}}{1+\left(3(N+\beta)^2+\frac{\eta^2}{(N+\beta)^2}\right)^2},$$

$$\xi_\beta^1 = \xi(N+\beta,\eta,\gamma) := \left(3(N+\beta)^2+\tfrac{\eta^2}{(N+\beta)^2}\right)\tau(N+\beta,\eta,\gamma).$$

Note that $\tau_\beta^1 \sim N^{-4}$ and $\xi_\beta^1 \sim N^{-2}$ (more precisely $\xi_\beta^1 = \frac{1}{12}N^{-2}+o(N^{-2})$). Further we note that the rectangle $R^1_{\beta,\eta}$ is situated between the following lines in the (τ,ξ) plane:

$$\tau = \left(3(N+\beta)^2+\tfrac{\eta^2}{(N+\beta)^2}\right)(\xi-N-\beta)+(N+\beta)^3-\tfrac{\eta^2}{N+\beta},$$

$$\tau = \left(3(N+\beta)^2+\tfrac{\eta^2}{(N+\beta)^2}\right)(\xi-N-\beta-\xi_\beta^1)+(N+\beta)^3-\tfrac{\eta^2}{N+\beta}-\tau_\beta^1.$$

Now we define the set $B_{\beta,\eta,K}$ as follows

$$B_{\beta,\eta,K} := \{(\tau,\xi) : -N+\beta-1 \le \xi \le -N+\beta-1+\gamma,\ \left|\tau-\xi^3+\frac{(\eta-K)^2}{\xi}\right| \le 2\}.$$

For $0 \le \beta \le 1$, $K = O(N)$ and $0 \le \eta \le N$, the set $B_{\beta,\eta,K}$ includes the rectangle $R^2_{\beta,\eta,K}$ of dimensions $cN^{-2} \times N^{3/2}$ with the longest side pointing in

$$\left(3(N+1-\beta-\gamma)^2+\frac{(\eta-K)^2}{(N+1-\beta-\gamma)^2}, 1\right)$$

direction of (τ,ξ) plane. The intersections of the planes $\{\eta = \text{const}\}$ with the set B in Lemma 4.1 will be chosen as subsets of $R^2_{\beta,\eta,K}$. The vertices of $R^2_{\beta,\eta,K}$ have the following coordinates:

$$\left(-(N+1-\beta-\gamma)^3+\tfrac{(\eta-K)^2}{N+1-\beta-\gamma}, -N-1+\beta+\gamma\right),$$

$$\left(-(N+1-\beta-\gamma)^3+\tfrac{(\eta-K)^2}{N+1-\beta-\gamma}+\tau_\beta^2, -N-1+\beta+\gamma-\xi_\beta^2\right),$$

$$\left(-(N+1-\beta)^3+\tfrac{(\eta-K)^2}{N+1-\beta}+1, -N-1+\beta\right),$$

$$\left(-(N+1-\beta)^3+\tfrac{(\eta-K)^2}{N+1-\beta}+1-\tau_\beta^2, -N-1+\beta+\xi_\beta^2\right),$$

where

$$\tau_\beta^2 = \tau(-N-1+\beta+\gamma,\eta-K,-\gamma), \quad \xi_\beta^2 = \xi(-N-1+\beta+\gamma,\eta-K,-\gamma).$$

Note that $\xi_\beta^2 = \frac{7}{12}N^{-2}+o(N^{-2})$. The rectangle $R^2_{\beta,\eta,K}$ is situated between the lines:

$$\begin{aligned}\tau \;=\; & \left(3(N+1-\beta-\gamma)^2+\tfrac{(\eta-K)^2}{(N+1-\beta-\gamma)^2}\right)(\xi+N+1-\beta-\gamma)- \\ & (N+1-\beta-\gamma)^3+\tfrac{(\eta-K)^2}{N+1-\beta-\gamma},\end{aligned}$$

$$\tau = \left(3(N+1-\beta-\gamma)^2 + \tfrac{(\eta-K)^2}{(N+1-\beta-\gamma)^2}\right)(\xi + N + 1 - \beta - \gamma + \xi_\beta^2) - (N+1-\beta-\gamma)^3 + \tfrac{(\eta-K)^2}{N+1-\beta-\gamma} + \tau_\beta^2$$

in the (τ, ξ) plane. For fixed η, the set W in Lemma 4.1 will be the intersection of suitable parallel translations of the rectangles $R^1_{\beta,\eta}$ and $R^2_{\beta,\eta,K}$. The relation between η and β will be chosen later. Now we define the parallelogram $R_{\beta,\eta,K}$ in $\mathbf{R}^2_{\tau,\xi}$ as the intersection of suitable translations of $R^1_{\beta,\eta+\eta_0}$ and $R^2_{\beta,\eta-\eta_0,K}$ with a proper choice of η_0. More precisely we set

$$R_{\beta,\eta,K} = \left\{R^1_{\beta,\eta+\eta_0} - \{\tau_0, \xi_0\}\right\} \cap \left\{R^2_{\beta,\eta-\eta_0,K} + \{\tau_0, \xi_0\}\right\},$$

where

$$\tau_0 := N^3 + \frac{3N^2}{2} + \frac{3N}{2} - \frac{3\sqrt{N}}{4} + \frac{5}{8} - \frac{K^2}{2N} + \frac{K^2}{2N^2}, \quad \xi_0 := N + \frac{1}{2}, \quad \eta_0 := \frac{K}{2}.$$

The parallelogram $R_{\beta,\eta,K}$ is determined by the following lines in (τ, ξ) plane:

$$(31)\qquad \left\{\ \tau = \left(3(N+\beta)^2 + \tfrac{(\eta+\frac{K}{2})^2}{(N+\beta)^2}\right)(\xi + \tfrac{1}{2} - \beta) + (N+\beta)^3 - \tfrac{(\eta+\frac{K}{2})^2}{N+\beta} - \tau_0,\right.$$

$$(32)\qquad \left\{\begin{array}{rcl} \tau &=& \left(3(N+\beta)^2 + \tfrac{(\eta+\frac{K}{2})^2}{(N+\beta)^2}\right)(\xi + \tfrac{1}{2} - \beta - \xi_\beta^3) \\ && +(N+\beta)^3 - \tfrac{(\eta+\frac{K}{2})^2}{N+\beta} - \tau_0 - \tau_\beta^3, \end{array}\right.$$

$$(33)\qquad \left\{\begin{array}{rcl} \tau &=& \left(3(N+1-\beta-\gamma)^2 + \tfrac{(\eta-\frac{K}{2})^2}{(N+1-\beta-\gamma)^2}\right)(\xi + \tfrac{1}{2} - \beta - \gamma) \\ && -(N+1-\beta-\gamma)^3 + \tfrac{(\eta-\frac{K}{2})^2}{N+1-\beta-\gamma} + \tau_0, \end{array}\right.$$

$$(34)\qquad \left\{\begin{array}{rcl} \tau &=& \left(3(N+1-\beta-\gamma)^2 + \tfrac{(\eta-\frac{K}{2})^2}{(N+1-\beta-\gamma)^2}\right)(\xi + \tfrac{1}{2} - \beta - \gamma + \xi_\beta^4) \\ && -(N+1-\beta-\gamma)^3 + \tfrac{(\eta-\frac{K}{2})^2}{N+1-\beta-\gamma} + \tau_0 + \tau_\beta^4, \end{array}\right.$$

where

$$\xi_\beta^3 = \xi(N+\beta, \eta + \frac{K}{2}, \gamma), \quad \xi_\beta^4 = \xi(-N-1+\beta+\gamma, \eta - \frac{K}{2}, -\gamma),$$
$$\tau_\beta^3 = \tau(N+\beta, \eta + \frac{K}{2}, \gamma), \quad \tau_\beta^4 = \tau(-N-1+\beta+\gamma, \eta - \frac{K}{2}, -\gamma).$$

In order to compute the ξ coordinates of the vertices of $R_{\beta,\eta,K}$ we need the following relations

$$(35)\quad (N+\beta)^3 + (N+1-\beta-\gamma)^3 = 2N^3 + 3N^2 - \frac{3}{2}N^{3/2} + 3(1+2\beta^2-2\beta)N + 3(\beta-1)\sqrt{N} + 1 + 3\left(\beta - \frac{1}{2}\right)^2 + O\left(\frac{1}{\sqrt{N}}\right),$$

$$(36)\qquad 3\gamma(N+1-\beta-\gamma)^2 = \frac{3}{2}N^{3/2} + 3(1-\beta)\sqrt{N} - \frac{3}{2} + O\left(\frac{1}{\sqrt{N}}\right),$$

$$\frac{(\eta+\frac{K}{2})^2}{N+\beta} = \left(\eta+\frac{K}{2}\right)^2\left(\frac{1}{N}-\frac{\beta}{N^2}\right)+O\left(\frac{1}{N}\right), \tag{37}$$

$$\frac{(\eta-\frac{K}{2})^2}{N+1-\beta-\gamma} = \left(\eta-\frac{K}{2}\right)^2\left(\frac{1}{N}-\frac{1-\beta}{N^2}\right)+O\left(\frac{1}{\sqrt{N}}\right), \tag{38}$$

$$\xi_\beta^3\left(3(N+\beta)^2+\frac{(\eta+\frac{K}{2})^2}{(N+\beta)^2}\right)+\tau_\beta^3 = \frac{1}{4}+O\left(\frac{1}{N}\right), \tag{39}$$

$$\xi_\beta^4\left(3(N+1-\beta-\gamma)^2+\frac{(\eta-\frac{K}{2})^2}{(N+1-\beta-\gamma)^2}\right)+\tau_\beta^4 = \frac{7}{4}+O\left(\frac{1}{N}\right). \tag{40}$$

Now using (35), (36), (37), (38), (39), (40) and the definition of τ_0, ξ_0 and η_0, we arrive at the following relations for the ξ coordinates of the intersection points of the lines (31) and (32) with the lines (33) and (34):

The intersection of (31) and (33):

$$\begin{aligned} d_\beta(\xi+\tfrac{1}{2}-\beta) &= \tfrac{3\sqrt{N}}{2}+6(\beta^2-\beta+\tfrac{K^2}{12N^2}-\tfrac{\eta^2}{3N^2})N-\tfrac{7}{4} \\ &\quad +3\left(\beta-\tfrac{1}{2}\right)^2+\tfrac{\eta^2}{N^2}+\tfrac{(2\beta-1)K\eta}{N^2}-\tfrac{3K^2}{4N^2}+O\left(\tfrac{1}{\sqrt{N}}\right), \end{aligned}$$

The intersection of (31) and (34):

$$\begin{aligned} d_\beta(\xi+\tfrac{1}{2}-\beta) &= \tfrac{3\sqrt{N}}{2}+6(\beta^2-\beta+\tfrac{K^2}{12N^2}-\tfrac{\eta^2}{3N^2})N-\tfrac{7}{4} \\ &\quad +3\left(\beta-\tfrac{1}{2}\right)^2+\tfrac{\eta^2}{N^2}+\tfrac{(2\beta-1)K\eta}{N^2}-\tfrac{3K^2}{4N^2}+\tfrac{7}{4}+O\left(\tfrac{1}{\sqrt{N}}\right), \end{aligned}$$

The intersection of (32) and (33):

$$\begin{aligned} d_\beta(\xi+\tfrac{1}{2}-\beta) &= \tfrac{3\sqrt{N}}{2}+6(\beta^2-\beta+\tfrac{K^2}{12N^2}-\tfrac{\eta^2}{3N^2})N-\tfrac{7}{4} \\ &\quad +3\left(\beta-\tfrac{1}{2}\right)^2+\tfrac{\eta^2}{N^2}+\tfrac{(2\beta-1)K\eta}{N^2}-\tfrac{3K^2}{4N^2}-\tfrac{1}{4}+O\left(\tfrac{1}{\sqrt{N}}\right), \end{aligned}$$

The intersection of (32) and (34):

$$\begin{aligned} d_\beta(\xi+\tfrac{1}{2}-\beta) &= \tfrac{3\sqrt{N}}{2}+6(\beta^2-\beta+\tfrac{K^2}{12N^2}-\tfrac{\eta^2}{3N^2})N-\tfrac{7}{4} \\ &\quad +3\left(\beta-\tfrac{1}{2}\right)^2+\tfrac{\eta^2}{N^2}+\tfrac{(2\beta-1)K\eta}{N^2}-\tfrac{3K^2}{4N^2}+\tfrac{7}{4}-\tfrac{1}{4}+O\left(\tfrac{1}{\sqrt{N}}\right), \end{aligned}$$

where

$$d_\beta = 6(1-2\beta)N-3\sqrt{N}+O(1).$$

Now we set

$$K = \sqrt{3}N(1-\gamma) \tag{41}$$

and choose the relation between η and β to be

$$\eta = \sqrt{3}N\left(\beta-\frac{1-\gamma}{2}\right) \tag{42}$$

in order to cancel the terms of order N. Note that the relation between η and β is linear. For the parameter β we suppose $0 \le \beta < 1/10$. Denote by R_β the parallelogram $R_{\beta,\eta,K}$, where η and K are obtained by (41) and (42). Now we have the following formulas for the vertices of R_β:

The intersection point between (31) and (33):

$$\begin{cases} \xi &= -\frac{1}{2}+\beta+\frac{1}{d_\beta}\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+O\left(\frac{1}{\sqrt{N}}\right)\right), \\ \tau &= \frac{3}{d_\beta}\left((N+\beta)^2+\frac{N^2\beta^2}{(N+\beta)^2}\right)\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+O\left(\frac{1}{\sqrt{N}}\right)\right) \\ & \quad +(N+\beta)^3-\frac{3N^2\beta^2}{N+\beta}-\tau_0, \end{cases}$$

The intersection point between (31) and (34):

$$\begin{cases} \xi &= -\frac{1}{2}+\beta+\frac{1}{d_\beta}\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+\frac{7}{4}+O\left(\frac{1}{\sqrt{N}}\right)\right), \\ \tau &= \frac{3}{d_\beta}\left((N+\beta)^2+\frac{N^2\beta^2}{(N+\beta)^2}\right)\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+\frac{7}{4}+O\left(\frac{1}{\sqrt{N}}\right)\right) \\ & \quad +(N+\beta)^3-\frac{3N^2\beta^2}{N+\beta}-\tau_0, \end{cases}$$

The intersection point between (32) and (33):

$$\begin{cases} \xi &= -\frac{1}{2}+\beta+\frac{1}{d_\beta}\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2-\frac{1}{4}+O\left(\frac{1}{\sqrt{N}}\right)\right), \\ \tau &= \frac{3}{d_\beta}\left((N+\beta)^2+\frac{N^2\beta^2}{(N+\beta)^2}\right)\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2-\frac{1}{4}+O\left(\frac{1}{\sqrt{N}}\right)\right) \\ & \quad +(N+\beta)^3-\frac{3N^2\beta^2}{N+\beta}-\tau_0-\frac{1}{4}+O\left(\frac{1}{N}\right), \end{cases}$$

The intersection point between (32) and (34):

$$\begin{cases} \xi &= -\frac{1}{2}+\beta+\frac{1}{d_\beta}\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+\frac{3}{2}+O\left(\frac{1}{\sqrt{N}}\right)\right), \\ \tau &= \frac{3}{d_\beta}\left((N+\beta)^2+\frac{N^2\beta^2}{(N+\beta)^2}\right)\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+\frac{3}{2}+O\left(\frac{1}{\sqrt{N}}\right)\right) \\ & \quad +(N+\beta)^3-\frac{3N^2\beta^2}{N+\beta}-\tau_0-\frac{1}{4}+O\left(\frac{1}{N}\right). \end{cases}$$

Finally the center of R_β has the following coordinates

$$\begin{cases} \xi &= -\frac{1}{2}+\beta+\frac{1}{d_\beta}\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+\frac{3}{4}+O\left(\frac{1}{\sqrt{N}}\right)\right), \\ \tau &= \frac{3}{d_\beta}\left((N+\beta)^2+\frac{N^2\beta^2}{(N+\beta)^2}\right)\left(\frac{3(1-2\beta)\sqrt{N}}{2}-4+12\left(\beta-\frac{1}{2}\right)^2+\frac{3}{4}+O\left(\frac{1}{\sqrt{N}}\right)\right) \\ & \quad +(N+\beta)^3-\frac{3N^2\beta^2}{N+\beta}-\tau_0-\frac{1}{8}+O\left(\frac{1}{N}\right). \end{cases}$$

When we shift β from 0 to $\frac{1}{100}$, the shift of d_β is $O(N)$. Recall that

$$d_\beta = 6(1-2\beta)N - 3\sqrt{N} + O(1).$$

But the difference with respect to ξ of the intersection points between (31)-(33) and (32)-(33) ((31)-(34) and (32)-(34)) is $O(\frac{1}{N})$. Hence we can assume

$$d_\beta = 6(1-2\beta)N.$$

Define the set V as

$$V = \left\{(\tau,\xi,\eta)\ :\ (\tau,\xi)\in R_\beta,\ \eta=\sqrt{3}N\left(\beta-\frac{1-\gamma}{2}\right),\ 0\le\beta\le\frac{1}{100}\right\}.$$

The projection of R_β on the ξ axis is with measure $O(N^{-1})$. The distance between the lines (31) and (32) is $O(N^{-2})$. The gradient of the lines (31) and (32) has coordinates $(O(N^2),1)$ in (τ,ξ) plane. Hence the measure of R_β is $O(N^{-1})O(N^{-2})O(N^2)=O(N^{-1})$ and moreover the measure of V is $O(1)$. Further if $(\tau,\xi,\eta)\in V$ then we can easily see that $|\tau|\sim N^2$, $|\xi|\sim O(1)$ and $\eta\sim N$ (cf. the definition of τ_0, ξ_0 and η_0). Hence

$$(\tau,\xi,\eta)\in V \Longrightarrow \left|\tau-\xi^3+\frac{\eta^2}{\xi}\right|\lesssim N^2.$$

Let V^* be the following subset of V

$$V^* = \{(\tau,\xi,\eta)\ :\ \eta=\sqrt{3}N\left(\beta-\frac{1-\gamma}{2}\right),\ \frac{49}{100^2}\le\beta\le\frac{51}{100^2},\ (\tau,\xi) \text{ belong to the parallelogram with the same center as } R_\beta, \text{ the sides being parallel to those of } R_\beta \text{ and the lengths of the sides are } \frac{1}{100} \text{ times the lengths of the sides of } R_\beta\}.$$

We easily see that

$$\mathrm{mes}(V)\sim\mathrm{mes}(V^*)\sim O(1).$$

Denote by $\zeta=(\tau,\xi,\eta)$ the coordinates in $\mathbf{R}^3_{\tau,\xi,\eta}$. Next for each $\zeta/2\in V^*$, we try to find a subset $V_\zeta\subset V$ such that

$$\mathrm{mes}(V_\zeta)=c\ \mathrm{mes}(V),\quad V_\zeta\subset\{\zeta_1^*\ :\ \zeta_1^*\in V,\ \zeta-\zeta_1^*\in V\},$$

where the constant c is independent of ζ. Let $\zeta/2\in V^*$ be on the parallelogram R_β in V. Up to a translation, we may assume that $\zeta/2$ is the center of R_β. We choose V_ζ as follows:

$$V_\zeta = \{(\tau,\xi,\eta)\ :\ \eta=\sqrt{3}N\left(\beta_1-\frac{1-\gamma}{2}\right),\ |\beta_1-\beta|\le\frac{1}{100^2},(\tau,\xi) \text{ belong to the parallelogram with the same center as } R_{\beta_1}, \text{ the sides being parallel to those of } R_{\beta_1} \text{ and the lengths of the sides are } \frac{1}{100} \text{ times the lengths of the sides of } R_{\beta_1}\}.$$

We have to check that for any $\zeta_1^*\in V_\zeta$ there exists $\zeta_2^*\in V$ such that

$$\zeta_2^*=\zeta-\zeta_1^*.$$

Assuming this claim, we have that for $\zeta/2\in V^*$,

$$\mathrm{mes}\{\zeta_1^*\ :\ \zeta_1^*\in V,\quad \zeta-\zeta_1^*\in V\}\ge\mathrm{mes}(V_\zeta)\sim\mathrm{mes}(V)=O(1).$$

Let $\zeta = (\tau, \xi, \eta)$, $\zeta_1 = (\tau_1, \xi_1, \eta_1)$, $\zeta_2 = (\tau_2, \xi_2, \eta_2)$. Since the points with coordinates $\zeta/2, \zeta_1, \zeta_2$ belong to V, then there exist pairs $(\beta, \eta), (\beta_1, \eta_1), (\beta_2, \eta_2)$ such that

$$\eta/2 = \sqrt{3}N\left(\beta - \frac{1-\gamma}{2}\right), \quad \eta_1 = \sqrt{3}N\left(\beta_1 - \frac{1-\gamma}{2}\right), \quad \eta_2 = \sqrt{3}N\left(\beta_2 - \frac{1-\gamma}{2}\right).$$

If $\zeta_2 = \zeta - \zeta_1$ then $\beta_2 = 2\beta - \beta_1$. Since

$$\frac{1}{d_\beta} - \frac{1}{d_{\beta_1}} = (\beta - \beta_1) O\left(\frac{1}{N}\right),$$

we can assume that d_β is independent of β and write just $d = d_\beta = O(N)$ for $0 \le \beta < \frac{1}{100}$. Let O_β be the center of R_β and $\vec{a}_\beta$, $\vec{b}_\beta$ be the half diagonals of R_β. We use the frame $(O_\beta, \vec{a}_\beta, \vec{b}_\beta)$ to parameterize R_β as follows:

$$\begin{cases} \xi &= -\frac{1}{2} + \beta + \frac{1}{4\sqrt{N}} + \frac{12\left(\beta - \frac{1}{2}\right)^2 - 4 + \frac{3}{4} + O\left(\frac{1}{\sqrt{N}}\right)}{d} + \frac{1}{4d}\left(4\theta_1 + 3\theta_2 + O\left(\frac{1}{\sqrt{N}}\right)\right) \\ &:= \bar{\xi}(\beta, \theta_1, \theta_2), \\ \tau &= 3\left((N+\beta)^2 + \frac{N^2\beta^2}{(N+\beta)^2}\right)\left(\frac{1}{4\sqrt{N}} + \frac{12\left(\beta - \frac{1}{2}\right)^2 - 4 + \frac{3}{4} + O\left(\frac{1}{\sqrt{N}}\right)}{d}\right) + (N+\beta)^3 - \frac{3N^2\beta^2}{N+\beta} \\ &\quad -\tau_0 - \frac{1}{8} + \frac{3}{4d}\left((N+\beta)^2 + \frac{N^2\beta^2}{(N+\beta)^2}\right)\left(4\theta_1 + 3\theta_2 + O\left(\frac{1}{\sqrt{N}}\right)\right) + \frac{\theta_1 - \theta_2}{8} + O\left(\frac{1}{N}\right) \\ &:= \bar{\tau}(\beta, \theta_1, \theta_2), \end{cases}$$

where $|\theta_1| + |\theta_2| \le 1$. We shall prove that for $\zeta_1^* \in V_\zeta$, there exists $\zeta_2^* \in V$ such that $\zeta_2^* = \zeta - \zeta_1^*$. Note that this is equivalent to prove that for $|\beta - \beta_1| \le \frac{1}{100^2}$ and $|\theta_1| + |\theta_2| \le \frac{1}{100}$, there exist θ_1' and θ_2' with $|\theta_1'| + |\theta_2'| \le 1$ such that $\theta_1, \theta_2, \theta_1', \theta_2'$ satisfy:

$$\text{(43)} \qquad \begin{cases} \bar{\xi}(2\beta - \beta_1, \theta_1', \theta_2') &= 2\bar{\xi}(\beta, 0, 0) - \bar{\xi}(\beta_1, \theta_1, \theta_2), \\ \bar{\tau}(2\beta - \beta_1, \theta_1', \theta_2') &= 2\bar{\tau}(\beta, 0, 0) - \bar{\tau}(\beta_1, \theta_1, \theta_2). \end{cases}$$

A straightforward computation shows that (43) is equivalent to

$$\text{(44)} \qquad \begin{cases} \theta_1' + \frac{3}{4}\theta_2' &= -\theta_1 - \frac{3}{4}\theta_2 - 24(\beta - \beta_1)^2 + O\left(\frac{1}{\sqrt{N}}\right), \\ (1 + c_1(N))\theta_1' + (\frac{3}{4} + c_2(N))\theta_2' &= -(1 + \bar{c}_1(N))\theta_1 - (\frac{3}{4} + \bar{c}_2(N))\theta_2 \\ &\quad -24(\beta - \beta_1)^2 + O\left(\frac{1}{\sqrt{N}}\right), \end{cases}$$

where

$$c_1(N) = \frac{2(2\beta - \beta_1)}{N} + \frac{d}{24N^2} + O\left(\frac{1}{N^2}\right), \quad c_2(N) = \frac{3(2\beta - \beta_1)}{2N} - \frac{d}{24N^2} + O\left(\frac{1}{N^2}\right),$$

$$\bar{c}_1(N) = \frac{2\beta_1}{N} + \frac{d}{24N^2} + O\left(\frac{1}{N^2}\right), \quad \bar{c}_2(N) = \frac{3\beta_1}{2N} - \frac{d}{24N^2} + O\left(\frac{1}{N^2}\right).$$

Clearly $c_1(N) \sim c_2(N) \sim \bar{c}_1(N) \sim \bar{c}_2(N) \sim O\left(\frac{1}{N}\right)$. Set

$$X = \begin{pmatrix} 1 & \frac{3}{4} \\ 1 + c_1(N) & \frac{3}{4} + c_2(N) \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & \frac{3}{4} \\ 1 + \bar{c}_1(N) & \frac{3}{4} + \bar{c}_2(N) \end{pmatrix},$$

$$\Theta = (\theta_1, \theta_2)^t, \quad \Theta' = (\theta'_1, \theta'_2)^t, \quad E = (1,1)^t.$$

Then (44) can be written in the form

$$X\Theta' = -Y\Theta + \left(-24(\beta - \beta_1)^2 + O\left(\frac{1}{\sqrt{N}}\right)\right) E.$$

We have that

$$\det(X) = c_2(N) - \frac{3}{4} c_1(N) = O\left(\frac{1}{N}\right).$$

Note that $c_2(N) - \frac{3}{4}c_1(N) \neq 0$, since $d \neq 0$ and hence $\det(X) \neq 0$. Therefore we have that

$$\Theta' = -X^{-1}Y\Theta + \left(-24(\beta - \beta_1)^2 + O\left(\frac{1}{\sqrt{N}}\right)\right) X^{-1}E.$$

We can easily obtain that

$$X^{-1}E = \frac{1}{\det(X)} \begin{pmatrix} c_2(N) \\ -c_1(N) \end{pmatrix} = \begin{pmatrix} O(1) \\ O(1) \end{pmatrix},$$

$$\begin{aligned} X^{-1}Y &= \frac{1}{\det(X)} \begin{pmatrix} c_2(N) - \frac{3}{4}\bar{c}_1(N) & \frac{3}{4}c_2(N) - \frac{3}{4}\bar{c}_2(N) \\ -c_1(N) + \bar{c}_1(N) & -\frac{3}{4}c_1(N) + \bar{c}_2(N) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\beta - \beta_1) \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}. \end{aligned}$$

Therefore we obtain that (43) can be written in the form

$$\begin{cases} \theta'_1 &= -\theta_1 + O(1)(\beta - \beta_1)(O(1)\theta_1 + O(1)\theta_2 + \beta - \beta_1) + O\left(\frac{1}{\sqrt{N}}\right), \\ \theta'_2 &= -\theta_2 + O(1)(\beta - \beta_1)(O(1)\theta_1 + O(1)\theta_2 + \beta - \beta_1) + O\left(\frac{1}{\sqrt{N}}\right). \end{cases}$$

Now it is clear that for $|\theta_1| + |\theta_2| \leq \frac{1}{100}$, we can choose θ'_1, θ'_2 with $|\theta'_1| + |\theta'_2| \leq 1$ in (43). Then we conclude that for $\zeta/2 \in V^*$

$$\text{mes}\{\zeta_1^* \in V \ : \ \zeta - \zeta_1^* \in V\} \geq \frac{1}{100^3}\text{mes}(V) \sim O(1). \tag{45}$$

We take A and B as follows

$$A = V + (\tau_0, \xi_0, \eta_0), \quad B = V - (\tau_0, \xi_0, \eta_0).$$

We have that

$$A \cap \{\eta = \eta^\star\} \subset R^1_{\beta^\star, \eta^\star}, \quad B \cap \{\eta = \eta^\star\} \subset R^2_{\beta^\star, \eta^\star, K},$$

where

$$K = \sqrt{3}N(1-\gamma), \quad \eta^\star = \sqrt{3}N\left(\beta^\star - \frac{1-\gamma}{2}\right), \quad |\eta^\star| \sim N.$$

Therefore

$$(\tau,\xi,\eta) \in \text{ supp } (A \cup B) \quad \Longrightarrow \quad \left|\tau - \xi^3 + \frac{\eta^2}{\xi}\right| \lesssim 1, \quad |\xi| \sim N.$$

Using (45) we obtain

$$\chi_A \star \chi_B(\zeta) \gtrsim \chi_{V^*}(\zeta/2).$$

Now we take $W := V^* + V^*$ in order to complete the proof of Lemma 4.1.

CHAPTER 6

The Cauchy problem for higher order KP equations

This Chapter essentially contains the joint paper with Jean-Claude Saut [49] *(The Cauchy problem for higher order KP equations, Journal of Differential Equations, 153 (1999), 196-222).*

Abstract.

We study the local well-posedness of higher order KP equations. Our well-posedness results make an essential use of a global smothing effect for the linearized equation established in [5], injected into the framework of Fourier transform restriction spaces introduced by Bourgain. Our ill-posedness results rely on the existence of solitary wave solutions and on scaling arguments. The method was first applied in the context of the KdV and Schrödinger equations (cf. [7], [8]).

AMS subject classification: 35Q53, 35Q51, 35A07.

1. Introduction

In this Chapter we shall study the Cauchy problem associated to Kadomtsev-Petviashvili (KP) equations having higher order dispersion in the main direction of propagation. Such equations occur naturally in the modeling of certain long dispersive waves (cf. [2],[35], [36]). The study of their solitary wave solutions was done in [10],[11]. Thus we consider the Cauchy problems in $\mathbf{R}^d$, $d = 2,3$

$$\begin{cases} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} = 0 \\ u(0,x,y) = \phi(x,y) \end{cases} \tag{1}$$

in the two dimensional case and

$$\begin{cases} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} + u_{zz} = 0 \\ u(0,x,y,z) = \phi(x,y,z) \end{cases} \tag{2}$$

in the three dimensional case. The "usual" KP equations correspond to $\beta = 0$ and $\alpha = -1$ (KP-I) or $\alpha = +1$ (KP-II). We are interested in the local well-posedness of the Cauchy problems (1) and (2). Following [7] and [24] we introduce the next notion for well-posedness which will be convenient for our purposes.

Definition 1. *The Cauchy problem (1) (resp. (2)) is locally well-posed in the space X if for any $\phi \in X$ there exists $T = T(\|\phi\|_X) > 0$ (T is a nondecreasing continuous function such that $\lim_{\rho\to 0} T(\rho) = \infty$) and a map F from X to $C([0,T];X)$ such that $u = F(\phi)$ solves the equation (1) (resp. (2)) and F is continuous in the sense that*

$$\|F(\phi_1) - F(\phi_2)\|_{L^\infty([0,T];X)} \le M(\|\phi_1 - \phi_2\|_X, R)$$

for some locally bounded function M from $\mathbf{R}^+ \times \mathbf{R}^+$ to $\mathbf{R}^+$ such that $M(S,R) \to 0$ for fixed R when $S \to 0$ and for $\phi_1, \phi_2 \in X$ such that $\|\phi_1\|_X + \|\phi_2\|_X \le R$.

We introduce the nonisotropic Sobolev spaces $H^{s_1,s_2}(\mathbf{R}^d)$ equipped with the norm

$$\|u\|^2_{H^{s_1,s_2}} = \int \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\widehat{u}(\xi,\eta)|^2 d\xi d\eta,$$

where $\langle . \rangle = (1+|.|^2)^{1/2}$ and $\eta = (\eta^1, \eta^2)$ in the case of three space dimensions. By $\dot{H}^{s_1,s_2}(\mathbf{R}^d)$ we shall denote the homogeneous nonisotropic Sobolev spaces equipped with the norm

$$\|u\|^2_{\dot{H}^{s_1,s_2}} = \int |\xi|^{2s_1} |\eta|^{2s_2} |\widehat{u}(\xi,\eta)|^2 d\xi d\eta,$$

Taking into account the specific structure of the KP-type equations we introduce the modified Sobolev space $\widetilde{H}^{s_1,s_2}(\mathbf{R}^d)$ equipped with the norm

$$\|u\|^2_{\widetilde{H}^{s_1,s_2}} = \int \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} (1+|\xi|^{-1})^2 |\widehat{u}(\xi,\eta)|^2 d\xi d\eta,$$

Note that any $u \in \widetilde{H}^{s_1,s_2}(\mathbf{R}^d)$ has (formally) a zero x mean value. We shall denote $\widetilde{H}^{0,0}$ by $\widetilde{L}^2$. For $b, s_1, s_2 \in \mathbf{R}$ we define $X^{b,s_1,s_2}(\mathbf{R}^{d+1})$ to be the completion of the functions of C_0^∞ with zero x mean value with respect to the norm

$$\|u\|^2_{X^{b,s_1,s_2}} = \int \langle \tau + p(\xi,\eta) \rangle^{2b} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} (1+|\xi|^{-1})^2 |\widehat{u}(\tau,\xi,\eta)|^2 d\tau d\xi d\eta,$$

where

$$p(\xi,\eta) = \beta\xi^5 - \alpha\xi^3 + \frac{|\eta|^2}{\xi}.$$

If α or β vanishes then the equations (1) and (2) are scale invariant. If $u(t,x,y)$ is a solution of (1) with $\beta = 0$ then so is $u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$ and $\|u_\lambda(t,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+2s_2+1/2} \|u(\lambda^3 t,.)\|_{\dot{H}^{s_1,s_2}}$. Hence one may expect local well-posedness (resp. ill-posedness) in H^{s_1,s_2} of (1) for $\beta = 0$ when $s_1 + 2s_2 > -1/2$ (resp. $s_1 + 2s_2 < -1/2$). Similarly if $u(t,x,y,z)$ is a solution of (2) with $\beta = 0$ then so is $u_\lambda(t,x,y,z) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y, \lambda^2 z)$ and $\|u_\lambda(t,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+2s_2-1/2} \|u(\lambda^3 t,.)\|_{\dot{H}^{s_1,s_2}}$. Hence one may expect local well-posedness (resp. ill-posedness) in H^{s_1,s_2} of (2) for $\beta = 0$ when $s_1 + 2s_2 > 1/2$ (resp. $s_1 + 2s_2 < 1/2$).

If $u(t,x,y)$ is a solution of (1) with $\alpha = 0$ then so is $u_\lambda(t,x,y) = \lambda^4 u(\lambda^5 t, \lambda x, \lambda^3 y)$ and $\|u_\lambda(t,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+3s_2+2} \|u(\lambda^5 t,.)\|_{\dot{H}^{s_1,s_2}}$. Hence one one may expect local well-posedness (resp. ill-posedness) in H^{s_1,s_2} of (1) for $\alpha = 0$ when $s_1 + 3s_2 > -2$ (resp. $s_1 + 2s_2 < -2$). Similarly if $u(t,x,y,z)$ is a solution of (2) with $\alpha = 0$ then so is $u_\lambda(t,x,y,z) = \lambda^4 u(\lambda^5 t, \lambda x, \lambda^3 y, \lambda^3 z)$ and $\|u_\lambda(t,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+3s_2+1/2} \|u(\lambda^5 t,.)\|_{\dot{H}^{s_1,s_2}}$. Hence one one may expect local well-posedness (resp. ill-posedness) in H^{s_1,s_2} of (2) for $\alpha = 0$ when $s_1 + 3s_2 > -1/2$ (resp. $s_1 + 3s_2 < -1/2$).

Now we state the well-posedness result. The proof uses the Fourier transform restriction norms introduced by Bourgain [**12**], [**13**], [**14**], in the context of Schrödinger, KdV or KP equations. In the nonlinear estimates we shall make an essential use of the smoothing effects for the linear group associated to (1) or (2) established in [**5**].

THEOREM 1. *Let $d = 2$ and $\beta < 0$, $\alpha \in \mathbf{R}$. Then for any $s_1 \geq -1/4$, $s_2 \geq 0$ the Cauchy problem (1) is locally well-posed in $\widetilde{H}^{s_1,s_2}$. Moreover there exists $b > 1/2$ such that the solution $u(t,x,y)$ satisfies $u \in X^{b,s_1,s_2}$.*

Let $d = 3$ and $\beta < 0$, $\alpha \in \mathbf{R}$. Then for any $s_1 \geq -1/8$, $s_2 \geq 0$ the Cauchy problem (2) is locally well-posed in $\widetilde{H}^{s_1,s_2}$. Moreover there exists $b > 1/2$ such that the solution $u(t,x,y,z)$ satisfies $u \in X^{b,s_1,s_2}$.

If $\beta = 0$ and $\alpha > 0$ (the "usual" KP-II equation) local well-posedness of (1) in $\widetilde{L}^2$ is established in [**14**]. The proof uses Fourier transform restriction norms, a dyadic decomposition related to the structure of the symbol of the linearized operator and can be performed for periodic initial data. Local well-posedness of KP-II in $\widetilde{H}^{s_1,s_2}$, $s_1 > -1/4$, $s_2 \geq 0$ is established in [**57**].

The sign of β is crucial in the proof of Theorem 1. We do not know of a similar result when $\beta > 0$ (KP-I type equations). In this case, however, it is easy to obtain global weak solutions by energy methods (cf. [**55**]). The uniqueness of such solutions is unknown[1].

In [**7**] it is shown that the solitary wave solutions can be used to construct examples proving local ill-posedness for the (generalized) KdV equation. One considers the limit of the solitary wave solutions as the propagation speed tends to infinity. In the favorable cases the initial data tends weakly to a nonzero distribution (for example the Dirac delta function for the KdV equation with cubic nonlinearity) while the solution at time $t > 0$ tends weakly to zero. Using the idea of [**7, 8**] together with the properties of the solitary waves of KP-I type equations (cf. [**10, 11**]), we can prove the next Theorem.

THEOREM 2. *Let $d = 2$. If $\alpha < 0$ and $\beta = 0$ then (1) is locally ill-posed in $H^{s,0}$ for $s < -1/2$. Let $d = 3$. If $\alpha < 0$ and $\beta = 0$ then (1) is locally ill-posed in $\dot{H}^{s_1,s_2}$ for $s_1 + 2s_2 = 1/2$, $s_1 \geq 0$, $s_2 \geq 0$. If $\alpha = 0$ and $\beta > 0$ then (2) is locally ill-posed in $H^{s,0}$ for $s < -1/2$.*

Note that our well-posedness or ill-posedness results do not contradict with the scaling argument. In fact the region $\{(s_1,s_2) : s_1 \geq -1/4, s_2 \geq 0\}$ is a subset of the region suggested by the scaling argument $\{(s_1,s_2) : s_1 + 3s_2 + 2 > 0\}$, where well-posedness is expected to hold in the case of two space dimensions when $\alpha = 0$, $\beta < 0$. The same holds for the region $\{(s_1,s_2) : s_1 \geq -1/8, s_2 \geq 0\}$ which is a subset of $\{(s_1,s_2) : s_1 + 3s_2 + 1/2 > 0\}$ in the case of three space dimensions. If $d = 2$, $\alpha < 0$ and $\beta = 0$ then ill-posedness holds for the pairs $(s,0)$, $s < -1/2$ which can achieve the "critical" line $s_1 + 2s_2 + 1/2 = 0$, suggested by the scaling argument. If $d = 3$, $\alpha < 0$ and $\beta = 0$ then ill-posedness holds for the pairs $\{(s_1,s_2) : s_1 + 2s_2 = 1/2, s_1 \geq 0, s_2 \geq 0\}$ which are a part of the line suggested by the scaling argument. If $d = 3$, $\alpha = 0$ and $\beta > 0$ then the pairs $(s,0)$, $s < -1/2$ can achieve the line $s_1 + 3s_2 + 1/2 = 0$, suggested by the scaling argument.

The method for obtaining local well-posedness works just for KP-II type equations (i.e. $\beta < 0$ or $\beta = 0$, $\alpha > 0$) since one can recuperate the derivative ∂_x in the non-linear term by the aid of an algebraic relation for the symbol of the linearized operator. (cf. (16) below).

[1] In the next Chapter we shall prove a global well-posedness result for the fifth order KP-I equations

In the case of KP-I type equations ($\beta > 0$ or $\beta = 0$, $\alpha < 0$) we can not gain the ∂_x derivative using the relation (16). However, using the parabolic regularisation method it is possible to prove local well-posedness of (1) in H^s, $s > 2$ and local well-posedness of (2) in H^s, $s > 5/2$ (cf. for instance [**31**] for an illustration of this method). The proof does not make use of the specific structure of the KP-type equations and could be performed for quite general evolution equations. The conditions for s are in order to control the L^∞ norm of the gradient of the solution and they seem to be very restrictive. Hence there is still no satisfactory theory for the local well-posedness of the Cauchy problem for KP-I type equations.

On the other hand our ill-posedness results are only valid for KP-I type equations since the existence of solitary wave solutions is used in an essential way. In [**10**] using Pohojaev type identities it is shown that the KP-II type equations do not possess localized solitary wave solutions. Note also that we are forced to take α or β zero in order to keep the scale invariance of the equation.

The rest of the Chapter is organized as follows. In Section 2 we recall the global smoothing effects for (1) and (2) established in [**5**]. Then we inject these estimates into the framework of the Bourgain spaces associated to (1) and (2). In Section 3 we prove the crucial nonlinear estimate. In Section 4 we perform a fixed point argument to complete our well-posedness result. Section 5 is devoted to the ill-posedness of (1) and (2). In Section 6 we remark that the ill-posedness of the (generalized) KdV equations does not depend on the special form of the solitary waves. Finally in an appendix we give the proof of a decay estimate for a fractional derivative of the "lump" solitary wave of KP-I equation.

2. Estimates for the linear equation

Let us consider the linear initial value problem associated to (1) or (2)

$$\begin{cases} i\partial_t u = p(D)u \\ u(0,x,y) = \phi \end{cases} \tag{3}$$

where $D = (D_1, D_2)$, when $d = 2$, $D = (D_1, D_2, D_3)$, when $d = 3$ $D_1 = -i\partial_x$, $D_2 = -i\partial_y$ and $D_3 = -i\partial_z$. We recall that $p(\xi,\eta) = \beta\xi^5 - \alpha\xi^3 + \frac{|\eta|^2}{\xi}$. We shall denote by $U(t) = \exp(-ip(D))$ the unitary group which generates the solutions of (3). We have the following representation for the solutions of (3), when $d = 3$

$$u(t,x,y,z) = U(t)\phi = G_t \star \phi,$$

where G_t is the oscillatory integral

$$G_t(x,y,z) = c\int_{\mathbf{R}^3} \exp(itp(\xi,\eta) + i(x\xi + y\eta^1 + z\eta^2))d\xi d\eta.$$

A similar representation for the solutions of (3) holds when $d = 2$. We have in fact a global smoothing effect for the solutions of (3).

LEMMA 2.1. *(cf. [5], Theorem 4.2) Let* $\delta(r) = d(\frac{1}{2} - \frac{1}{r})$. *Then the following estimates hold*

$$\||D_x|^{\frac{\delta(r)}{2}} U(t)\phi\|_{L_t^q(L^r_{(x,y)})} \le c\|\phi\|_{L^2}, \quad (d = 2),$$

$$\||D_x|^{\frac{\delta(r)}{6}} U(t)\phi\|_{L_t^q(L^r_{(x,y,z)})} \le c\|\phi\|_{L^2}, \quad (d = 3),$$

provided

$$\frac{2}{q} = \delta(r).$$

Next we shall prove an inequality which will be used in the proof of the nonlinear estimates.

LEMMA 2.2. *Let* $d = 2$ *and*

$$0 < \theta \le 1, \quad \epsilon_1 > 0, \quad 0 \le b \le 1/2 + \epsilon_1.$$

Then the following inequality holds

$$\|\mathcal{F}^{-1}(|\xi|^\lambda \langle \tau + p(\xi,\eta)\rangle^{-b} |\widehat{u}(\tau,\xi,\eta)|); L_t^q(L^r_{(x,y)})\| \le c\|u\|_{L^2}, \tag{4}$$

where

$$\frac{2}{q} = 1 - \frac{\theta b}{1/2 + \epsilon_1}, \quad \delta(r) = \frac{(1-\theta)b}{1/2 + \epsilon_1}, \quad \lambda = \frac{\delta(r) b}{2(1/2 + \epsilon_1)}.$$

Let $d = 3$ *and*

$$0 < \theta \le 1, \quad \epsilon_1 > 0, \quad 0 \le b \le 1/2 + \epsilon_1.$$

Then the following inequality holds

$$\|\mathcal{F}^{-1}(|\xi|^\lambda \langle \tau + p(\xi,\eta)\rangle^{-b} |\widehat{u}(\tau,\xi,\eta)|); L_t^q(L^r_{(x,y,z)})\| \le c\|u\|_{L^2}, \tag{5}$$

where

$$\frac{2}{q} = 1 - \frac{\theta b}{1/2 + \epsilon_1}, \quad \delta(r) = \frac{(1-\theta)b}{1/2 + \epsilon_1}, \quad \lambda = \frac{\delta(r) b}{6(1/2 + \epsilon_1)}.$$

Proof. Let $d = 3$. Using Lemma 2.1 and [23], Lemma 3.3 we obtain

$$\||D_x|^{\frac{\delta(r)}{6}} u; L_t^q(L^r_{(x,y,z)})\| \le c\|u; X^{1/2+\epsilon_1,0,0}\|, \tag{6}$$

provided

$$\frac{2}{q} = \delta(r).$$

Interpolating between (6) and

$$\|u; L_t^2(L^2_{(x,y)})\| \le \|u; X^{0,0,0}\|,$$

we obtain

$$\||D_x|^\lambda u; L_t^q(L^r_{(x,y)})\| \le c\|u; X^{b,0,0}\|, \tag{7}$$

provided

$$\frac{2}{q} = 1 - \frac{\theta b}{1/2 + \epsilon_1}, \quad \delta(r) = \frac{(1-\theta)b}{1/2 + \epsilon_1}, \quad \lambda = \frac{\delta(r) b}{6(1/2 + \epsilon_1)}.$$

But (7) is equivalent to (4) which completes the proof of Lemma 2.2 when $d = 3$. If $d = 2$ then the arguments are the same.

COROLLARY 1. *Let $d = 2$. Then the following inequalities hold*

$$\|\mathcal{F}^{-1}(|\xi|^{1/4}\langle\tau + p(\xi,\eta)\rangle^{-b}|\widehat{u}(\tau,\xi,\eta)|); L_t^4(L^4_{(x,y)})\| \le c\|u\|_{L^2}, \tag{8}$$

where $b > 1/2$.

$$\|\mathcal{F}^{-1}(\langle\tau + p(\xi,\eta)\rangle^{-b}|\widehat{u}(\tau,\xi,\eta)|); L_t^q(L^2_{(x,y)})\| \le c\|u\|_{L^2}, \tag{9}$$

where $2 \le q \le \infty$, $b \ge 0$ and $\frac{bq}{q-2} > \frac{1}{2}$.

Let $d = 3$. Then the following inequalities hold

$$\|\mathcal{F}^{-1}(|\xi|^{1/8}\langle\tau + p(\xi,\eta)\rangle^{-b}|\widehat{u}(\tau,\xi,\eta)|); L_t^{8/3}(L^4_{(x,y,z)})\| \le c\|u\|_{L^2}, \tag{10}$$

where $b > 1/2$.

$$\|\mathcal{F}^{-1}(\langle\tau + p(\xi,\eta)\rangle^{-b}|\widehat{u}(\tau,\xi,\eta)|); L_t^q(L^2_{(x,y,z)})\| \le c\|u\|_{L^2}, \tag{11}$$

where $2 \le q \le \infty$, $b \ge 0$ and $\frac{bq}{q-2} > \frac{1}{2}$.

Proof. The proof of (8) follows from Lemma 2.2 ($d = 2$) applied with $\theta = 1/2$ and $b = 1/2 + \epsilon_1$. Lemma 2.2 ($d = 3$) applied with $\theta = 1/4$ and $b = 1/2 + \epsilon_1$ yields (10). To prove (9) and (11) we apply Lemma 2.2 with $\theta = 1$.

3. An estimate for the nonlinear term

LEMMA 3.1. *Let $d = 2$ and $s_1 \ge -1/4$, $s_2 \ge 0$. Then for sufficiently small ϵ we have*

$$\|uu_x\|_{X^{-1/2+3\epsilon,s_1,s_2}} \le c\|u\|^2_{X^{1/2+2\epsilon,s_1,s_2}}. \tag{12}$$

Let $d = 3$ and $s_1 \ge -1/8$, $s_2 \ge 0$. Then for sufficiently small ϵ we have

$$\|uu_x\|_{X^{-1/2+3\epsilon,s_1,s_2}} \le c\|u\|^2_{X^{1/2+2\epsilon,s_1,s_2}}. \tag{13}$$

Proof. We shall give the proof only when $d = 3$. In the case of two space dimensions the arguments are similar. We set

$$\widehat{w}(\tau,\xi,\eta) = \langle\tau + p(\xi,\eta)\rangle^{1/2+2\epsilon}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\widehat{u}(\tau,\xi,\eta),$$

$$\sigma := \sigma(\tau,\xi,\eta) = \tau + p(\xi,\eta), \quad \sigma_1 := \sigma(\tau_1,\xi_1,\eta_1), \quad \sigma_2 := \sigma(\tau - \tau_1, \xi - \xi_1, \eta - \eta_1).$$

Let $\zeta = (\xi,\eta)$. Then (13) is equivalent to

$$\|\langle\sigma\rangle^{-1/2+3\epsilon}\langle\xi\rangle^{1+s_1}\langle\eta\rangle^{s_2}\int\int K(\tau,\zeta,\tau_1,\zeta_1)\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)d\tau_1 d\zeta_1\|_{L^2_{(\tau,\zeta)}} \le$$

$$c\{\|w\|^2_{L^2} + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|^2_{L^2}\}, \tag{14}$$

where

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}\langle\eta_1\rangle^{-s_2}\langle\eta-\eta_1\rangle^{-s_2}}{\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}}$$

Hence by the self-duality of L^2 we obtain that (14) is equivalent to

$$|\int\int K_1(\tau,\zeta,\tau_1,\zeta_1)\widehat{w}(\tau_1,\zeta_1)\widehat{w}(\tau-\tau_1,\zeta-\zeta_1)\widehat{v}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta| \le$$

$$c\{\|w\|^2_{L^2} + \|w\|_{L^2}\||D_x|^{-1}w\|_{L^2} + \||D_x|^{-1}w\|^2_{L^2}\}\|v\|_{L^2}, \tag{15}$$

where

$$K_1(\tau,\zeta,\tau_1,\zeta_1)=\frac{\langle\xi\rangle^{1+s_1}\langle\xi_1\rangle^{-s_1}\langle\xi-\xi_1\rangle^{-s_1}}{\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/2-3\epsilon}}\frac{\langle\eta\rangle^{s_2}}{\langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2}}$$

Since for $s_2\geq 0$

$$\frac{\langle\eta\rangle^{s_2}}{\langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2}}\leq \text{const},$$

we shall suppose that $s_2=0$ from now on. Without loss of generality we can assume that $\widehat{w}\geq 0$ and $\widehat{v}\geq 0$. We have the following relation, where $\eta=(\eta^1,\eta^2)$ and $\eta_1=(\eta_1^1,\eta_1^2)$

$$\sigma_1+\sigma_2-\sigma=-5\beta\xi_1\xi(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2)+3\alpha\xi_1\xi(\xi-\xi_1)+$$

$$\frac{(\xi_1\eta^1-\xi\eta_1^1)^2}{\xi_1\xi(\xi-\xi_1)}+\frac{(\xi_1\eta^2-\xi\eta_1^2)^2}{\xi_1\xi(\xi-\xi_1)} \tag{16}$$

One has the next Lemma.

LEMMA 3.2. *There exists a positive constant c_0 such that*

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq c|\xi_1(\xi-\xi_1)\xi^3|,\quad \textit{for}\quad |\xi|\geq c_0. \tag{17}$$

Proof. Since $\beta<0$ we have that

$$3\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq|\xi_1\xi(\xi-\xi_1)|(-\frac{15}{4}\beta\xi^2+3\alpha),$$

where we used the elementary inequality $\xi^2-\xi\xi_1+\xi_1^2\geq\frac{3}{4}\xi^2$. If $\alpha>0$ then we obtain (17) with $c_0=0$. If $\alpha<0$ then for $-\frac{15}{8}\beta\xi^2+3\alpha\geq 0$ we have

$$3\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq|\xi_1\xi(\xi-\xi_1)|(-\frac{15}{8}\beta\xi^2),$$

which completes the proof of Lemma 3.2.

We shall denote by J the integral in the left-hand side of (15). We shall divide the domain of integration taking into account which term dominates in the left hand side of (17). By symmetry we can assume that

$$|\sigma_1|\geq|\sigma_2|.$$

Case 1. $|\xi|\leq\max(2,c_0)$. We denote by J_1 the restriction of J on this region. In this case we have that

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{\langle\xi_1\rangle^{1/8}\langle\xi-\xi_1\rangle^{1/8}}{\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/2-3\epsilon}}$$

Case 1.1. $|\xi_1|\geq 1$, $|\xi-\xi_1|\geq 1$. We denote by J_{11} the restriction of J on this region. In this case we have that

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{|\xi_1|^{1/8}|\xi-\xi_1|^{1/8}}{\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/2-3\epsilon}}$$

Hence using Hölder inequality (10) and (11) we obtain

$$\begin{aligned} J_{11} &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2+3\epsilon}\widehat{v}(\tau,\zeta))\|_{L^4_t(L^2_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi_1|^{1/8}\langle\sigma_1\rangle^{-1/2-2\epsilon}\widehat{w}(\tau_1,\zeta_1))\|_{L^{8/3}_t(L^4_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi-\xi_1|^{1/8}\langle\sigma_2\rangle^{-1/2-2\epsilon}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^{8/3}_t(L^4_{(x,y,z)})} \\ &\leq c\|w\|^2_{L^2}\|v\|_{L^2}. \end{aligned}$$

Case 1.2. $|\xi_1|\leq 1$. We denote by J_{12} the restriction of J on this region. In this case we have that

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{|\xi_1|^{1/8}|\xi-\xi_1|^{1/8}}{|\xi_1|\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/2-3\epsilon}}$$

Hence using Hölder inequality (10) and (11) we obtain

$$\begin{aligned} J_{12} &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/2+3\epsilon}\widehat{v}(\tau,\zeta))\|_{L^4_t(L^2_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi_1|^{1/8}\langle\sigma_1\rangle^{-1/2-2\epsilon}|\xi_1|^{-1}\widehat{w}(\tau_1,\zeta_1))\|_{L^{8/3}_t(L^4_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi-\xi_1|^{1/8}\langle\sigma_2\rangle^{-1/2-2\epsilon}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^{8/3}_t(L^4_{(x,y,z)})} \\ &\leq c\|w\|_{L^2}\||D_x|^{-1}w\|_{L^2}\|v\|_{L^2}. \end{aligned}$$

Case 1.3. $|\xi-\xi_1|\leq 1$. This case can be treated similarly to Case 1.2.

Case 2. $|\xi|\geq\max(2,c_0)$, $|\sigma|\geq|\sigma_1|$. We denote by J_2 the restriction of J on this region. In this case we have that

$$|\sigma|^{1/4-4\epsilon}\geq c|\xi|^{3/4-12\epsilon}|\xi_1|^{1/4-4\epsilon}|\xi-\xi_1|^{1/4-4\epsilon}.$$

Hence

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{\langle\xi\rangle^{1/4+s_1+12\epsilon}\langle\xi_1\rangle^{-s_1+4\epsilon}\langle\xi-\xi_1\rangle^{-s_1+4\epsilon}}{|\xi_1|^{1/4}|\xi-\xi_1|^{1/4}\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/4}+\epsilon}$$

Case 2.1. $|\xi_1|\geq 1$, $|\xi-\xi_1|\geq 1$. We denote by J_{21} the restriction of J on this region. In this case we have that

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{|\xi_1|^{1/8}|\xi-\xi_1|^{1/8}}{\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/4}+\epsilon}$$

Using (10), (11) and Hölder inequality we obtain

$$\begin{aligned} J_{21} &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/4-\epsilon}\widehat{v}(\tau,\zeta))\|_{L_t^4(L^2_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi_1|^{1/8}\langle\sigma_1\rangle^{-1/2-2\epsilon}\widehat{w}(\tau_1,\zeta_1))\|_{L_t^{8/3}(L^4_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi-\xi_1|^{1/8}\langle\sigma_2\rangle^{-1/2-2\epsilon}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1))\|_{L_t^{8/3}(L^4_{(x,y,z)})} \\ &\leq c\|w\|^2_{L^2}\|v\|_{L^2}. \end{aligned}$$

Case 2.2. $|\xi_1|\leq 1$. We denote by J_{22} the restriction of J on this region. In this case we have that

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{|\xi_1|^{1/8}|\xi-\xi_1|^{1/8}}{|\xi_1|\langle\sigma_1\rangle^{1/2+2\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/4}+\epsilon}$$

Using (10), (11) and Hölder inequality we obtain

$$\begin{aligned} J_{22} &\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-1/4-\epsilon}\widehat{v}(\tau,\zeta))\|_{L_t^4(L^2_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi_1|^{1/8}\langle\sigma_1\rangle^{-1/2-2\epsilon}|\xi_1|^{-1}\widehat{w}(\tau_1,\zeta_1))\|_{L_t^{8/3}(L^4_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi-\xi_1|^{1/8}\langle\sigma_2\rangle^{-1/2-2\epsilon}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1))\|_{L_t^{8/3}(L^4_{(x,y,z)})} \\ &\leq c\|w\|_{L^2}\||D_x|^{-1}w\|_{L^2}\|v\|_{L^2}. \end{aligned}$$

Case 2.3. $|\xi-\xi_1|\leq 1$. This case can be treated similarly to Case 2.2.

Case 3. $|\xi|\geq\max(2,c_0)$, $|\sigma_1|\geq|\sigma|$. We denote by J_3 the restriction of J on this region. In this case we have that

$$|\sigma_1|^{1/4-4\epsilon}\geq c|\xi|^{3/4-12\epsilon}|\xi_1|^{1/4-4\epsilon}|\xi-\xi_1|^{1/4-4\epsilon}.$$

Hence

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{\langle\xi\rangle^{1/4+s_1+12\epsilon}\langle\xi_1\rangle^{-s_1+4\epsilon}\langle\xi-\xi_1\rangle^{-s_1+4\epsilon}}{|\xi_1|^{1/4}|\xi-\xi_1|^{1/4}\langle\sigma_1\rangle^{1/4+6\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/2-3\epsilon}}$$

Case 3.1. $|\xi_1|\geq 1$, $|\xi-\xi_1|\geq 1$. We denote by J_{31} the restriction of J on this region. In this case we have that

$$K_1(\tau,\zeta,\tau_1,\zeta_1)\leq\frac{\langle\xi\rangle^{12\epsilon}\langle\xi_1\rangle^{4\epsilon}\langle\xi-\xi_1\rangle^{4\epsilon}}{\langle\sigma_1\rangle^{1/4+6\epsilon}\langle\sigma_2\rangle^{1/2+2\epsilon}\langle\sigma\rangle^{1/2-3\epsilon}}$$

Using Lemma 2.2 and Hölder inequality we obtain

$$\begin{aligned} J_{31} &\le \|\mathcal{F}^{-1}(|\xi|^{\frac{1/2-3\epsilon}{1/2+\epsilon_1}\frac{\delta(r_1)}{6}}\langle\sigma\rangle^{-1/2+3\epsilon}\widehat{v}(\tau,\zeta))\|_{L_t^{q_1}(L^{r_1}_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi_1|^{\frac{1/4+6\epsilon}{1/2+\epsilon_1}\frac{\delta(r_2)}{6}}\langle\sigma_1\rangle^{-1/4-6\epsilon}\widehat{w}(\tau_1,\zeta_1))\|_{L_t^{q_2}(L^{r_2}_{(x,y,z)})} \\ &\quad\times\|\mathcal{F}^{-1}(|\xi-\xi_1|^{\frac{1/2+2\epsilon}{1/2+\epsilon_1}\frac{\delta(r_3)}{6}}\langle\sigma_2\rangle^{-1/2-2\epsilon}\widehat{w}(\tau-\tau_1,\zeta-\zeta_1))\|_{L_t^{q_3}(L^{r_3}_{(x,y,z)})} \\ &\le c\|w\|_{L^2}^2\|v\|_{L^2}, \end{aligned}$$

provided

$$\begin{aligned} \frac{2}{q_1} &= 1-\frac{\theta(1/2-3\epsilon)}{1/2+\epsilon_1}, \quad \delta(r_1)=\frac{(1-\theta)(1/2-3\epsilon)}{1/2+\epsilon_1}, \\ \frac{2}{q_2} &= 1-\frac{\theta(1/4+6\epsilon)}{1/2+\epsilon_1}, \quad \delta(r_2)=\frac{(1-\theta)(1/4+6\epsilon)}{1/2+\epsilon_1}, \\ \frac{2}{q_3} &= 1-\frac{\theta(1/2+2\epsilon)}{1/2+\epsilon_1}, \quad \delta(r_3)=\frac{(1-\theta)(1/2+2\epsilon)}{1/2+\epsilon_1}, \\ \frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3} &= 1, \quad \delta(r_1)+\delta(r_2)+\delta(r_3)=\frac{3}{2}. \end{aligned}$$

Now it is sufficient to take $\epsilon_1 = 2\epsilon$ and $\theta = 2/5$ to ensure the restrictions of the Hölder inequality.

Case 3.2. $|\xi_1| \le 1$. We denote by J_{32} the restriction of J on this region. Using the arguments of Case 3.1, we similarly obtain

$$J_{32} \le c\|w\|_{L^2}\|\|\||D_x|^{-1}w\|_{L^2}\|\|\|v\|_{L^2}.$$

Case 3.2. $|\xi-\xi_1| \le 1$. This case can be treated similarly to Case 3.2.

This completes the proof of Lemma 3.1.

4. Proof of Theorem 1

In this section we shall give the proof of Theorem 1. Note that the equations (1) or (2) are equivalent to the integral equation

$$u(t) = U(t)\phi - \int_0^t U(t-t')u(t')\partial_x u(t')dt'. \tag{18}$$

Let ψ be a cut-off function such that

$$\psi \in C_0^\infty(\mathbf{R}),\ \text{supp}\ \psi \subset [-2,2], \psi = 1 \text{ over the interval } [-1,1].$$

We truncate (18)

$$u(t) = \psi(t)U(t)u(0) - \psi(t/T)\int_0^t U(t-t')u(t')\partial_x u(t')dt'. \tag{19}$$

We shall solve (19) globally in time. To the solutions of (19) will correspond local solutions of (18) in the time interval $[-T, T]$. We shall apply a fixed point argument to solve (19) in $X^{1/2+\epsilon, s_1, s_2}$ for sufficiently small ϵ. Now we state the estimates for the two terms in the left-hand side of (19).

LEMMA 4.1. *The following estimates hold for sufficiently small* $\epsilon > 0$

$$\|\psi(t)U(t)\phi\|_{X^{1/2+2\epsilon, s_1, s_2}} \leq c\|\phi\|_{\widetilde{H}^{s_1, s_2}}, \tag{20}$$

$$\|\psi(t/T)\int_0^t U(t-t')u(t')\partial_x u(t')dt'\|_{X^{1/2+2\epsilon, s_1, s_2}} \leq cT^{\epsilon}\|uu_x\|_{X^{-1/2+3\epsilon, s_1, s_2}}. \tag{21}$$

Proof. To prove (20) it is sufficient to note that

$$\|u\|_{X^{b, s_1, s_2}} = \|U(-t)u\|_{\widetilde{H}^{b, s_1, s_2}},$$

where $\widetilde{H}^{b, s_1, s_2}$ is equipped with the norm

$$\|u\|^2_{\widetilde{H}^{b, s_1, s_2}} = \int \langle\tau\rangle^b \langle\xi\rangle^{s_1} \langle\eta\rangle^{s_2} (1 + |\xi|^{-1})^2 |\widehat{u}(\tau, \xi, \eta)|^2 d\tau d\xi d\eta.$$

The proof of (21) is a direct consequence of [**23**], Lemma 3.2 applied with $b = 1/2 + 2\epsilon$ and $b' = -1/2 + 3\epsilon$. This completes the proof of Lemma 4.1.

Now we define an operator L

$$Lu(t) := \psi(t)U(t)u(0) - \frac{1}{2}\psi(t/T)\int_0^t U(t-t')u(t')\partial_x u(t'))dt'.$$

We obtain from Lemma 3.1 and Lemma 4.1

$$\|Lu\|_{X^{1/2+2\epsilon, s_1, s_2}} \leq c\|\phi\|_{\widetilde{H}^s} + cT^{\epsilon}\|u\|^2_{X^{1/2+2\epsilon, s_1, s_2}}. \tag{22}$$

Similarly we obtain

$$\|Lu - Lv\|_{X^{1/2+2\epsilon, s_1, s_2}} \leq cT^{\epsilon}\|u + v\|_{X^{1/2+2\epsilon, s_1, s_2}}\|u - v\|_{X^{1/2+2\epsilon, s_1, s_2}} \tag{23}$$

Using (22) and (23) we can apply the contraction mapping principle for sufficiently small T which completes the proof of Theorem 1.

5. Proof of Theorem 2

It is known (cf. [**10**]) that in some cases (1) (resp. (2)) possesses solitary wave solutions. We recall that a solitary wave is a "localized" solution of (1) (resp. (2)) of the form $u(x - ct, y)$ (resp. $u(x - ct, y, z)$). More precisely we have the following Theorem.

THEOREM 5.1. *(cf. [**10, 11**]). Let* $\beta > 0$ *and* $\alpha \leq 0$. *Then (1) (resp. (2)) possesses a solitary wave solution* $u(x - ct, y)$ *(resp.* $u(x - ct, y, z)$*) such that*

$$u \in L^2_y(L^1_x) \cap_{s=0}^{\infty} H^s, \quad (\text{ resp. } u \in L^2_{(y,z)}(L^1_x) \cap_{s=0}^{\infty} H^s).$$

The same statement holds when $\beta = 0$ *and* $\alpha < 0$.

We have (cf. [11]) that $r^\delta u \in L^2(\mathbf{R}^d)$, for $\delta < \frac{d}{2}$, where $r^2 = x^2 + y^2$, when $d = 2$ and $r^2 = x^2+y^2+z^2$, when $d = 3$. Hence using Cauchy-Schwarz inequality we obtain immediately

$$u \in L^2_y(L^1_x) \quad (d=2), \quad u \in L^2_{(y,z)}(L^1_x) \quad (d=3).$$

The rest of the proof of Theorem 5.1 is contained in [10, 11]. Note that when $\beta = 0$, $\alpha < 0$ an explicit form of a solitary wave can be derived by the inverse scattering method and we will use it below. In the $3D$ case we will use essentially scaling arguments and no explicit form of the solitary waves is needed. Actually, solitary waves for (1) or (2) (cf. [10, 11]) can be obtained by the aid of the concentration compactness principle for some pure power nonlinearities and the method for obtaining ill-posedness could be extended to these cases (see the end of the section).

- Let $d = 2$, $\beta = 0$ and $\alpha < 0$. Then (1) has a solitary wave solution of the form

$$u_c(t, x, y) = \phi_c(x - ct, y), \quad c > 0$$

where

$$u_c(0, x, y) = \phi_c(x, y) = c\phi_1(c^{1/2}x, cy) := c\phi(c^{1/2}x, cy).$$

Here u_c is the "lump"solitary wave of KP-I equation. Recall that

$$u_c(t, x, y) = \frac{8c(1 - \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)}{(1 + \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)^2}$$

is a solitary wave solution of (1) with $\alpha = -1$ and $\beta = 0$. We will need a decay estimate of a fractional derivative of ϕ_c. Obviously $x\phi_c$ does not belong to $L^2(\mathbf{R}^2)$, but

LEMMA 5.1. *(cf. Appendix) Let* $\epsilon > 0$. *Then*

$$|D_x|^\epsilon(x\phi_c) \in L^2(\mathbf{R}^2).$$

We shall denote by $\mathcal{F}_x$ the partial Fourier transform with respect to x. One has

$$\mathcal{F}_x(\phi_c)(\xi, y) = c^{1/2}\mathcal{F}_x(\phi)(\frac{\xi}{c^{1/2}}, cy).$$

Further we have

$$\|\phi_c\|^2_{H^{s,0}} = \int_{\mathbf{R}^2} \langle\xi\rangle^{2s} |\mathcal{F}_x(\phi)(\frac{\xi}{c^{1/2}}, y)|^2 d\xi dy.$$

Hence for $s < -1/2$ Lebesgue theorem yields

$$\lim_{c\to\infty} \|\phi_c\|^2_{H^{s,0}} = c_s\|\mathcal{F}_x(\phi)(0, .)\|^2_{L^2} = c_s\|\phi\|^2_{L^2_y(L^1_x)},$$

where

$$c_s = \int_{-\infty}^{\infty} \langle\xi\rangle^{2s} d\xi.$$

Similarly we obtain

$$(\phi_{c_1}, \phi_{c_2})_s = (\frac{c_2}{c_1})^{1/2} \int_{\mathbf{R}^2} \langle\xi\rangle^{2s} \mathcal{F}_x(\phi)(\frac{\xi}{c_1^{1/2}}, y)\overline{\mathcal{F}_x(\phi)(\frac{\xi}{c_2^{1/2}}, \frac{c_2}{c_1}y)} d\xi dy,$$

where $(.,.)_s$ stays for the $H^{s,0}$ scalar product. Hence

$$\lim_{c_1,c_2\to\infty, c_1/c_2\to 1} (\phi_{c_1}, \phi_{c_2})_s = c_s\|\mathcal{F}_x(\phi)(0, .)\|^2_{L^2}$$

and

$$\lim_{c_1,c_2\to\infty,c_1/c_2\to 1} \|\phi_{c_1}-\phi_{c_2}\|_{H^{s,0}}=0.$$

Now we have that

$$\mathcal{F}_x(u_c)(t,\xi,y)=c^{1/2}\exp(-itc\xi)\mathcal{F}_x(\phi)(\xi/c^{1/2},cy).$$

The presence of the oscillatory term $\exp(-itc\xi)$ makes the expression $(u_{c_1}(t,.),u_{c_2}(t,.))_s$ tend to zero as $n\to\infty$ by taking $c_1=n^2$ and $c_2=(n+1)^2$. More precisely we have by integration by parts

$$(u_{n^2}(t,.),u_{(n+1)^2}(t,.))_s=$$

$$c(n,t)\int_{\mathbf{R}^2}\partial_\xi(e^{it(2n+1)\xi})\langle\xi\rangle^{2s}\mathcal{F}_x(\phi)(\frac{\xi}{n},y)\overline{\mathcal{F}_x(\phi)(\frac{\xi}{n+1},\frac{(n+1)^2}{n^2}y)}d\xi dy=$$

$$2sc(n,t)\int_{\mathbf{R}^2}e^{it(2n+1)\xi}\langle\xi\rangle^{2s-1}\xi\mathcal{F}_x(\phi)(\frac{\xi}{n},y)\overline{\mathcal{F}_x(\phi)(\frac{\xi}{n+1},\frac{(n+1)^2}{n^2}y)}d\xi dy+$$

$$\frac{c(n,t)n^\epsilon}{in}\int_{\mathbf{R}^2}e^{it(2n+1)\xi}\frac{\langle\xi\rangle^{2s}}{|\xi|^\epsilon}\mathcal{F}_x(|D_x|^\epsilon(x\phi))(\frac{\xi}{n},y)\overline{\mathcal{F}_x(\phi)(\frac{\xi}{n+1},\frac{(n+1)^2}{n^2}y)}d\xi dy+$$

$$\frac{c(n,t)n^\epsilon}{i(n+1)}\int_{\mathbf{R}^2}e^{it(2n+1)\xi}\frac{\langle\xi\rangle^{2s}}{|\xi|^\epsilon}\mathcal{F}_x(\phi)(\frac{\xi}{n},y)\overline{\mathcal{F}_x(|D_x|^\epsilon(x\phi))(\frac{\xi}{n+1},\frac{(n+1)^2}{n^2}y)}d\xi dy:=$$

$$(1)+(2)+(3),$$

where

$$c(n,t)=\frac{n+1}{in(2n+1)t}.$$

Using Cauchy-Schwarz inequality and the estimate $\|\widehat{\phi}\|_{L^\infty}\le\|\phi\|_{L^1}$ we obtain

$$\begin{aligned}|(1)| &\le 2sc(n,t)\int_{\mathbf{R}^2}\langle\xi\rangle^{2s-1}\|\phi(.,y)\|_{L^1}\|\phi(.,\frac{(n+1)^2}{n^2}x')\|_{L^1}d\xi dy\\ &\le c(n,t)\frac{cn}{n+1}\|\phi\|^2_{L^2_y(L^1_x)}.\end{aligned}$$

It is clear that the last expression tends to zero as n tends to infinity. Further we have

$$\begin{aligned} |(2)| &\le \frac{n^\epsilon c(n,t)}{n}\int_{\mathbf{R}^2}\frac{\langle\xi\rangle^{2s}}{|\xi|^\epsilon}|\mathcal{F}_x(|D_x|^\epsilon(x\phi))(\frac{\xi}{n},y)|\|\phi(.,\frac{(n+1)^2}{n^2}y)\|_{L^1}d\xi dy \\ &\le \frac{n^\epsilon c(n,t)}{n}\frac{n}{n+1}\|\phi\|_{L^2_y(L^1_x)}\{\int_{\mathbf{R}^2}|\mathcal{F}_x(|D_x|^\epsilon(x\phi))(\frac{\xi}{n},y)|^2 d\xi dy\}^{1/2} \\ &\le \frac{n^\epsilon c(n,t)}{n}\frac{n^{3/2}}{n+1}\|\phi\|_{L^2_y(L^1_x)}\||D_x|^\epsilon(x\phi)\|_{L^2_{(x,y)}}, \end{aligned}$$

where we used that $|\xi|^{-\epsilon}\langle\xi\rangle^{2s}\in L^2_\xi$ and Lemma 5.1. Hence we obtain that for $\epsilon<1/2$ $|(2)|$ tends to zero as n tends to infinity. The term $|(3)|$ can be estimated in a similar fashion. Hence

$$\lim_{n\to\infty}(u_{(n+1)^2}(t,.),u_{n^2}(t,.))_s=0.$$

Moreover

$$\lim_{n\to\infty}\|u_{(n+1)^2}(t,.)-u_{n^2}(t,.)\|^2_{H^{s,0}}=2c_s\|\mathcal{F}_x(\phi)(0,.)\|^2_{L^2}.$$

Therefore we obtain that (1) is locally ill-posed in $H^{s,0}(\mathbf{R}^2)$ for $\beta=0$ and $\alpha<0$, when $s<-1/2$.

- Let $d=3$, $\beta>0$ and $\alpha=0$. Then (2) has a solitary wave solution of the form

$$u_c(t,x,y,z)=\phi_c(x-ct,y,z),\quad c>0$$

where

$$u_c(0,x,y,z)=\phi_c(x,y,z)=c\phi_1(c^{1/4}x,c^{3/4}y,c^{3/4}z):=c\phi(c^{1/4}x,c^{3/4}y,c^{3/4}z)$$

One obtain easily that

$$\mathcal{F}_x(\phi_c)(\xi,y,z)=c^{3/4}\mathcal{F}_x(\phi)(\frac{\xi}{c^{1/4}},c^{3/4}y,c^{3/4}z).$$

Further we have

$$\|\phi_c\|^2_{H^{s,0}}=\int_{\mathbf{R}^3}\langle\xi\rangle^{2s}|\mathcal{F}_x(\phi)(\frac{\xi}{c^{1/2}},y,z)|^2 d\xi dydz$$

Hence for $s<-1/2$ Lebesgue theorem yields

$$\lim_{c\to\infty}\|\phi_c\|^2_{H^{s,0}}=c_s\|\mathcal{F}_x(\phi)(0,.)\|^2_{L^2}=c_s\|\phi\|^2_{L^2_{(y,z)}(L^1_x)}.$$

Similarly we obtain

$$(\phi_{c_1},\phi_{c_2})_s=(\frac{c_2}{c_1})^{3/4}\int_{\mathbf{R}^2}\langle\xi\rangle^{2s}\mathcal{F}_x(\phi)(\frac{\xi}{c_1^{1/4}},y,z)\overline{\mathcal{F}_x(\phi)(\frac{\xi}{c_2^{1/4}},\frac{c_2}{c_1}y,\frac{c_2}{c_1}z)}d\xi dydz,$$

where $(.,.)_s$ stays for the $H^{s,0}$ scalar product. Hence

$$\lim_{c_1,c_2\to\infty,c_1/c_2\to1}(\phi_{c_1},\phi_{c_2})_s=c_s\|\mathcal{F}_x(\phi)(0,.)\|^2_{L^2}$$

and

$$\lim_{c_1,c_2\to\infty,c_1/c_2\to1}\|\phi_{c_1}-\phi_{c_2}\|_{H^{s,0}}=0.$$

Now we have that

$$\mathcal{F}_x(u_c)(t,\xi,y,z) = c^{3/4}\exp(-itc\xi)\mathcal{F}_x(\phi)(\xi/c^{1/4}, cy, cz).$$

The presence of the oscillatory term $\exp(-itc\xi)$ makes the expression $(u_{c_1}(t,.), u_{c_2}(t,.))_s$ tend to zero as $n \to \infty$ by taking $c_1 = n^2$ and $c_2 = (n+1)^2$. Let $x' = (y,z)$. Then

$$(u_{n^2}(t,.), u_{(n+1)^2}(t,.))_s =$$

$$c(n,t)\int_{\mathbf{R}^3} \partial_\xi(e^{it(2n+1)\xi})\langle\xi\rangle^{2s}\mathcal{F}_x(\phi)(\frac{\xi}{n^{1/2}}, x')\overline{\mathcal{F}_x(\phi)(\frac{\xi}{(n+1)^{1/2}}, \frac{(n+1)^2}{n^2}x')}d\xi dx' =$$

$$2sc(n,t)\int_{\mathbf{R}^3} e^{it(2n+1)\xi}\langle\xi\rangle^{2s-2}\xi\mathcal{F}_x(\phi)(\frac{\xi}{n^{1/2}}, x')\overline{\mathcal{F}_x(\phi)(\frac{\xi}{(n+1)^{1/2}}, \frac{(n+1)^2}{n^2}x')}d\xi dx' +$$

$$\frac{c(n,t)}{in^{1/2}}\int_{\mathbf{R}^3} e^{it(2n+1)\xi}\langle\xi\rangle^{2s}\mathcal{F}_x(x\phi)(\frac{\xi}{n^{1/2}}, x')\overline{\mathcal{F}_x(\phi)(\frac{\xi}{(n+1)^{1/2}}, \frac{(n+1)^2}{n^2}x')}d\xi dx' +$$

$$\frac{c(n,t)}{i(n+1)^{1/2}}\int_{\mathbf{R}^3} e^{it(2n+1)\xi}\langle\xi\rangle^{2s}\mathcal{F}_x(\phi)(\frac{\xi}{n^{1/2}}, x')\overline{\mathcal{F}_x(x\phi)(\frac{\xi}{(n+1)^{1/2}}, \frac{(n+1)^2}{n^2}x')}d\xi dx' :=$$

$$(1)+(2)+(3),$$

where

$$c(n,t) = \frac{(n+1)^{3/2}}{in^{3/2}(2n+1)t}.$$

Using Cauchy-Schwarz inequality and the estimate $\|\widehat{\phi}\|_{L^\infty} \le \|\phi\|_{L^1}$ we obtain

$$\begin{aligned} |(1)| &\le 2sc(n,t)\int_{\mathbf{R}^3}\langle\xi\rangle^{2s-1}\|\phi(.,x')\|_{L^1}\|\phi(., \frac{(n+1)^2}{n^2}x')\|_{L^1}d\xi dx' \\ &\le c(n,t)\frac{cn^2}{(n+1)^2}\|\phi\|^2_{L^2_{x'}(L^1_x)}. \end{aligned}$$

It is clear that the last expression tends to zero as n tends to infinity. Further we have

$$\begin{aligned} |(2)| &\le \frac{c(n,t)}{n^{1/2}}\int_{\mathbf{R}^3}\langle\xi\rangle^{2s}|\mathcal{F}_x(x\phi)(\frac{\xi}{n^{1/2}}, x')|\|\phi(., \frac{(n+1)^2}{n^2}x')\|_{L^1}d\xi dx' \\ &\le \frac{c(n,t)}{n^{1/2}}\frac{n^2}{(n+1)^2}\|\phi\|_{L^2_{x'}(L^1_x)}\{\int_{\mathbf{R}^3}|\mathcal{F}_x(x\phi)(\frac{\xi}{n^{1/2}}, x')|^2 d\xi dx'\}^{1/2} \\ &\le \frac{c(n,t)n^{1/4}}{n^{1/2}}\frac{n^2}{(n+1)^2}\|\phi\|_{L^2_{x'}(L^1_x)}\|x\phi\|_{L^2_{(x,x')}}, \end{aligned}$$

where we used that $x\phi \in L^2(\mathbf{R}^3)$, which follows from the decay properties of the solitary waves. Hence we obtain that $|(2)|$ tends to zero as n tends to infinity. The term $|(3)|$ can be estimated in a similar fashion. Hence

$$\lim_{n\to\infty}(u_{(n+1)^2}(t,.),u_{n^2}(t,.))_s = 0.$$

Moreover

$$\lim_{n\to\infty}\|u_{(n+1)^2}(t,.) - u_{n^2}(t,.)\|^2_{H^{s,0}} = 2c_s\|\mathcal{F}_x(\phi)(0,.)\|^2_{L^2}.$$

Therefore we obtain that (2) is locally ill-posed in $H^{s,0}(\mathbf{R}^3)$ for $\beta > 0$ and $\alpha = 0$, when $s < -1/2$.

- Let $d = 3$, $\beta = 0$ and $\alpha < 0$. Then (2) has a solitary wave solution of the form

$$u_c(t,x,y,z) = \phi_c(x-ct,y,z),$$

where

$$u_c(0,x,y,z) = \phi_c(x,y) = c\phi(c^{1/2}x,cy,cz).$$

We have that

$$\widehat{\phi_c}(\xi,\eta) = c^{-3/2}\widehat{\phi}(\frac{\xi}{c^{1/2}},\frac{\eta}{c}).$$

Further we have

$$\|\phi_c\|^2_{\dot{H}^{s_1,s_2}} = c^{s_1+2s_2-1/2}\int_{\mathbf{R}^3}|\xi|^{2s_1}|\eta|^{2s_2}|\widehat{\phi}(\xi,\eta)|^2 d\xi d\eta = c^{s_1+s_2-1/2}\|\phi\|^2_{\dot{H}^{s_1,s_2}}.$$

Hence when $s_1 + 2s_2 = 1/2$, $s_1 \geq 0$, $s_2 \geq 0$ we have that $\|\phi_c\|_{\dot{H}^{s_1,s_2}} = \|\phi\|_{\dot{H}^{s_1,s_2}}$. Similarly we obtain

$$(\phi_{c_1},\phi_{c_2})_{s_1,s_2} = (\frac{c_2}{c_1})^{-3/2}\int_{\mathbf{R}^3}|\xi|^{2s_1}|\eta|^{2s_2}\widehat{\phi}(\xi,\eta)\overline{\widehat{\phi}((\frac{c_1}{c_2})^{1/2}\xi,\frac{c_1}{c_2}\eta)}d\xi d\eta,$$

where now $(.,.)_{s_1,s_2}$ stays for the H^{s_1,s_2} scalar product. Hence using Lebesgue theorem we obtain

$$\lim_{c_1,c_2\to\infty,c_1/c_2\to 1}(\phi_{c_1},\phi_{c_2})_{s_1,s_2} = \|\phi\|^2_{\dot{H}^{s_1,s_2}}.$$

and

$$\lim_{c_1,c_2\to\infty,c_1/c_2\to 1}\|\phi_{c_1}-\phi_{c_2}\|_{\dot{H}^{s_1,s_2}} = 0.$$

Now we have that

$$\widehat{u_c}(t,\xi,\eta) = c^{-3/2}\exp(-itc\xi)\widehat{\phi}(\frac{\xi}{c^{1/2}},\frac{\eta}{c}).$$

and furthermore

$$(u_{c_1}(t,.),u_{c_2}(t,.))_{s_1,s_2} =$$

$$(\frac{c_2}{c_1})^{-3/2}\int_{\mathbf{R}^3}\exp(-it\xi(c_1-c_2))|\xi|^{2s_1}|\eta|^{2s_2}\widehat{\phi}(\xi,\eta)\overline{\widehat{\phi}((\frac{c_1}{c_2})^{1/2}\xi,\frac{c_1}{c_2}\eta)}d\xi d\eta.$$

Hence using Riemann-Lebesgue lemma arguments we obtain as above

$$\lim_{n\to\infty}(u_{(n+1)^2}(t,.),u_{n^2}(t,.))_{s_1,s_2} = 0.$$

Moreover

$$\lim_{n\to\infty}\|u_{(n+1)^2}(t,.) - u_{n^2}(t,.)\|^2_{\dot{H}^{s_1,s_2}} = \|\phi\|^2_{\dot{H}^{s_1,s_2}}$$

Therefore we obtain that (2) is locally ill-posed in $\dot{H}^{s_1,s_2}(\mathbf{R}^2)$ for $\beta = 0$ and $\alpha < 0$, when $s_1 + 2s_2 = 1/2$ and $s_1 \geq 0$, $s_2 \geq 0$. This completes the proof of Theorem 2.

There are some difficulties to extend the results of Theorem 2 to the two dimensional case with higher order term. If $d = 2$, $\beta > 0$ and $\alpha = 0$ then (1) has a solitary wave solution of the form

$$u_c(t,x,y) = \phi_c(x - ct, y), \quad c > 0$$

where

$$u_c(0,x,y) = \phi_c(x,y) = c\phi_1(c^{1/4}x, c^{3/4}y) := c\phi(c^{1/4}x, c^{3/4}y).$$

One has

$$\widehat{\phi}_c(\xi,\eta) = \widehat{\phi}(\frac{\xi}{c^{1/4}}, \frac{\eta}{c^{3/4}}).$$

and furthermore

$$\|\phi_c\|^2_{\dot{H}^{s_1,s_2}} = \int_{\mathbf{R}^2} \langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}|\widehat{\phi}(\frac{\xi}{c^{1/4}}, \frac{\eta}{c^{3/4}})|^2 d\xi d\eta. \tag{24}$$

It is easy to see (cf. [11]) that ϕ does not belong to $L^1_{(x,y)}$. Hence one can not define the trace of $\widehat{\phi}(\xi,\eta)$ at zero and therefore it is impossible to pass to the limit as c tends to infinity in (24).

We conclude this section with some remarks for the generalized KP-I equation.

$$\begin{cases} (u_t + \alpha u_{xxx} + \beta u_{xxxxx} + u^p u_x)_x + u_{yy} + u_{zz} = 0 \\ u(0,x,y,z) = \phi(x,y,z) \end{cases} \tag{25}$$

If $\beta = 0$ and $u(t,x,y,z)$ is a solution of (25) then so is

$$u_\lambda(t,x,y,z) = \lambda^{2/p} u(\lambda^3 t, \lambda x, \lambda^2 y, \lambda^2 z)$$

and

$$\|u_\lambda(t,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+2s_2+(5p-4)/2p}\|u(\lambda^3 t,.)\|_{\dot{H}^{s_1,s_2}}.$$

In [10] it is shown that for $1 \leq p < 4/3$ and $\alpha < 0$ (25) possesses nontrivial localized solitary waves of the form

$$u(t,x,y,z) = \phi_c(x - ct, y, z), \quad c > 0,$$

where $\phi_c(x,y,z) = c^{1/p}\phi(c^{1/2}x, cy, cz)$, Using the arguments of the proof of Theorem 2 one can prove that (25) is locally ill-posed in $\dot{H}^{s_1,s_2}$ for $s_1 + 2s_2 + (5p-4)/2p = 0$, $s_1 \geq 0, s_2 \geq 0$. If $\alpha = 0$ and $u(t,x,y,z)$ is a solution of (25) then so is

$$u_\lambda(t,x,y,z) = \lambda^{4/p} u(\lambda^5 t, \lambda x, \lambda^3 y, \lambda^3 z)$$

and

$$\|u_\lambda(t,.)\|_{\dot{H}^{s_1,s_2}} = \lambda^{s_1+3s_2+(8-7p)/2p}\|u(\lambda^5 t,.)\|_{\dot{H}^{s_1,s_2}}.$$

In [10] it is shown that for $1 \leq p < 8/3$ and $\beta > 0$ (25) possesses nontrivial localized solitary waves of the form

$$u(t,x,y,z) = \phi_c(x - ct, y, z), \quad c > 0,$$

where $\phi_c(x,y,z) = c^{1/p}\phi(c^{1/4}x, c^{3/4}y, c^{3/4}z)$, Using the arguments of the proof of Theorem 2 one can prove that (25) is locally ill-posed in $\dot{H}^{s_1,s_2}$ for $s_1 + 2s_2 + (8-7p)/2p = 0$, $s_1 \geq 0, s_2 \geq 0$. In particilar when $p = 8/7$ (25) is locally ill-posed in L^2.

Finally we note that when $\alpha < 0$ and $\beta > 0$, the existence of solitary waves is known for a suitable range of $p's$ (cf. [10]) but the scaling argument leading to ill-posedness does not work.

6. A remark on the (generalized) KdV equation

In this section we shall show that the example providing ill-posedness for the (generalized) KdV equations uses just scaling properties and hence the special form of the solitary waves is not needed. Consider the (generalized) KdV equation

$$u_t + u_{xxx} + u^p u_x = 0 \tag{26}$$

The local ill-posedness of (26) is studied in [7], [8]. It is known that (26) possesses solitary wave solutions of the form

$$u_{c,p}(t,x) = \phi_{c,p}(x - ct),$$

where

$$u_{c,p}(0,x) = c^{\frac{1}{p}}\phi_p(c^{\frac{1}{2}}x), \quad \phi_p(x) = \{\frac{1}{2}(p+2)\text{sech}^2(\frac{1}{2}px)\}^{1/p}.$$

Since the solitary waves of (26) decays exponentialy at infinity one can define the trace of its Fourier transforms at zero and then prove local ill-posedness of (26) for $p = 2$ in H^s, $s < -1/2$ which is the result of [8]. Actually if $p = 2$ then $\widehat{\phi}_{c,2}(\xi) = \widehat{\phi_2}(\xi/c^{1/2})$ and one obtains that for $s < -1/2$

$$\lim_{c\to\infty} \|\phi_{c,2}\|^2_{H^s} = \lim_{c\to\infty} \int \langle\xi\rangle^s \widehat{\phi_2}(\xi/c^{1/2})d\xi = c_s\widehat{\phi_2}(0)$$

and therefore one can easily obtain

$$\lim_{c_1,c_2\to\infty, c_1/c_2\to 1} (\phi_{c_1}, \phi_{c_2})_s = c_s\widehat{\phi_2}(0)$$

and moreover

$$\lim_{c_1,c_2\to\infty, c_1/c_2\to 1} \|\phi_{c_1,2} - \phi_{c_2,2}\|_{H^s} = 0.$$

Further we have

$$\mathcal{F}(u_{c,2})(t,\xi) = \exp(-itc\xi)\widehat{\phi_2}(\xi/c^{1/2}).$$

Due to the Riemann-Lebesgue lemma, the presence of the oscillatory term $\exp(-itc\xi)$ makes the expression $(u_{c_1,2}(t,.), u_{c_2,2}(t,.))_s$ tend to zero as $n \to \infty$ by taking $c_1 = n^2$ and $c_2 = (n+1)^2$. Therefore we obtain that (26) is locally ill-posed in H^s for $p = 2$ when $s < -1/2$. In fact one can easlily show that the sequence $\phi_{c,2}$ converge strongly in H^s, $s < -1/2$ to $\sqrt{2\pi}\delta$ as c tends to infinity. Here δ stays for Dirac delta function. Similarly we can see that $u_{c,2}(t,.)$ converge weakly to zero in H^s. Now the boundedness of $\|\phi_c\|_{H^s}$, $s < -1/2$ provides the locall ill-posedness.

Similarly when $p = 1$ (the "usual" KdV equation) one can prove that $\phi_{c,1}(x)$ converge weakly in H^s, $s < - - 3/2$ to $\sqrt{2\pi}\partial_x\delta(x)$ as c tends to infinity. We also have that $u_{c,1}(t,.)$ converge weakly to zero as c tends to infinity. We also have that

$$\|u_{c,1}(t,.)\|_{H^s} = \|\phi_{c,1}\|_{H^s} = c^{1/2}\|\phi_1\|_{H^s}.$$

Since $\partial_x\delta(x) \in H^s$, $s < -3/2$ the last arguments strongly suggest local ill-posedness of (26) in H^s, $s < -3/2$ when $p = 1$. On the other hand, J. Bourgain has shown in [15] that if one requires that the map data solution $u_0 \mapsto u(t)$ be of class C^3 from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$ then the rsult of [42] for the KdV equation ($s > -3/4$) is sharp [2].

7. Appendix

In this appendix we shall give the proof of Lemma 5.1. By scaling it suffices to prove that $|D_x|^\epsilon(x\phi) \in L^2(\mathbf{R}^2)$, where

$$\phi(x,y) = \frac{1 - x^2 + y^2}{(1 + x^2 + y^2)^2}.$$

We have

$$\phi(x,y) = \frac{1}{1+x^2+y^2} - \frac{2x^2}{(1+x^2+y^2)^2} := \phi_1(x,y) + \phi_2(x,y).$$

A simple computation shows that

$$\begin{aligned}\widehat{\phi_1}(\xi,y) := \mathcal{F}_x(\phi_1)(\xi,y) &= \frac{1}{(1+y^2)^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-i(1+y^2)^{1/2}x\xi}}{1+x^2} dx \\ &= \frac{1}{(1+y^2)^{1/2}} e^{-(1+y^2)^{1/2}|\xi|}\end{aligned}$$

Hence

$$\frac{\partial \widehat{\phi_1}}{\partial \xi}(\xi,y) = -\mathrm{sgn}\xi \cdot e^{-(1+y^2)^{1/2}|\xi|}$$

and

$$\begin{aligned}\int_{\mathbf{R}^2} |\xi|^{2\epsilon} |\frac{\partial \widehat{\phi_1}}{\partial \xi}(\xi,y)|^2 d\xi dy &= \int_{\mathbf{R}^2} |\xi|^{2\epsilon} e^{-2(1+y^2)^{1/2}|\xi|} d\xi dy \\ &= \int_{\mathbf{R}^2} \frac{|X|^{2\epsilon}}{(1+y^2)^{1/2+\epsilon}} e^{-2|X|} dX dy < \infty,\end{aligned}$$

since $\epsilon > 0$. Let now $\psi(x,y) = -\frac{1}{2}\phi_2(x,y)$. A simple calculation yields

$$\widehat{\psi}(\xi,y) := \mathcal{F}_x(\psi)(\xi,y) = \frac{1}{(1+y^2)^{1/2}} \int_{\mathbf{R}^2} \frac{x^2 e^{-i(1+y^2)^{1/2}x\xi}}{(1+x^2)^2} dx.$$

Write $\frac{x^2}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{1}{(1+x^2)^2}$ and recall that (cf. [1])

$$\mathcal{F}(\frac{1}{1+x^2})(\xi) = e^{-|\xi|}$$

and

$$\mathcal{F}(\frac{1}{(1+x^2)^2})(\xi) = c|\xi|^{3/2} K_{3/2}(|\xi|),$$

where $K_{3/2}$ is the modified Bessel function of order 3/2 (cf. [1])

$$K_{3/2}(z) = \sqrt{\frac{\pi}{2z}} \cdot e^{-z}(1 + \frac{1}{z}).$$

[2]In Chapter 8 we shall prove that C^2 regularity of the map data-solution suffices in order to construct the example providing the local ill-posedness

Therefore

$$\begin{aligned}
\widehat{\psi}(\xi,y) &= \frac{1}{(1+y^2)^{1/2}}e^{-(1+y^2)^{1/2}|\xi|} + \frac{1}{(1+y^2)^{1/2}}\mathcal{F}(\frac{1}{(1+x^2)})((1+y^2)^{1/2}\xi) \\
&= \frac{1}{(1+y^2)^{1/2}}e^{-(1+y^2)^{1/2}|\xi|} - \sqrt{\frac{\pi}{2}}\cdot\frac{c}{(1+y^2)^{1/2}}(1+|\xi|(1+y^2)^{1/2})e^{-(1+y^2)^{1/2}|\xi|} \\
&= \frac{1-c\sqrt{\frac{\pi}{2}}}{(1+y^2)^{1/2}}e^{-(1+y^2)^{1/2}|\xi|} - c\sqrt{\frac{\pi}{2}}\cdot|\xi|e^{-(1+y^2)^{1/2}|\xi|} \\
&:= \widehat{\psi_1}(\xi,y) + \widehat{\psi_2}(\xi,y)
\end{aligned}$$

The term $\widehat{\psi_1}(\xi,y)$ leads to a computation similar to that of ϕ_1. Now

$$\begin{aligned}
\frac{\partial\widehat{\psi_2}}{\partial\xi}(\xi,y) &= -c\sqrt{\frac{\pi}{2}}\cdot\mathrm{sgn}\xi\cdot(e^{-(1+y^2)^{1/2}|\xi|} - |\xi|(1+y^2)^{1/2}e^{-(1+y^2)^{1/2}|\xi|}) \\
&:= \widehat{\psi_{21}}(\xi,y) + \widehat{\psi_{22}}(\xi,y)
\end{aligned}$$

Now we have that

$$\begin{aligned}
\int_{\mathbf{R}^2}|\xi|^{2\epsilon}|\widehat{\psi_{21}}(\xi,y)|^2d\xi dy &= c\int_{\mathbf{R}^2}|\xi|^{2\epsilon}e^{-2(1+y^2)^{1/2}|\xi|}d\xi dy \\
&= c\int_{\mathbf{R}^2}\frac{|X|^{2\epsilon}}{(1+y^2)^{1/2+\epsilon}}e^{-2|X|}dX\,dy < \infty,
\end{aligned}$$

since $\epsilon > 0$ and furthermore

$$\begin{aligned}
\int_{\mathbf{R}^2}|\xi|^{2\epsilon}|\widehat{\psi_{22}}(\xi,y)|^2d\xi dy &= c\int_{\mathbf{R}^2}|\xi|^{1+2\epsilon}(1+y^2)^{1/2}e^{-2(1+y^2)^{1/2}|\xi|}d\xi dy \\
&= c\int_{\mathbf{R}^2}\frac{|X|^{1+2\epsilon}}{(1+y^2)^{1/2+\epsilon}}e^{-2|X|}dX\,dy < \infty.
\end{aligned}$$

This completes the proof.

Remark. We conjecture that the result of Lemma 5.1 is valid for any localized solitary wave of KP-I type equation.

CHAPTER 7

The Cauchy problem for the fifth order KP equations

This Chapter essentially contains the joint paper with Jean-Claude Saut [**50**] *(The Cauchy problem for the fifth order KP equations).*

Abstract

We study fifth order KP equations. In 2D the global well-posedness of the Cauchy problem in the energy space for the fifth order KP-I is obtained despite the "bad sign" in the algebraic relation related to the symbol (cf. (7) below). In the case of the fifth order KP-II, global solution with data in $L^2(\mathbf{R}^2)$ for the corresponding integral equation are obtained, removing the additional condition on the data imposed in [**49**]. The case of periodic boundary conditions is also considered. In 2D the local existence for data in Sobolev spaces below $L^2(\mathbf{T}^2)$ is obtained and in particular the global well-posedness for data in $L^2(\mathbf{T}^2)$. In 3D the local well-posedness for data in Sobolev spaces of low order is proven.

AMS subject classification: 35 Q 53, 35 Q 51, 35 A 07.

Résumé

On étudie les équations de KP d'ordre 5. En dimension 2, on montre que le problème de Cauchy est localement bien posé dans l'espace d'énergie pour l'équation de KP-I d'ordre 5, malgré le "mauvais signe" dans la relation algébrique liée au symbole (voir (7) ci-dessous). Pour l'équation de KP-II d'ordre 5, on obtient des solutions globales correspondant à des données initiales dans $L^2(\mathbf{R}^2)$ pour l'équation intégrale associée, sans la condition supplémentaire pour la donnée initiale imposée dans [**49**]. On considère aussi le cas des données initiales périodiques. En dimension 2, on montre que le problème de Cauchy pour l'équation de KP-II d'ordre 5 est localement bien posé pour des données initiales dans des espaces de Sobolev plus gros que $L^2(\mathbf{T}^2)$. En dimension 3, on établit le caractère localement bien posé du problème de Cauchy pour des données appartenant à des espaces de Sobolev d'ordre petit.

Code Matière AMS: 35 Q 53, 35 Q 51, 35 A 07.

1. Introduction

In this Chapter we continue the study of the fifth order Kadomtsev-Petviashvili (KP) equations started in the previous Chapter. These equations occur naturally in the modeling of certain long dispersive waves. For waves propagating in one direction, the fifth order

Korteweg de Vries (KdV) equation

$$\partial_t v + v\partial_x v + \alpha\partial_x^3 v + \partial_x^5 v = 0, \quad \alpha = \pm 1, 0, \tag{1}$$

has been derived in many physical situations when higher order dispersive effects are to be taken into account. Kawahara [**38**] has introduced it (equation (1) is often called the Kawahara equation) in order to model solitary waves with an oscillatory structure, which can not be obtained from the usual KdV equation. More specifically, (1) with $\alpha = \pm 1$ has been derived to model one dimensional gravity-capillary waves when the Bond number which measures the surface tension effects is close to the critical value 1/3 (cf. [**28, 26, 30**]), or to model surface water waves under the presence of an elastic ice plate [**46**]. On the other hand, (1) with $\alpha = 0$ arises as the approximation to small amplitude, long waves at the surface of shallow water having the critical depth 0.54 cm.

Taking into account weak transverse effects in the y direction leads (cf. [**33**]) to KP type equations of the form

$$(\partial_t u + \alpha\partial_x^3 u - \partial_x^5 u + uu_x)_x + \epsilon u_{yy} = 0. \tag{2}$$

Here $\epsilon = -1$ corresponds to the "focusing" case (KP-I type), while $\epsilon = 1$ corresponds to the defocusing case (KP-II type). The solitary waves of these equations have been recently studied in [**2, 10, 11, 27, 29, 34, 35, 36**]. We will deal here with the Cauchy problem. Thus in the case of two space dimensions we shall study the following equations

$$(\partial_t u - \partial_x^5 u + \alpha\partial_x^3 u + uu_x)_x - u_{yy} = 0, \quad \text{(fifth order KP-I)} \tag{3}$$

and

$$(\partial_t u - \partial_x^5 u + \alpha\partial_x^3 u + uu_x)_x + u_{yy} = 0, \quad \text{(fifth order KP-II)}\ . \tag{4}$$

Here u is a real valued function and $(t, x, y) \in \mathbf{R}^3$. Both equations (3) and (4) are completed by the initial condition

$$u(0, x, y) = \phi(x, y). \tag{5}$$

The equations (3) and (4) are infinite dimensional Hamiltonian systems with Hamiltonian

$$H(\phi) = \frac{1}{2}\int |\partial_x^2\phi|^2 + \frac{\alpha}{2}\int |\partial_x\phi|^2 \pm \frac{1}{2}\int |\partial_x^{-1}\partial_y\phi|^2 - \frac{1}{6}\int \phi^3,$$

where the sign + (resp. −) corresponds to (3) (resp. (4)). The Hamiltonians are (at least formally) constant along the trajectories of (3), (4), i.e.

$$H(u(t)) = H(\phi).$$

Note that the main contribution of the quadratic leading part of the Hamiltonian is positive only in the case of KP-I. The L^2 norms are also conserved along the trajectories due to the gauge invariance, i.e.

$$\int |u(t)|^2 = \int |\phi|^2.$$

The equations (3), (4) can be written in the form

$$\sigma_\pm(D_t, D_x, D_y)u = -iuu_x, \tag{6}$$

where $D_t = -i\partial_t$, $D_x = -i\partial_x$, $D_y = -i\partial_y$ and $\sigma_\pm(D_t, D_x, D_y)$ are Fourier multipliers with symbols

$$\sigma_\pm(\tau,\xi,\eta) = \tau - \xi^5 - \alpha\xi^3 \mp \frac{\eta^2}{\xi}.$$

The symbol σ_+ corresponds to KP-I and σ_- to KP-II. We have the following algebraic identity for $\sigma_\pm$, which was first introduced in the context of the "usual" KP-II equation by J. Bourgain (cf. [**14**])

$$\sigma_\pm(\tau_1,\xi_1,\eta_1) + \sigma_\pm(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1) - \sigma_\pm(\tau,\xi,\eta) = \tag{7}$$
$$\xi_1\xi(\xi-\xi_1)\left\{5(\xi^2-\xi\xi_1+\xi_1^2) \mp \left(\frac{\xi_1\eta-\xi\eta_1}{\xi_1\xi(\xi-\xi_1)}\right)^2 + 3\alpha\right\}$$

Note that in the KP-II case (7) permits to give a lower bound for the maximum modulus of the three terms in the the left-hand side of (7). This bound enables one to compensate the derivative loss in the nonlinear term uu_x (cf [**49**]).

In [**49**] the local well posedness of (4) in Sobolev spaces of negative indices is obtained. There are two main ingredients in the proof. The first one is a global smoothing effect for the linearized equation (cf. [**5**]) injected into the framework of the Fourier transform restriction spaces introduced by J. Bourgain. The second is the essential use of the algebraic relation (7). In the case of σ_+ (i.e. KP-I) the relation (7) does not give a priori a lower bound. Nevertheless, in this Chapter we shall be able to overcome this difficulty, mainly because of the higher dispersion (cf. Theorem 1 below). Unfortunately our methods do not work at the moment in the case of the "usual" KP-I equation

$$(\partial_t u + \partial_x^3 u + uu_x)_x - u_{yy} = 0. \tag{8}$$

On the other hand, using the parabolic regularisation method it is possible to prove local well-posedness of (8) in H^s, $s > 2$ (cf. [**31**]). The proof does not use the specific structure of the KP equations and could be performed for quite general evolution equations. The condition for s is in order to control the L^∞ norm of the x derivative of the solution and it seems to be very restrictive. Much deeper results concerning the KP-II equation have recently appeared. The "usual" KP-II equation has the form

$$(\partial_t u + \partial_x^3 u + uu_x)_x + u_{yy} = 0. \tag{9}$$

In [**14**] local solutions of (9) in L^2 are obtained with emphasize on the case of periodic boundary conditions. The proof uses dyadic decompositions related to the symbol of the linearized operator. The full space problem is also considered. A short proof of local well-posedness in $H^s(\mathbf{R}^2)$, $s > 0$, which uses only Strichartz inequalities, is done in [**56**]. The same result is obtained in [**32**], where the nonlinear estimates make use of the simple calculus techniques introduced by C. Kenig, G. Ponce and L. Vega in the context of the KdV equation (cf. [**42**]). In [**57**] the local well-posedness of (9) in Sobolev spaces with negative indices with respect to x variable is obtained. However, an additional condition on the initial data is imposed. More precisely, it is assumed that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in L^2$, which is rather restrictive. In [**52**] and [**58**] this restriction on the data is removed. Due to the conservation of the L^2 norm the solutions are extended globally in time for L^2 initial data. Global existence for data below L^2 is established in [**53**] and [**58**]. The proof uses the local existence techniques and a new idea of decomposing the initial data in high and low Fourier modes due to J. Bourgain in the context of NLS (cf. [**15**]).

The equation (3) possesses solitary wave solutions, i.e. solutions of type $u(x-ct,y)$, where c is the propagation speed (cf. [**10**],[**11**]). Actually by the aid of concentration compactness principle solitary waves for some generalization of (3) could be obtained. Namely one is allowed to consider $u^p u_x$ instead of uu_x as a nonlinear interaction in a suitable range for p. In [**49**] solitary wave solutions were used in an essential way to prove local ill-posedness results for some KP-I type equations (including (4) with $\alpha = 0$).

In this Chapter we are going to prove the local well-posedness of (3) for initial data in a suitable anisotropic Sobolev space. Further we extend the solutions globally in time due to the conservation of the energy and thus we obtain the global well-posedness of the fifth order KP-I equation in the natural energy space. Note that our result are the first of this kind for KP-I equations. In [**55**] M. Tom got by energy methods the existence of global weak solutions in the energy space but uniqueness was not proven. The proof of the local well-posedness result uses the Fourier transform restriction method due to J. Bourgain. The global solutions are obtained for data in the energy space, i.e such that $H(\phi) < \infty$. In the case (4) local and global well-posedness in L^2 are obtained, removing the restriction on the initial data imposed in [**49**]. Now we state our results in the continuous case. A precise formulation will be given in Theorems 4.1, 4.2, 5.1.

THEOREM 1. (local well-posedness for higher order KP-I) *The initial value problem (3)-(5) is locally well-posed for initial data satisfying:*

$$\|\phi\|_{L^2} + \||D_x|^s\phi\|_{L^2} + \||D_y|^k\phi\|_{L^2} < \infty, \quad s \geq 1,\ k \geq 0, \quad |\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbf{R}^2).$$

THEOREM 2. (global well-posedness for higher order KP-I) *The initial value problem (3)-(5) is globally well-posed for initial data $\phi \in L^2(\mathbf{R}^2)$ satisfying $H(\phi) < \infty$.*

The next result removes the constraint on ϕ which was needed in [**49**].

THEOREM 3. (global well-posedness for higher order KP-II) *The initial value problem (4)-(5) is globally well-posed for initial data satisfying:*

$$\phi \in L^2, \quad |\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbf{R}^2).$$

Remarks. There are several extensions of the Theorems above that we can think of. One may conjecture that global well-posedness of (3)-(5) (KP-I) holds for data in L^2. For that purpose we need a further extension of the arguments of Case 3 of the proof of Theorem 2.1 below. On the other hand local well-posedness of (4)-(5) (KP-II) can be obtained in Sobolev spaces with negative indices with respect to the x variable even without the restriction on the data imposed in [**49**]. Then one could use the same arguments as in [**58**] in order to obtain global well-posedness of (4)-(5) below L^2.

Now we turn to the periodic boundary conditions. We shall consider only the higher order KP-II equation

$$(\partial_t u - \partial_x^5 u + uu_x)_x + \Delta_\perp u = 0, \tag{10}$$

where $\Delta_\perp = \partial_y^2$ in 2D and $\Delta_\perp = \partial_y^2 + \partial_z^2$ in 3D. The equation (10) is completed with the initial condition

$$u(0) = \phi \;:\; \mathbf{T}^d \mapsto \mathbf{R}, \quad d = 2, 3. \tag{11}$$

Here $\mathbf{T}^d$ stays for the d-dimensional torus. One of the main difficulties in the case of periodic boundary conditions is the absence of Strichartz inequalities. The point is that the free evolution does not possess any dispersion property. However in the case of periodic KdV or Schrödinger equations there is a partial knowledge of Strichartz inequalities (cf. [**12, 13**]). Similar inequalities in the context of KP are not known. Nevertheless in [**14**] J. Bourgain proves the local well-posedness for the 2D KP-II with periodic data by an analysis of multiple Fourier series. In this Chapter we shall adapt this approach for the case of higher order KP-II. In order to state our well-posedness results we define the Sobolev type space $Y^s(\mathbf{T}^d)$, $d = 2, 3$ measuring the regularity with respect to x, equipped with the norm

$$\|\phi\|_{Y^s} = \|\langle m\rangle^s \widehat{\phi}(m, .)\|_{L^2(\mathbf{T}^d)}.$$

Now we state the result in 2D.

THEOREM 4. *Let $d = 2$. Then the Cauchy problem (10)-(11) is locally well-posed for initial data:*

$$\phi \in Y^s(\mathbf{T}^2),\; s \geq -1/8, \quad \int_{\mathbf{T}^1} \phi(x, y)dx = const.$$

In particular when $s = 0$ we have global well-posedness due to the L^2 conservation law.

In 3D we shall prove the following result.

THEOREM 5. *Let $d = 3$. Then the Cauchy problem (10)-(11) is locally well-posed for initial data:*

$$\phi \in Y^s(\mathbf{T}^3),\; s > 0, \quad \int_{\mathbf{T}^1} \phi(x, y, z)dx = const.$$

Notation. By $\hat{}$ or $\mathcal{F}$ we denote the Fourier transform and by $\mathcal{F}^{-1}$ the inverse transform. $\|.\|_{L^p}$ denotes the norm in the Lebesgue space L^p. The notation $a\pm$ means $a \pm \varepsilon$ for arbitrary small $\varepsilon > 0$. Constants are denoted by c and may change from line to line. $A \sim B$ means that there exists a constant $c \geq 1$ such that $\frac{1}{c}|A| \leq |B| \leq c|A|$. For any positive A and B the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there exists a positive constant c such that $A \leq cB$ (resp. $A \geq cB$). For $A, B \in \mathbf{R}$ we set $A \vee B = \max\{A, B\}$ and $A \wedge B = \min\{A, B\}$. By mes$(A)$ we denote the measure of a set A.

2. Preliminaries

In order to prove Theorem 1, Theorem 2 and Theorem 3, we shall apply a Picard fixed point argument in a suitable functional space to the integral equation corresponding to (3) or (4)

$$u(t) = U^{\pm}(t)\phi - \frac{1}{2}\int_0^t U^{\pm}(t - t')\partial_x(u^2(t'))dt'. \tag{12}$$

Here $U^{\pm}(t)$ are the unitary groups which define the free evolutions of (3) and (4), i.e.

$$U^{\pm}(t) = \exp(-itp^{\pm}(D_x, D_y)),$$

where $p^{\pm}(D_x, D_y)$ are Fourier multipliers with symbols

$$p^+(\xi,\eta) = -\xi^5 - \alpha\xi^3 - \frac{\eta^2}{\xi}, \text{ for the fifth order KP-I}$$

and

$$p^-(\xi,\eta) = -\xi^5 - \alpha\xi^3 + \frac{\eta^2}{\xi}, \text{ for the fifth order KP-II.}$$

We define an anisotropic Sobolev space $H^{s,k}_{x,y}(\mathbf{R}^2)$ by

$$H^{s,k}_{x,y}(\mathbf{R}^2) = \left\{\phi \in \mathcal{S}'(\mathbf{R}^2) : \|\phi\|_{H^{s,k}_{x,y}} < \infty\right\},$$

where

$$\|\phi\|_{H^{s,k}_{x,y}} = \|(1-\partial_x^2)^{s/2}(1-\partial_y^2)^{k/2}\phi\|_{L^2_{x,y}}$$

The spaces $H^{s,k}_{x,y}(\mathbf{R}^2)$ are natural ones for the initial data of KP type equations since their homogeneous versions are invariant under scale transformations which preserve the KP equations. Further we define the Bourgain type spaces associated to fifth order KP equations. Let $X^{b,s,k}$ be the Sobolev space equipped with the norm

$$\|u; X^{b,s,k}(\mathbf{R}^3)\| = \|\langle\tau\rangle^b\langle\xi\rangle^s\langle\eta\rangle^k\widehat{u}(\tau,\xi,\eta); L^2_{\tau,\xi,\eta}\|,$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. We denote by $B^{b,s,k}_{\pm}(\mathbf{R}^3)$ the spaces equipped with the norm

$$\|u; B^{b,s,k}_{\pm}\| = \|U^{\pm}(-t)u; X^{b,s,k}\|. \tag{13}$$

Since $\mathcal{F}(U^{\pm}(-t)u)(\tau,\xi,\eta) = \widehat{u}(\tau - p^{\pm}(\xi,\eta),\xi,\eta)$ we obtain that, in terms of the Fourier transform variables, the norm of $B^{b,s,k}_{\pm}$ can be expressed as

$$\|u; B^{b,s,k}_{\pm}\| = \|\langle\tau + p^{\pm}(\xi,\eta)\rangle^b\langle\xi\rangle^s\langle\eta\rangle^k\widehat{u}(\tau,\xi,\eta); L^2_{(\tau,\xi,\eta)}\|.$$

Let $I \subset \mathbf{R}$ be an interval. Then we define a localized Bourgain space $B^{b,s,k}_{\pm}(I)$ equipped with the norm

$$\|u\|_{B^{b,s,k}_{\pm}(I)} = \inf_{w\in B^{b,s,k}_{\pm}}\{\|w\|_{B^{b,s,k}_{\pm}}, w(t) = u(t) \text{ on } I\}.$$

The following inequality, proven in [49] is a version of the smoothing effect established in [5]

PROPOSITION 1. (cf. [49], Corollary 1 of Lemma 3). *The following inequality holds*

$$\|\mathcal{F}^{-1}(|\xi|^{\frac14}\langle\tau + p^{\pm}(\xi,\eta)\rangle^{-\frac12-}|\widehat{u}(\tau,\xi,\eta)|)\|_{L^4_{(t,x,y)}} \lesssim \|u\|_{L^2}. \tag{14}$$

We shall make use of simple calculus inequalities.

PROPOSITION 2. *Let $\gamma > 1$. Then for any $a \in \mathbf{R}$ the following inequalities hold*

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^\gamma\langle t-a\rangle^\gamma} \lesssim \langle a\rangle^{-\gamma}, \tag{15}$$

$$\int_{-\infty}^{\infty} \frac{dt}{\langle t\rangle^\gamma|t-a|^{1/2}} \lesssim \langle a\rangle^{-1/2}. \tag{16}$$

Let ψ be a cut-off function such that

$$\psi \in C_0^\infty(\mathbf{R}), \text{ supp } \psi \subset [-2,2], \psi = 1 \text{ on the interval } [-1,1].$$

We consider a cut-off version of (12)

$$u(t) = \psi(t)U^{\pm}(t)\phi - \frac{1}{2}\psi(t/T)\int_0^t U^{\pm}(t-t')\partial_x(u^2(t'))dt'. \tag{17}$$

We shall solve (17) globally in time. To the solutions of (17) will correspond local solutions of (12) in the time interval $[-T,T]$. We have the following estimates for the two terms in the right-hand side of (17).

PROPOSITION 3. (linear estimates) *Let* $-\frac{1}{2} < b' \leq 0 \leq b \leq b'+1$, $s \geq 0$ *and* $k \geq 0$. *Then the following inequalities hold*

$$\|\psi(t)U^{\pm}(t)\phi; B^{b,s,k}\| \lesssim \|\phi\|_{H^{s,k}_{x,y}}, \tag{18}$$

$$\|\psi(t/T)\int_0^t U^{\pm}(t-t')\partial_x(u^2(t'))dt'; B^{b,s,k}\| \lesssim T^{1-b+b'}\|uu_x; B^{b',s,k}\|. \tag{19}$$

Proof. The proof of (18) is a direct consequence of definition (13). Clearly (19) is equivalent to

$$\|Lg; H_t^b\| \lesssim T^{1-b+b'}\|g; H_t^{b'}\|, \tag{20}$$

where the operator L is defined by

$$(Lg)(t) = \psi(t/T)\int_0^t g(t')dt'.$$

Note that (20) is one dimensional and does not depend on the special structure of the equations (3), (4). In addition since $b-b' \leq 1$, (20) asserts that the integration gains one derivative, which is natural. For the proof of (20) we refer to [**23**], Lemma 3.2.

We shall apply Proposition 3 with $b = \frac{1}{2}+$ and $b' = -\frac{1}{2}+$. A small factor will appear in the right-hand side of (19). In order to apply a fixed point argument we need the following crucial bilinear estimates which will be proved in the next sections.

THEOREM 2.1. (KP-I) *Let* $s \geq 1$ *and* $k \geq 0$. *Then*

$$\|uu_x\|_{B_+^{-\frac{1}{2}+,s,k}} \lesssim \|u\|^2_{B_+^{\frac{1}{2}+,s,k}}. \tag{21}$$

THEOREM 2.2. (KP-II) *The following inequality holds*

$$\|uu_x\|_{B_-^{-\frac{1}{2}+,0,0}} \lesssim \|u\|^2_{B_-^{\frac{1}{2}+,0,0}}. \tag{22}$$

3. The bilinear estimates

3.1. The fifth order KP-I equation. Let $\zeta = (\xi, \eta)$ and set

$$\sigma := \sigma_+(\tau, \zeta) = \tau - \xi^5 - \alpha\xi^3 - \frac{\eta^2}{\xi}, \quad \sigma_1 := \sigma_+(\tau_1, \zeta_1), \quad \sigma_2 := \sigma_+(\tau - \tau_1, \zeta - \zeta_1).$$

By duality, (21) is equivalent to

$$J \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}, \tag{23}$$

where

$$J = \left| \int \int K(\tau, \zeta, \tau_1, \zeta_1)\widehat{u}(\tau_1, \zeta_1)\widehat{v}(\tau - \tau_1, \zeta - \zeta_1)\widehat{w}(\tau, \zeta) d\tau_1 d\zeta_1 d\tau d\zeta \right|$$

and

$$K(\tau, \zeta, \tau_1, \zeta_1) = \frac{|\xi|\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi - \xi_1\rangle^s\langle\sigma\rangle^{\frac{1}{2}-}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{\frac{1}{2}+}} \frac{\langle\eta\rangle^k}{\langle\eta_1\rangle^k\langle\eta - \eta_1\rangle^k}$$

Since for $k \geq 0$ the quantity

$$\frac{\langle\eta\rangle^k}{\langle\eta_1\rangle^k\langle\eta - \eta_1\rangle^k}$$

is bounded we shall suppose that $k = 0$ hereafter. Without loss of generality we can assume that $\widehat{u} \geq 0$, $\widehat{v} \geq 0$ and $\widehat{w} \geq 0$. We consider several cases for $(\tau, \zeta, \tau_1, \zeta_1)$.

Case 1. $|\xi| \leq 100$.

Case 1.1. $|\xi_1| \geq 200$. In this case $|\xi - \xi_1| \geq 100$. By the aid of Proposition 2.1 we estimate the contribution to J in this case by

$$\begin{aligned} &\|\mathcal{F}^{-1}(|\xi_1|^{\frac{1}{4}}\langle\sigma_1\rangle^{-\frac{1}{2}-}\widehat{u}(\tau_1, \zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(|\xi - \xi_1|^{\frac{1}{4}}\langle\sigma_2\rangle^{-\frac{1}{2}-}\widehat{v}(\tau - \tau_1, \zeta - \zeta_1))\|_{L^4}\|w\|_{L^2} \\ \lesssim\ & \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \end{aligned}$$

Case 1.2. $|\xi_1| \leq 200$. Denote by J_{12} the contribution of this region to J. Cauchy-Schwarz inequality yields

$$J_{12} \leq \int I_{12}(\tau, \zeta) \left\{ \int \int |\widehat{u}(\tau_1, \zeta_1)\widehat{v}(\tau - \tau_1, \zeta - \zeta_1)|^2 d\tau_1 d\zeta_1 \right\}^{\frac{1}{2}} \widehat{w}(\tau, \zeta) d\tau d\zeta.$$

where by the aid of (15) we obtain the following bound for $I_{12}(\tau, \zeta)$

$$I_{12}(\tau, \zeta) \lesssim \frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}} \left\{ \int \frac{d\zeta_1}{\langle\sigma_1 + \sigma_2\rangle^{1+}} \right\}^{\frac{1}{2}}.$$

We perform the change of variables $\eta_1 \mapsto \nu$

$$\nu = \sigma_1 + \sigma_2.$$

Then

$$\left|\frac{\partial \nu}{\partial \eta_1}\right| = \frac{2|\xi(\xi_1\eta - \xi\eta_1)|}{|\xi\xi_1(\xi-\xi_1)|}$$

$$= \frac{2|\xi|\,|\sigma + \xi\xi_1(\xi-\xi_1)(5\xi^2 - 5\xi\xi_1 + 5\xi_1^2 + 3\alpha) - \nu|^{\frac{1}{2}}}{|\xi\xi_1(\xi-\xi_1)|^{\frac{1}{2}}}$$

$$\gtrsim |\xi|^{\frac{1}{2}}\,|\sigma + \xi\xi_1(\xi-\xi_1)(5\xi^2 - 5\xi\xi_1 + 5\xi_1^2 + 3\alpha) - \nu|^{\frac{1}{2}}$$

Now using (16) we arrive at

$$I_{12}(\tau,\zeta) \lesssim \frac{|\xi|^{\frac{3}{4}}}{\langle\sigma\rangle^{\frac{1}{2}-}}\left\{\int\!\!\int \frac{d\xi_1 d\nu}{|\sigma + \xi\xi_1(\xi-\xi_1)(5\xi^2 - 5\xi\xi_1 + 5\xi_1^2 + 3\alpha) - \nu|^{\frac{1}{2}}\,\langle\nu\rangle^{1+}}\right\}^{\frac{1}{2}}$$

$$\lesssim \left\{\int \frac{d\xi_1}{\langle\sigma + \xi\xi_1(\xi-\xi_1)(5\xi^2 - 5\xi\xi_1 + 5\xi_1^2 + 3\alpha) - \nu\rangle^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$$

$$\leq \text{const},$$

since $|\xi_1|$ is bounded. Hence using Cauchy-Schwarz we obtain

$$J_{12} \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 2. $|\xi| \geq 100$, $\min\{|\xi_1|, |\xi - \xi_1|\} \leq 1$. Denote by J_2 the contribution of this region to J. Cauchy-Schwarz inequality yields

$$J_2 \leq \int I_2(\tau,\zeta)\left\{\int |\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1\right\}^{\frac{1}{2}} \widehat{u}(\tau,\zeta)d\tau d\zeta,$$

where due to (15)

$$I_2(\tau,\zeta) \lesssim \frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}}\left\{\int \frac{d\zeta_1}{\langle\sigma_1+\sigma_2\rangle^{1+}}\right\}^{\frac{1}{2}}.$$

We perform the change of variables $(\xi_1,\eta_1) \mapsto (\mu,\nu)$

$$\mu = \xi\xi_1(\xi-\xi_1)(5\xi^2 - 5\xi\xi_1 + 5\xi_1^2 + 3\alpha), \quad \nu = \sigma_1 + \sigma_2.$$

Then

$$\left|\frac{\partial \mu}{\partial \xi_1}\right| = |\xi(\xi - 2\xi_1)||5\xi^2 - 10\xi\xi_1 + 10\xi_1^2 + 3\alpha|$$

$$\gtrsim |\xi|^3|\xi - 2\xi_1| \text{ (since } |\xi| \gg 1)$$

$$\sim |\xi|^{\frac{3}{2}}\sqrt{|\xi|^5 - |\xi|^3\xi_1(\xi-\xi_1)}$$

$$\gtrsim |\xi|^{\frac{3}{2}}\sqrt{|\xi|^5 - 2|\mu|}.$$

and

$$\left|\frac{\partial \nu}{\partial \eta_1}\right| = \frac{2|\xi||\sigma+\mu-\nu|^{\frac{1}{2}}}{|\xi\xi_1(\xi-\xi_1)|^{\frac{1}{2}}} \gtrsim \frac{|\xi|^2|\sigma+\mu-\nu|^{\frac{1}{2}}}{|\mu|^{\frac{1}{2}}}$$

Hence

$$\left|\frac{\partial(\xi_1,\eta_1)}{\partial(\mu,\nu)}\right| \lesssim \frac{|\mu|^{\frac{1}{2}}}{|\xi|^{\frac{7}{2}}|\sigma+\mu-\nu|^{\frac{1}{2}}(|\xi|^5-2|\mu|)^{\frac{1}{2}}}$$

and moreover

$$I_2(\tau,\zeta) \lesssim \frac{1}{|\xi|^{\frac{3}{4}}\langle\sigma\rangle^{\frac{1}{2}-}}\left\{\int\int \frac{|\mu|^{\frac{1}{2}}d\mu d\nu}{|\sigma+\mu-\nu|^{\frac{1}{2}}(|\xi|^5-2|\mu|)^{\frac{1}{2}}\langle\nu\rangle^{1+}}\right\}^{\frac{1}{2}}$$

Now a use of (16) yields

$$I_2(\tau,\zeta) \lesssim \frac{1}{|\xi|^{\frac{3}{4}}\langle\sigma\rangle^{\frac{1}{2}-}}\left\{\int \frac{|\mu|^{\frac{1}{2}}d\mu}{\langle\sigma+\mu\rangle^{\frac{1}{2}}(|\xi|^5-2|\mu|)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

Since $\min\{|\xi_1|,|\xi-\xi_1|\} \leq 1$ we have that $|\mu| \leq 30|\xi|^4$ and therefore

$$\begin{aligned} I_2(\tau,\zeta) &\lesssim \frac{1}{|\xi|^{\frac{3}{4}}\langle\sigma\rangle^{\frac{1}{2}-}}\frac{|\xi|}{|\xi|^{\frac{5}{4}}}\left\{\int_{|\mu|\leq c|\xi|^4} \frac{d\mu}{\langle\sigma+\mu\rangle^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \\ &\lesssim \frac{1}{|\xi|\langle\sigma\rangle^{\frac{1}{2}-}}\left\{\langle\sigma\rangle^{\frac{1}{4}}+|\xi|\right\} \\ &\leq \text{const.} \end{aligned}$$

Hence using Cauchy-Schwarz we arrive at

$$J_2 \lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Case 3. $|\xi| \geq 100$, $\frac{1}{100}|\xi| \leq \min\{|\xi_1|,|\xi-\xi_1|\}$. Denote by J_3 the contribution of this region J. Clearly

$$\frac{|\xi|\langle\xi\rangle^s}{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s} \leq \text{ const } \leq |\xi_1|^{\frac{1}{4}}|\xi-\xi_1|^{\frac{1}{4}}.$$

Therefore as in Case 1.1 we obtain a bound for J_3

$$\begin{aligned} &\|\mathcal{F}^{-1}(|\xi_1|^{\frac{1}{4}}\langle\sigma_1\rangle^{-\frac{1}{2}-}\widehat{u}(\tau_1,\zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(|\xi-\xi_1|^{\frac{1}{4}}\langle\sigma_2\rangle^{-\frac{1}{2}-}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^4}\|w\|_{L^2} \\ \lesssim\ &\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}. \end{aligned}$$

Thus in the rest of the proof of (23) we can assume:

$$(24) \qquad |\xi| \geq 100, \quad 1 \leq \min\{|\xi_1|,|\xi-\xi_1|\} \leq \frac{1}{100}|\xi|.$$

Case 4. (24) and $|\sigma_1+\sigma_2-\sigma| \geq \frac{1}{100}|\xi|^4$. Denote by J_4 the contribution of this region to J. On the support of J_4 one has

$$(25) \qquad \max\{|\sigma_1|,|\sigma_2|,|\sigma|\} \gtrsim |\xi|^4.$$

Suppose that $|\sigma|$ dominates in (25). Then

$$K(\tau,\zeta,\tau_1,\zeta_1) \lesssim \frac{|\xi_1|^{\frac{1}{4}}|\xi-\xi_1|^{\frac{1}{4}}}{\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_1\rangle^{\frac{1}{2}+}}$$

and therefore we bound J_4 by

$$\|\mathcal{F}^{-1}(|\xi_1|^{\frac{1}{4}}\langle\sigma_1\rangle^{-\frac{1}{2}-}\widehat{u}(\tau_1,\zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(|\xi-\xi_1|^{\frac{1}{4}}\langle\sigma_2\rangle^{-\frac{1}{2}-}\widehat{v}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^4}\|w\|_{L^2}$$

$$\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

If $|\sigma_1|$ dominates then $|\sigma_1|^{-\frac{1}{2}-}|\sigma|^{-\frac{1}{2}+} \le |\sigma_1|^{-\frac{1}{2}+}|\sigma|^{-\frac{1}{2}-}$ and we can use the same argument as when $|\sigma|$ dominates just replacing σ with σ_1. A similar argument could be performed when $|\sigma_2|$ dominates.

Case 5. (24) and $|\sigma_1+\sigma_2-\sigma| \le \frac{1}{100}|\xi|^4$. Denote by J_5 the contribution of this region to J. By symmetry we can assume that $|\xi_1| \le \frac{1}{100}|\xi|$. Consider the dyadic levels

(26) $$|\xi_1| \sim K, \quad K-\text{ dyadic }, \quad K \ge 1.$$

Denote by J_5^K the contribution of (26) to J_5. Since

$$J_5 \lesssim \sum_K J_5^K$$

we need an estimate for J_5^K. Cauchy-Schwarz inequality yields

$$J_5^K \le \int I_5^K(\tau,\zeta)\left\{\int\int |\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)|^2 d\tau_1 d\zeta_1\right\}^{\frac{1}{2}} \widehat{w}(\tau,\zeta)d\tau d\zeta.$$

Since $\langle\xi\rangle^s\langle\xi_1\rangle^{-s}\langle\xi-\xi_1\rangle^{-s} \le cK^{-s}$ we have the following bound for $I_5^K(\tau,\zeta)$

$$I_5^K(\tau,\zeta) \lesssim \frac{|\xi|}{K^s}\left\{\int\int \frac{d\zeta_1}{\langle\sigma_1+\sigma_2\rangle^{1+}}\right\}^{\frac{1}{2}}.$$

Perform a change of variables $(\xi_1,\eta_1) \mapsto (\mu,\nu)$

$$\mu = \xi\xi_1(\xi-\xi_1)(5\xi^2-5\xi\xi_1+5\xi_1^2+3\alpha), \quad \nu = \sigma_1+\sigma_2-\sigma.$$

We have similarly to Case 2

$$\left|\frac{\partial\mu}{\partial\xi_1}\right| \gtrsim |\xi|^{\frac{3}{2}}(\xi-2|\mu|)^{\frac{1}{2}}, \quad \left|\frac{\partial\nu}{\partial\eta_1}\right| \gtrsim \frac{|\xi|^2|\mu-\nu|^{\frac{1}{2}}}{|\mu|^{\frac{1}{2}}}.$$

Hence

$$\left|\frac{\partial(\xi_1,\eta_1)}{\partial(\mu,\nu)}\right| \lesssim \frac{|\mu|^{\frac{1}{2}}}{|\xi|^{\frac{7}{2}}|\mu-\nu|^{\frac{1}{2}}(|\xi|^5-2|\mu|)^{\frac{1}{2}}}.$$

Therefore we arrive at the following bound for $I^K(\tau,\zeta)$

$$I_5^K(\tau,\zeta) \lesssim \frac{1}{|\xi|^{\frac{3}{4}}K^s}\left\{\int\int \frac{|\mu|^{\frac{1}{2}}d\mu d\nu}{|\mu-\nu|^{\frac{1}{2}}(|\xi|^5-2|\mu|)^{\frac{1}{2}}\langle\nu+\sigma\rangle^{1+}}\right\}^{\frac{1}{2}}$$

Since $|\mu| \sim K|\xi|^4$ and $|\nu| \leq \frac{1}{100}|\xi|^4$ we obtain

$$\frac{|\mu|^{\frac{1}{2}}}{|\mu-\nu|^{\frac{1}{2}}} \lesssim 1, \quad \frac{1}{(|\xi|^5 - 2|\mu|)^{\frac{1}{2}}} \lesssim \frac{1}{|\xi|^{\frac{5}{2}}}.$$

Hence

$$\begin{aligned} I_5^K(\tau,\zeta) &\lesssim \frac{1}{|\xi|^2 K^s}\left\{\int_{|\mu|\sim K|\xi|^4}\int_{-\infty}^{\infty} \frac{d\mu d\nu}{\langle \nu+\sigma\rangle^{1+}}\right\}^{\frac{1}{2}} \\ &\lesssim \frac{1}{|\xi|^2 K^s}\left\{\int_{|\mu|\sim K|\xi|^4} d\mu\right\}^{\frac{1}{2}} \\ &\lesssim \frac{1}{K^{s-\frac{1}{2}}} \end{aligned}$$

and furthermore

$$J_5^K \lesssim \frac{1}{K^{s-\frac{1}{2}}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Summing over K provides a bound for J_5. This completes the proof of (23).

3.2. The fifth order KP-II equation. Set

$$\sigma := \sigma_-(\tau,\zeta) = \tau - \xi^5 - \alpha\xi^3 + \frac{\eta^2}{\xi}, \quad \sigma_1 := \sigma_-(\tau_1,\zeta_1), \quad \sigma_2 := \sigma_-(\tau-\tau_1,\zeta-\zeta_1).$$

Due to the algebraic relation (7) we obtain

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \gtrsim |\xi_1||\xi-\xi_1||\xi|^3. \tag{27}$$

A duality argument writes (22) in the form

$$\left|\int\int K(\tau,\zeta,\tau_1,\zeta_1)\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}(\tau,\zeta)d\tau_1 d\zeta_1 d\tau d\zeta\right| \lesssim$$

$$\|u\|_{L^2}\|v\|_{L^2},\|w\|_{L^2},$$

where

$$K(\tau,\zeta,\tau_1,\zeta_1) = \frac{|\xi|}{\langle\sigma\rangle^{\frac{1}{2}-}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{\frac{1}{2}+}}$$

Without loss of generality we can assume that $\widehat{u} \geq 0$, $\widehat{v} \geq 0$ and $\widehat{w} \geq 0$. The cases when $|\xi|$, $|\xi_1|$ or $|\xi-\xi_1|$ are near to zero can be treated exactly as in the proof of Theorem 2.1, Case 1 and Case 2. When $|\xi|$, $|\xi_1|$ and $|\xi-\xi_1|$ are away from zero then we are in position to apply the arguments of Case 4 of the proof of Theorem 2.1 since in this case (27) provides the bound (25) needed for the proof (which is not always available in the case of higher order KP-I).

4. The local well-posedness

In this Section we shall prove the local existence results concerning fifth order KP equations with initial data defined on $\mathbf{R}^2$. We state the complete version of Theorem 1.

THEOREM 4.1. *Let $s \geq 1$ and $k \geq 0$. Then for any $\phi \in H^{s,k}_{x,y}(\mathbf{R}^2)$, such that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbf{R}^2)$ there exist a positive $T = T(\|\phi\|_{H^{s,k}_{x,y}})$ ($\lim_{\rho\to 0} T(\rho) = \infty$) and a unique solution $u(t,x,y)$ of the initial value problem (3)-(5) on the time interval $I = [-T,T]$ such that*

$$u \in C(I, H^{s,k}_{x,y}(\mathbf{R}^2)) \cap B_{+}^{\frac{1}{2}+,s,k}(I).$$

Now we state the result for KP-II equation.

THEOREM 4.2. *For any $\phi \in L^2(\mathbf{R}^2)$, such that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'(\mathbf{R}^2)$ there exist a positive $T = T(\|\phi\|_{L^2})$ ($\lim_{\rho\to 0} T(\rho) = \infty$) and a unique solution $u(t,x,y)$ of the initial value problem (4)-(5) on the time interval $I = [-T,T]$ such that*

$$u \in C(I, L^2(\mathbf{R}^2)) \cap B_{-}^{\frac{1}{2}+,0,0}(I).$$

Remark. The assertion of Theorem 3 is a direct consequence Theorem 4.2 due to the conservation of the L^2 norm.

Proof of Theorem 4.1 and Theorem 4.2. Define the operators $L^{\pm}$

$$L^{\pm}u(t) = \psi(t)U^{\pm}(t)\phi - \frac{1}{2}\psi(t/T)\int_0^t U^{\pm}(t-t')\partial_x(u^2(t'))dt'.$$

Then by the aid of Proposition 3 and Theorem 2.1 we obtain

$$\|L^+u\|_{B_+^{\frac{1}{2}+,s,k}} \lesssim \left(\|\phi\|_{H^{s,k}_{x,y}} + T^{0+}\|u\|^2_{B_+^{\frac{1}{2}+,s,k}}\right)$$

Similarly using Proposition 3 and Theorem 2.2 we obtain

$$\|L^-u\|_{B_-^{\frac{1}{2}+,0,0}} \lesssim \left(\|\phi\|_{L^2} + T^{0+}\|u\|^2_{B_-^{\frac{1}{2}+,0,0}}\right)$$

and furthermore

$$\|L^+u - L^+v\|_{B_+^{\frac{1}{2}+,s,k}} \lesssim T^{0+}\|u+v\|_{B_+^{\frac{1}{2}+,s,k}}\|u-v\|_{B_+^{\frac{1}{2}+,s,k}},$$

$$\|L^-u - L^-v\|_{B_-^{\frac{1}{2}+,0,0}} \lesssim T^{0+}\|u+v\|_{B_-^{\frac{1}{2}+,0,0}}\|u-v\|_{B_-^{\frac{1}{2}+,0,0}}.$$

Now we can use the contraction mapping principle for sufficiently small T to prove the local well-posedness of the integral equation (12) on the time interval $[-T,T]$. Let u be a solution of (12). Then u is a solution of the original equation (3) (resp. (4)) with initial data ϕ only if an additional condition on ϕ is imposed. This is because of the singularity of the symbols $p^{\pm}(\tau,\xi,\eta)$ at $\xi = 0$. In order to have a well defined time derivative of $U^{\pm}(t)\phi$ we should be able to give a sense of $|\xi|^{-1}\widehat{\phi}(\xi,\eta)$. Therefore if we suppose that $|\xi|^{-1}\widehat{\phi}(\xi,\eta) \in \mathcal{S}'$ then the solutions of (12) are as well solutions of (3) (resp. (4)) with initial data ϕ.

5. The global well-posedness in the energy space (KP-I)

5.1. Local well-posedness in the energy space. In this Section we shall prove global well-posedness of the fifth order KP-I equation in the energy space. Because of the specific structure of the energy density for KP type equations, we shall prove first a local existence result which requires more regularity on the data and in this sense weaker than Theorem 1. But on the other hand we shall consider data which satisfy the constraints needed to give a sense to the energy. Denote by $E(\mathbf{R}^2)$ the space equipped with the norm

$$\|\phi\|_E = \|(1+|\xi|^2+|\xi|^{-1}|\eta|)\widehat{\phi}(\xi,\eta)\|_{L^2_{\xi,\eta}}$$

The space $E(\mathbf{R}^2)$ is related to the energy of the fifth order KP-I equation. Recall that

$$H(\phi) = \frac{1}{2}\int |\partial_x^2\phi|^2 + \frac{\alpha}{2}\int |\partial_x\phi|^2 \pm \frac{1}{2}\int |\partial_x^{-1}\partial_y\phi|^2 - \frac{1}{6}\int \phi^3,$$

is the energy for (3). Note that if $\phi \in E(\mathbf{R}^2)$ then $H(\phi)$ is finite. Denote by $Y^{b,s,k}(\mathbf{R}^3)$ the space equipped with the norm

$$\|u\|_{Y^{b,s,k}} = \|u\|_{B_+^{b,s,0}} + \left\| \left\langle \tau - \xi^5 - \alpha\xi^3 - \frac{\eta^2}{\xi} \right\rangle^b |\xi|^{-1}|\eta|^k \widehat{u}(\tau,\xi,\eta) \right\|_{L^2_{\tau,\xi,\eta}} \tag{28}$$

$$:= \|u\|_{Y_1^{b,s}} + \|u\|_{Y_2^{b,k}}$$

Similarly to $B_\pm^{b,s,k}(I)$ we define the space $Y^{b,s,k}(I)$, where $I \subset \mathbf{R}$ is an interval. We remark that $Y^{b,2,1}(\mathbf{R}^3)$ is the Bourgain space associated to $E(\mathbf{R}^2)$, i.e. the following relation holds

$$\|u\|_{Y^{b,2,1}} \sim \|U^+(-t)u\|_{H_t^b(E)}.$$

In order to state the local existence result we define the space $\widetilde{H}^{s,k}(\mathbf{R}^2)$ equipped with the norm

$$\|\phi\|_{\widetilde{H}^{s,k}} = \|(1+|\xi|^s+|\xi|^{-1}|\eta|^k)\widehat{\phi}(\xi,\eta)\|_{L^2_{\xi,\eta}}.$$

Clearly $\widetilde{H}^{2,1}(\mathbf{R}^2) = E(\mathbf{R}^2)$. We have the following Theorem.

THEOREM 5.1. *Let $s \geq 1$ and $k \geq 0$. Then for any $\phi \in \widetilde{H}^{s,k}(\mathbf{R}^2)$, there exist a positive $T = T(\|\phi\|_{\widetilde{H}^{s,k}})$ ($\lim_{\rho\to 0} T(\rho) = \infty$) and a unique solution $u(t,x,y)$ of the initial value problem (3)-(5) on the time interval $I = [-T,T]$ such that*

$$u \in C(I, \widetilde{H}^{s,k}(\mathbf{R}^2)) \cap Y^{\frac{1}{2}+,s,k}(I).$$

The proof of Theorem 5.1 is in the spirit of the previous section once we obtain the crucial estimate

$$\|\partial_x(uv)\|_{Y^{-\frac{1}{2}+,s,k}} \lesssim \|u\|_{Y^{\frac{1}{2}+,s,k}}\|v\|_{Y^{\frac{1}{2}+,s,k}} \tag{29}$$

Due to Theorem 2.1 we obtain

$$\|\partial_x(uv)\|_{Y_1^{-\frac{1}{2}+,s}} \lesssim \|u\|_{Y_1^{\frac{1}{2}+,s}}\|v\|_{Y_1^{\frac{1}{2}+,s}} \tag{30}$$

In the next Proposition we give a bound for the the $Y_2^{\frac{1}{2}+,k}(\mathbf{R}^3)$ norm of the bilinear expression $\partial_x(uv)$ in terms of $Y_1^{\frac{1}{2}+,s}(\mathbf{R}^3)$ and $Y_2^{\frac{1}{2}+,k}(\mathbf{R}^3)$ norms of u and v.

PROPOSITION 4. *Let $s \geq 1$ and $k \geq 0$. Then the following estimate holds*

$$\|\partial_x(uv)\|_{Y_2^{-\frac{1}{2}+,k}} \lesssim \left\{ \|u\|_{Y_1^{\frac{1}{2}+,s}} \|v\|_{Y_2^{\frac{1}{2}+,k}} + \|u\|_{Y_2^{\frac{1}{2}+,k}} \|v\|_{Y_1^{\frac{1}{2}+,s}} \right\} \tag{31}$$

Proof of (31). We have

$$\|\partial_x(uv)\|_{Y_2^{-\frac{1}{2}+,k}} = \|I(\tau,\xi,\eta)\|_{L^2_{\tau,\xi,\eta}},$$

where

$$\begin{aligned} I(\tau,\xi,\eta) &= \frac{\langle\eta\rangle^k}{\langle\sigma\rangle^{\frac{1}{2}-}} \int \widehat{u}(\tau_1,\xi_1,\eta_1)\widehat{v}(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)d\tau_1 d\xi_1 d\eta_1 \\ &= \frac{\langle\eta\rangle^k}{\langle\sigma\rangle^{\frac{1}{2}-}} \left\{ \int_{|\eta|\leq 2|\eta_1|} \cdots + \int_{|\eta|\geq 2|\eta_1|} \cdots \right\} \\ &:= I_1(\tau,\xi,\eta) + I_2(\tau,\xi,\eta) \end{aligned}$$

We claim that

$$\|I_1(\tau,\xi,\eta)\|_{L^2_{\tau,\xi,\eta}} \lesssim \|u\|_{Y_2^{\frac{1}{2}+,k}} \|v\|_{Y_1^{\frac{1}{2}+,s}} \tag{32}$$

and

$$\|I_2(\tau,\xi,\eta)\|_{L^2_{\tau,\xi,\eta}} \lesssim \|u\|_{Y_1^{\frac{1}{2}+,s}} \|v\|_{Y_2^{\frac{1}{2}+,k}} \tag{33}$$

Since if $|\eta| \geq 2|\eta_1|$ then $|\eta| \leq 2|\eta-\eta_1|$ the proof of (33) is essentially the same as that of (32) via a symmetry argument. A duality argument writes (32) in the form

$$\int\int_{|\eta|\leq 2|\eta_1|} \frac{\langle\eta\rangle^k |\xi_1| f(\tau_1,\zeta_1) g(\tau-\tau_1,\zeta-\zeta_1) h(\tau,\zeta) d\tau d\zeta d\tau_1 d\zeta_1}{\langle\sigma\rangle^{\frac{1}{2}-}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{\frac{1}{2}+}\langle\xi-\xi_1\rangle^s\langle\eta_1\rangle^k} \tag{34}$$

$$\lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2} \quad (\text{ recall that } \zeta = (\xi,\eta),\ \zeta_1 = (\xi_1,\eta_1)).$$

The proof of (34) is similar to that of Theorem 2.1. Denote by J the left-hand side of (34). Since on the support of J one has $|\eta| \leq 2|\eta_1|$ the quantity $\langle\eta\rangle^k\langle\eta_1\rangle^{-k}$ is bounded. Consider different cases in order to prove (34). Let $|\xi_1| \leq 1$. If $|\xi - \xi_1| \leq 1$ then we can perform the arguments of Case 1.2 of the proof of Theorem 2.1. If $|\xi - \xi_1| \geq 1$ then we can use Proposition 2.1 as in Case 1.1 of the proof of Theorem 2.1. Hence we can assume that $|\xi_1| \geq 1$. Let $1 \leq |\xi_1| \leq 2|\xi - \xi_1|$. Then by the aid of Proposition 2.1 the contribution of this region J is bounded by

$$\begin{aligned} &\|\mathcal{F}^{-1}(|\xi_1|^{\frac{1}{4}}\langle\sigma_1\rangle^{-\frac{1}{2}-}\widehat{f}(\tau_1,\zeta_1))\|_{L^4}\|\mathcal{F}^{-1}(|\xi-\xi_1|^{\frac{1}{4}}\langle\sigma_2\rangle^{-\frac{1}{2}-}\widehat{g}(\tau-\tau_1,\zeta-\zeta_1))\|_{L^4}\|h\|_{L^2} \\ \lesssim\ &\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \end{aligned}$$

Let $|\xi_1| \geq 1$ and $|\xi_1| \geq 2|\xi - \xi_1|$. Denote by $\bar{J}$ the contribution of this region to J. In this case we have

$$|\xi| \leq \frac{3}{2}|\xi_1| \leq 3|\xi| \quad => \quad |\xi| \sim |\xi_1|.$$

Hence we have to prove that

$$\int\int_{|\eta|\leq 2|\eta_1|} \frac{|\xi| f(\tau_1,\zeta_1) g(\tau-\tau_1,\zeta-\zeta_1) h(\tau,\zeta) d\tau d\zeta d\tau_1 d\zeta_1}{\langle\sigma\rangle^{\frac{1}{2}-}\langle\sigma_1\rangle^{\frac{1}{2}+}\langle\sigma_2\rangle^{\frac{1}{2}+}\langle\xi-\xi_1\rangle^{s}} \lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.$$

Let $|\xi-\xi_1|\leq 1$. We can estimate the contribution of this region as in Case 2 of the proof of Theorem 2.1. Let $|\xi-\xi_1|\geq 1$ and $|\sigma_1+\sigma_2-\sigma|\geq \frac{1}{100}|\xi|^4$. We can estimate the contribution of that region to $\bar{J}$ as in Case 4 of the proof of Theorem 2.1. Let finally $|\xi-\xi_1|\geq 1$ and $|\sigma_1+\sigma_2-\sigma|\leq \frac{1}{100}|\xi|^4$. We are in a position to apply the argument of Case 5 of the proof of Theorem 2.1. This completes the proof of (31). Then (29) follows from (30) and (31) and hence applying a contraction argument we can complete the proof of Theorem 5.1.

5.2. Global extension of the solutions. Let $\phi\in L^2(\mathbf{R}^2)$ be such that $H(\phi)<\infty$. Then $\phi\in E(\mathbf{R}^2)$. Applying Theorem 5.1 with $s=2$ and $k=1$ we obtain a local solution of (3)-(5) on the time interval $[-T,T]$ where T depends on

$$\|u(t)\|_{L^2}+\|u_{xx}(t)\|_{L^2}+\|\partial_x^{-1}\partial_y u(t)\|_{L^2}.$$

Since $\|u(t)\|_{L^2}=\|\phi\|_{L^2}$, we need to prove that the quantity

$$Q(t)=\frac{1}{2}\int_{\mathbf{R}^2}\{|\partial_x^2 u(t)|^2+|\partial_x^{-1}\partial_y u(t)|^2\}$$

remains bounded along the trajectories. Then a standard continuation argument provides the global well-posedness. Recall that the Hamiltonian of (3) is (at least formally) constant along the trajectories, i.e.

$$\frac{1}{2}\int_{\mathbf{R}^2}\left\{|\partial_x^2 u(t)|^2+|\partial_x^{-1}\partial_y u(t)|^2-\frac{1}{3}u^3(t)\right\}=H(\phi). \tag{35}$$

In order to justify (35) we need some additional arguments. Let $Y(\mathbf{R}^2)$ be an auxiliary space equipped with the norm

$$\|\phi\|_Y=\|(1+|\xi|^4+|\xi|^{-2}|\eta|^2)\widehat{\phi}(\xi,\eta)\|_{L^2_{\xi,\eta}}$$

Then Lemma 3.2 of [**47**] yields that $Y(\mathbf{R}^2)$ is dense in $E(\mathbf{R}^2)$. Hence we can find a sequence of smooth functions ϕ_n converging to ϕ in $E(\mathbf{R}^2)$ and in addition $\phi_n\in Y(\mathbf{R}^2)$. Let u_n be the solution of (3) with data ϕ_n. Since $\phi_n\in Y(\mathbf{R}^2)$, Theorem 3.1 of [**48**] yields

$$H(u_n(t))=H(\phi_n(t)).$$

Since ϕ_n converges to ϕ in $E(\mathbf{R}^2)$ we have

$$H(\phi_n)\longmapsto H(\phi).$$

On the other hand the local well-posedness (Theorem 5.1) yields

$$\int|\partial_x^2 u_n(t)|^2+\int|\partial_x^{-1}\partial_y u_n(t)|^2\longmapsto\int|\partial_x^2 u(t)|^2+\int|\partial_x^{-1}\partial_y u(t)|^2 \tag{36}$$

Further we have to prove that

$$\int u_n^3(t)\longmapsto\int u^3(t) \tag{37}$$

In order to prove (37), write

$$\begin{aligned}\left|\int u_n^3(t) - \int u^3(t)\right| &\le \|u_n(t) - u(t)\|_{L^2}\left(\|u_n(t)\|_{L^4}^2 + \|u(t)\|_{L^4}^2\right)\\ &\le \|u_n(t) - u(t)\|_{L^2}(\|u_n(t)\|_{L^2}\|\partial_x^2 u_n(t)\|_{L^2}^{\frac12}\|\partial_x^{-1}\partial_y u_n(t)\|_{L^2}^{\frac12}\\ &\quad +\|u(t)\|_{L^2}\|\partial_x^2 u(t)\|_{L^2}^{\frac12}\|\partial_x^{-1}\partial_y u(t)\|_{L^2}^{\frac12}),\end{aligned}$$

where we used the Sobolev inequality (cf. [6]):

$$\|u\|_{L^4} \lesssim \|u\|_{L^2}^{\frac12}\|\partial_x^2 u\|_{L^2}^{\frac14}\|\partial_x^{-1}\partial_y u\|_{L^2}^{\frac14}, \quad u \in E(\mathbf{R}^2).$$

Clearly $u_n(t)$ converges to $u(t)$ in $L^2(\mathbf{R}^2)$ and hence

$$\left|\int u_n^3(t) - \int u^3(t)\right| \longmapsto 0,$$

which proves (37). A use of (36) and (37) shows that $H(u_n(t))$ converges to $H(u(t))$ and hence $H(u(t)) = H(\phi)$. Now via a Sobolev inequality (cf. [6]) we obtain

$$\begin{aligned}\left|\int u^3(t)\right| &\le 2\|u(t)\|_{L^2}^2\|\partial_x^2 u(t)\|_{L^2}^{\frac12}\|\partial_x^{-1}\partial_y u(t)\|_{L^2}^{\frac12}\\ &\le 4\|\phi\|_{L^2}^4 + \frac14 Q(t).\end{aligned}$$

and finally we arrive at

$$Q(t) \le \frac{24}{23}H(\phi) + \frac{16}{23}\|\phi\|_{L^2}^4.$$

Hence $Q(t)$ is bounded by a quantity which remains constant along the trajectories. This completes the proof of Theorem 2.

We conclude this section with a remark on the global behavior of the solutions obtained in Theorem 2 and Theorem 3. Due to the local existence argument we have that for any finite interval $I \subset \mathbf{R}$ the global solution u of (3)-(5) belongs to the localized Bourgain space $Y^{\frac12+,2,1}(I)$ associated to the energy space $E(\mathbf{R}^2)$. One may ask whether the solution u belongs to the global (in time) Bourgain space $Y^{\frac12+,2,1}(\mathbf{R}^3)$. The answer to this question is negative, since in [10] solitary wave solutions of (3) are obtained and clearly they do not belong to any global Bourgain space. On the other hand similar a statement is not valid in the context of KP-II type equations because it easily results from Pohojaev type identities that there are no localized solitary waves for KP-II type equations. We do not know if the global solutions of KP-II type equations belong to the corresponding global Bourgain spaces. Finally we remark that in [16], J. Bourgain proves that the absence of solitary waves (defocusing case) leads to global properties of the solution in the context of the H^1-critical 3D semi-linear Schrödinger equation.

6. The case of periodic boundary conditions - 2D

We suppose that

$$\int_{\mathbf{T}^1} \phi(x,y)dx = 0. \tag{38}$$

Actually we can always suppose (38). In the general case a lower order perturbation of the equation appears which does not affect the analysis (cf. [**13, 14**]). Now we define a Fourier transform restriction space where the solutions shall be obtained. Denote by $X_0^{b,s}(\mathbf{R}\times\mathbf{T}^2)$ the completion of $C_0^\infty(\mathbf{R}\times\mathbf{T}^2)$ with zero x mean value with respect to the norm

$$\|u\|^2_{X_0^{b,s}} = \sum_{\substack{(m,n)\in\mathbf{Z}^2\\ m\neq 0}} \int d\tau \left\langle \tau - m^5 + \frac{n^2}{m}\right\rangle^{2b} \langle m\rangle^{2s} |\widehat{u}(\tau,m,n)|^2$$

$$+ \sum_{\substack{(m,n)\in\mathbf{Z}^2\\ m\neq 0}} \int d\tau \frac{\left\langle \tau - m^5 + \frac{n^2}{m}\right\rangle^{2b+\frac14}}{|m|} \langle m\rangle^{2s} |\widehat{u}(\tau,m,n)|^2.$$

The next estimate is crucial in the proof of Theorem 4.

PROPOSITION 5. *Let $u\in X_0^{\frac12+,s}(\mathbf{R}\times\mathbf{T}^2)$, $s\geq -1/8$. Then the following inequality holds*

$$\|uu_x\|_{X_0^{-\frac12+,s}} \lesssim \|u\|^2_{X_0^{\frac12+,s}}. \tag{39}$$

Once we obtain (39), the proof of Theorem 4 results from a contraction argument slightly different from that in the proof of Theorem 1 because of the second term in the definition of $X_0^{b,s}(\mathbf{R}\times\mathbf{T}^2)$. This argument will be presented at the end of the next section.

6.1. Proof of (39).

Set $\zeta = (m,n)\in\mathbf{Z}^2$ and

$$\sigma = \sigma(\tau,\zeta) = \tau - m^5 + \frac{n^2}{m}, \quad \sigma_1 = \sigma(\tau_1,\zeta_1), \quad \sigma_2 = \sigma(\tau-\tau_1,\zeta-\zeta_1).$$

$$\theta = \frac{\langle\sigma\rangle^{\frac18}}{|m|^{\frac12}}, \quad \theta_1 = \theta(\tau_1,\zeta_1), \quad \theta_2 = \theta(\tau-\tau_1,\zeta-\zeta_1).$$

By duality, (39) is equivalent to

$$\sum_{\zeta_1\in\mathbf{Z}^2}\sum_{\zeta\in\mathbf{Z}^2}\int d\tau_1\int d\tau \frac{|m|\langle m\rangle^s\langle\theta\rangle\widehat{u}(\tau_1,\zeta_1)\widehat{v}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}(\tau,\zeta)}{\langle m_1\rangle^s\langle m-m_1\rangle^s\langle\theta_1\rangle\langle\theta_2\rangle\langle\sigma\rangle^{\frac12-}\langle\sigma_1\rangle^{\frac12+}\langle\sigma_2\rangle^{\frac12+}}$$

$$\lesssim \|u\|_{L^2(\mathbf{R}\times\mathbf{T}^2)}\|v\|_{L^2(\mathbf{R}\times\mathbf{T}^2)}\|w\|_{L^2(\mathbf{R}\times\mathbf{T}^2)}, \tag{40}$$

where $\widehat{u}(\tau_1,\zeta_1), \widehat{v}(\tau-\tau_1,\zeta-\zeta_1)$ and $\widehat{w}(\tau,\zeta)$ can be assumed to be positive. Consider the dyadic levels in $(\tau,\zeta,\tau_1,\zeta_1)$ space:

$$D^{KK_1K_2}_{MM_1M_2} = \begin{cases} \langle\sigma\rangle \sim K, \quad \langle\sigma_1\rangle \sim K_1, \quad \langle\sigma_2\rangle \sim K_2, \\ |m| \sim M, \quad |m_1| \sim M_1, \quad |m-m_1| \sim M_2 \end{cases}$$

Due to the zero x mean value assumption, we can suppose that $M \geq 1$, $M_1 \geq 1$, $M_2 \geq 1$. Denote by J the left-hand side of (40) and by $J^{KK_1K_2}_{MM_1M_2}$ the contribution of $D^{KK_1K_2}_{MM_1M_2}$ to J. Then

$$J \lesssim \sum_{K,K_1,K_2,M,M_1,M_2} J^{KK_1K_2}_{MM_1M_2}.$$

Let

$$\widehat{u}_{K_1M_1}(\tau_1,\zeta_1) = \begin{cases} \widehat{u}(\tau_1,\zeta_1), & \text{when } \langle\sigma_1\rangle \sim K_1, \quad |m_1| \sim M_1 \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly one can define the localized $\widehat{v}$ and $\widehat{w}$ denoted respectively by:

$$\widehat{v}_{K_2M_2}(\tau-\tau_1,\zeta-\zeta_1) \quad \text{and} \quad \widehat{w}_{KM}(\tau,\zeta).$$

Hence we have

$$J^{KK_1K_2}_{MM_1M_2} \lesssim \frac{M^{1+s}\left(1+\frac{K^{\frac{1}{8}}}{M^{\frac{1}{2}}}\right)}{K^{\frac{1}{2}-}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}} \int_{D^{KK_1K_2}_{MM_1M_2}} \frac{\widehat{u}_{K_1M_1}(\tau_1,\zeta_1)\widehat{v}_{K_2M_2}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}_{KM}(\tau,\zeta)}{M_1^sM_2^s\left(1+\frac{K_1^{\frac{1}{8}}}{M_1^{\frac{1}{2}}}\right)\left(1+\frac{K_2^{\frac{1}{8}}}{M_2^{\frac{1}{2}}}\right)},$$

where $\int_{D^{KK_1K_2}_{MM_1M_2}}$ means integration with respect to τ, τ_1 and summation with respect to ζ, ζ_1 on the range of $D^{KK_1K_2}_{MM_1M_2}$. In order to estimate the expression

$$\int_{D^{KK_1K_2}_{MM_1M_2}} \widehat{u}_{K_1M_1}(\tau_1,\zeta_1)\widehat{v}_{K_2M_2}(\tau-\tau_1,\zeta-\zeta_1)\widehat{w}_{KM}(\tau,\zeta)$$

we shall use two main tools. The first one is the algebraic relation for the symbols σ, σ_1 and σ_2, where the KP-II nature is essentially used. The second one is a convolution lemma in the spirit of [14]. State the algebraic relation

$$\sigma_1+\sigma_2-\sigma = 5m_1m(m-m_1)(m^2-mm_1+m_1^2) + \frac{(m_1n-mn_1)^2}{m_1m(m-m_1)}.$$

Hence on $D^{KK_1K_2}_{MM_1M_2}$ we have

$$\max\{K,K_1,K_2\} \gtrsim MM_1^2M_2^2, \tag{41}$$

$$\max\{K,K_1,K_2\} \gtrsim M_2M^2M_1^2, \tag{42}$$

$$\max\{K,K_1,K_2\} \gtrsim M^3M_1M_2. \tag{43}$$

Now we state the convolution estimate.

PROPOSITION 6. *Let $u_1(\tau,m,n)$ and $u_2(\tau,m,n)$ be two functions defined on $\mathbf{R}\times\mathbf{Z}^2$ with the following support properties*

$$\text{If} \quad (\tau,m,n)\in \text{supp } u_j \quad \text{then} \quad |m|\sim M_j, \quad \left|\tau - m^5 + \frac{n^2}{m}\right| \sim K_j, \quad j=1,2,$$

where $M_j \geq 1$. Then the following inequality holds

$$\|(u_1 * u_2)(\tau,m,n)\|_{L^2(\tau,|m|\sim M,n)} \lesssim \tag{44}$$

$$(K_1\wedge K_2)^{\frac{1}{2}}(K_1\vee K_2)^{\frac{1}{4}}(M_1\wedge M_2)^{\frac{1}{2}}\frac{(M_1M_2)^{\frac{1}{4}}}{M^{\frac{1}{4}}}\|u_1\|_{L^2}\|u_2\|_{L^2}$$

The proof of Proposition 6 will be performed in the next section. In order to estimate $J^{KK_1K_2}_{MM_1M_2}$ we distinguish the cases taking into account which of the symbols $|\sigma|$, $|\sigma_1|$, $|\sigma_2|$ dominates. By symmetry we can assume that $|\sigma_1|\geq|\sigma_2|$. Note also that $M\lesssim(M_1+M_2)$. Hence

$$M \lesssim (M_1\vee M_2). \tag{45}$$

The inequality (45) will be used intensively hereafter without explicit mention. We shall consider only the cases $s=0$ and $s=-1/8$. Then the proof for $s\in(-1/8,0)$ results from the three lines theorem. The case $s>0$ can be treated as the case $s=0$ below.

- Let $s=0$, $|\sigma|$ dominates and $K^{\frac{1}{8}}\leq M^{\frac{1}{2}}$. Then due to Proposition 6 and (41) we estimate the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$ by

$$\frac{cM}{K^{\frac{1}{2}-}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\langle \widehat{u}_{K_1M_1}\star\widehat{v}_{K_2M_2},\widehat{w}_{KM}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{3}{4}}(M_1\wedge M_2)^{\frac{1}{2}}(M_1M_2)^{\frac{1}{4}}(K_1\wedge K_2)^{\frac{1}{2}}(K_1\vee K_2)^{\frac{1}{4}}}{M^{\frac{1}{2}-}(M_1M_2)^{1-}K^{0+}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K\gtrsim M^3M_1M_2$, $K\gtrsim K_1$ and $K\gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

• Let $s = 0$, $|\sigma|$ dominates and $K^{\frac{1}{8}} \geq M^{\frac{1}{2}}$. Then due to Proposition 6 and (41) we estimate the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$ by

$$\frac{cM^{\frac{1}{2}}}{K^{\frac{3}{8}-}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\langle \widehat{u}_{K_1M_1} \star \widehat{v}_{K_2M_2}, \widehat{w}_{KM}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{1}{4}}(M_1 \wedge M_2)^{\frac{1}{2}}(M_1M_2)^{\frac{1}{4}}(K_1 \wedge K_2)^{\frac{1}{2}}(K_1 \vee K_2)^{\frac{1}{4}}}{M^{\frac{3}{8}-}(M_1M_2)^{\frac{3}{4}-}K^{0+}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K \gtrsim M^3M_1M_2$, $K \gtrsim K_1$ and $K \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

The additional term in the definition of the Fourier transform restriction space $X_0^{b,s}(\mathbf{R} \times \mathbf{T}^2)$ is introduced in order to deal with the cases when $|\sigma_1|$ dominates. The advantage of this term is an additional small factor $K_1^{-\frac{1}{8}}$ (or $M^{-\frac{1}{2}}$) in the estimate. On the other hand a factor $M_1^{\frac{1}{2}}$ also appears. That factor is easily canceled by using the convolution lemma.

• Let $s = 0$, $|\sigma_1|$ dominates and $K^{\frac{1}{8}} \leq M^{\frac{1}{2}}$. Note that in this case

$$\frac{1 + \frac{K^{\frac{1}{8}}}{M^{\frac{1}{2}}}}{1 + \frac{K_1^{\frac{1}{8}}}{M_1^{\frac{1}{2}}}} \lesssim \frac{M_1^{\frac{1}{2}}}{K_1^{\frac{1}{8}}}$$

Then due to Proposition 6 and (42) we obtain the following bound for the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$

$$\frac{cMM_1^{\frac{1}{2}}}{K^{\frac{1}{2}+}K_1^{\frac{5}{8}-}K_2^{\frac{1}{2}+}}\langle \widehat{w}_{KM} \star \widehat{v}_{K_2M_2}, \widehat{u}_{K_1M_1}\rangle_{L^2}$$

$$\lesssim \frac{MM_1^{\frac{1}{2}}(M \wedge M_2)^{\frac{1}{2}}(MM_2)^{\frac{1}{4}}(K \wedge K_2)^{\frac{1}{2}}(K \vee K_2)^{\frac{1}{4}}}{M_1^{\frac{1}{4}}M_1^{\frac{5}{8}-}(MM_2)^{\frac{5}{4}-}K^{\frac{1}{2}+}K_1^{0+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K_1^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K_1 \gtrsim M^3M_1M_2$, $K_1 \gtrsim K$ and $K_1 \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $s = 0$, $|\sigma_1|$ dominates and $K^{\frac{1}{8}} \geq M^{\frac{1}{2}}$. Note that in this case

$$\frac{1 + \frac{K^{\frac{1}{8}}}{M^{\frac{1}{2}}}}{1 + \frac{K_1^{\frac{1}{8}}}{M_1^{\frac{1}{2}}}} \lesssim \frac{M_1^{\frac{1}{2}}}{M^{\frac{1}{2}}}$$

Then due to Proposition 6 and (42) we obtain the following bound for the contribution of this region to $J_{MM_1M_2}^{KK_1K_2}$

$$\frac{cM^{\frac{1}{2}}M_1^{\frac{1}{2}}}{K^{\frac{1}{2}+}K_1^{\frac{1}{2}-}K_2^{\frac{1}{2}+}}\langle \widehat{w}_{KM} \star \widehat{v}_{K_2M_2}, \widehat{u}_{K_1M_1}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{1}{2}}M_1^{\frac{1}{2}}(M \wedge M_2)^{\frac{1}{2}}(MM_2)^{\frac{1}{4}}(K \wedge K_2)^{\frac{1}{2}}(K \vee K_2)^{\frac{1}{4}}}{M_1^{\frac{1}{4}}M_1^{\frac{1}{2}-}(MM_2)^{1-}K^{\frac{1}{2}+}K_1^{0+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K_1^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K_1 \gtrsim M^3M_1M_2$, $K_1 \gtrsim K$ and $K_1 \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $s = -1/8$, $|\sigma|$ dominates and $K^{\frac{1}{8}} \leq M^{\frac{1}{2}}$. Then due to Proposition 6 and (41) we estimate the contribution of this region to $J_{MM_1M_2}^{KK_1K_2}$ by

$$\frac{cM^{\frac{7}{8}}(M_1M_2)^{\frac{1}{8}}}{K^{\frac{1}{2}-}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\langle \widehat{u}_{K_1M_1} \star \widehat{v}_{K_2M_2}, \widehat{w}_{KM}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{5}{8}}(M_1 \wedge M_2)^{\frac{1}{2}}(M_1M_2)^{\frac{3}{8}}(K_1 \wedge K_2)^{\frac{1}{2}}(K_1 \vee K_2)^{\frac{1}{4}}}{M^{\frac{1}{2}-}(M_1M_2)^{1-}K^{0+}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K \gtrsim M^3M_1M_2$, $K \gtrsim K_1$ and $K \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $s=-1/8$, $|\sigma|$ dominates and $K^{\frac{1}{8}} \geq M^{\frac{1}{2}}$. Then due to Proposition 6 and (41) we estimate the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$ by

$$\frac{cM^{\frac{3}{8}}(M_1M_2)^{\frac{1}{8}}}{K^{\frac{3}{8}-}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\langle \widehat{u}_{K_1M_1} \star \widehat{v}_{K_2M_2}, \widehat{w}_{KM}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{1}{8}}(M_1\wedge M_2)^{\frac{1}{2}}(M_1M_2)^{\frac{3}{8}}(K_1\wedge K_2)^{\frac{1}{2}}(K_1\vee K_2)^{\frac{1}{4}}}{M^{\frac{3}{8}-}(M_1M_2)^{\frac{3}{4}-}K^{0+}K_1^{\frac{1}{2}+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K \gtrsim M^3M_1M_2$, $K \gtrsim K_1$ and $K \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $s=-1/8$, $|\sigma_1|$ dominates and $K^{\frac{1}{8}} \leq M^{\frac{1}{2}}$. Note that in this case

$$\frac{1+\frac{K^{\frac{1}{8}}}{M^{\frac{1}{2}}}}{1+\frac{K_1^{\frac{1}{8}}}{M_1^{\frac{1}{2}}}} \lesssim \frac{M_1^{\frac{1}{2}}}{K_1^{\frac{1}{8}}}$$

Then due to Proposition 6 and (42) we obtain the following bound for the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$

$$\frac{cM^{\frac{7}{8}}M_1^{\frac{1}{2}}(M_1M_2)^{\frac{1}{8}}}{K^{\frac{1}{2}+}K_1^{\frac{5}{8}-}K_2^{\frac{1}{2}+}}\langle \widehat{w}_{KM} \star \widehat{v}_{K_2M_2}, \widehat{u}_{K_1M_1}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{7}{8}}M_1^{\frac{1}{2}}(M_1M_2)^{\frac{1}{8}}(M\wedge M_2)^{\frac{1}{2}}(MM_2)^{\frac{1}{4}}(K\wedge K_2)^{\frac{1}{2}}(K\vee K_2)^{\frac{1}{4}}}{M_1^{\frac{1}{4}}M_1^{\frac{5}{8}-}(MM_2)^{\frac{3}{4}-}K^{\frac{1}{2}+}K_1^{0+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K_1^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K_1 \gtrsim M^3M_1M_2$, $K_1 \gtrsim K$ and $K_1 \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $s=-1/8$, $|\sigma_1|$ dominates and $K^{\frac{1}{8}} \geq M^{\frac{1}{2}}$. Note that in this case

$$\frac{1+\frac{K^{\frac{1}{8}}}{M^{\frac{1}{2}}}}{1+\frac{K_1^{\frac{1}{8}}}{M_1^{\frac{1}{2}}}} \lesssim \frac{M_1^{\frac{1}{2}}}{M^{\frac{1}{2}}}$$

Then due to Proposition 6 and (42) we obtain the following bound for the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$

$$\frac{cM^{\frac{3}{8}}M_1^{\frac{1}{2}}(M_1M_2)^{\frac{1}{8}}}{K^{\frac{1}{2}+}K_1^{\frac{1}{2}-}K_2^{\frac{1}{2}+}}\langle \widehat{w}_{KM}\star\widehat{v}_{K_2M_2},\widehat{u}_{K_1M_1}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{3}{8}}M_1^{\frac{1}{2}}(M_1M_2)^{\frac{1}{8}}(M\wedge M_2)^{\frac{1}{2}}(MM_2)^{\frac{1}{4}}(K\wedge K_2)^{\frac{1}{2}}(K\vee K_2)^{\frac{1}{4}}}{M_1^{\frac{1}{4}}M_1^{\frac{1}{2}-}(MM_2)^{1-}K^{\frac{1}{2}+}K_1^{0+}K_2^{\frac{1}{2}+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K_1^{0+}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K_1\gtrsim M^3M_1M_2$, $K_1\gtrsim K$ and $K_1\gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

This completes the proof of (39)

Proof of Theorem 4. Let $Y_0^{b,s}(\mathbf{R}\times\mathbf{T}^2)$ be the completion of $C_0^\infty(\mathbf{R}\times\mathbf{T}^2)$ with zero x mean value with respect to the norm

$$\|u\|^2_{Y_0^{b,s}} = \sum_{\substack{(m,n,l)\in\mathbf{Z}^3\\ m\neq 0}}\int d\tau\left\langle \tau-m^5+\frac{n^2}{m}\right\rangle^{2b}\langle m\rangle^{2s}|\widehat{u}(\tau,m,n)|^2.$$

Then clearly

$$\|u\|_{X_0^{b,s}}\sim\|u\|_{Y_0^{b,s}}+\|u\|_{Y_0^{b+\frac{1}{8},s-\frac{1}{2}}} \tag{46}$$

We shall apply a Picard fixed point argument to the following integral equation corresponding to the Cauchy problem (10)-(11)

$$u(t)=\psi(t)S(t)\phi-\frac{1}{2}\psi(t)\int_0^t S(t-t')\psi(t/T)\partial_x(u^2(t'))dt'. \tag{47}$$

Here $S(t)=\exp(-t(-\partial_x^5+\partial_x^{-1}\partial_y^2))$ is the unitary group describing the free evolution of the fifth order KP-II equation with periodic boundary conditions and $\psi(t)$ is the cut-off function used in the proof of Theorem 1. Clearly a solution of (47) on $\mathbf{R}$ corresponds to a solution of (10)-(11) on $[-T,T]$. We shall apply the contraction mapping principle to (47) in the spaces $X_0^{b,s}(\mathbf{R}\times\mathbf{T}^2)$. We start we the following Proposition which helps to provide a small factor in the iteration scheme.

PROPOSITION 7. *Let $1/2>b'>b''>0$. Then there exists $\theta>0$ such that for $T\in(0,1)$ and $s\in\mathbf{R}$ the following estimate holds*

$$\|\psi(t/T)u\|_{X_0^{-b',s}}\lesssim T^\theta\|u\|_{X_0^{-b'',s}}. \tag{48}$$

Proof of Proposition 7. If $0 < b < a < 1/2$ and $T \in (0,1)$ then there exists $\theta > 0$ such that

$$\|\psi(t/T)u\|_{Y_0^{-a,s}} \lesssim T^\theta \|u\|_{Y_0^{-b,s}}. \tag{49}$$

The inequality (49) is proven in **[42]** in the context of the KdV equation (cf. (3.29) in **[42]**). Actually the proof does not depend on the particular choice of the unitary group the Bourgain spaces are associated to. Now we obtain via (46)

$$\begin{aligned}\|\psi(t/T)u\|_{X_0^{-b',s}} &\sim \|\psi(t/T)u\|_{Y_0^{-b',s}} + \|\psi(t/T)u\|_{Y_0^{-b'+\frac{1}{8},s-\frac{1}{2}}}\\ &\lesssim T^{\theta_1}\|u\|_{Y_0^{-b'',s}} + T^{\theta_2}\|u\|_{Y_0^{-b''+\frac{1}{8},s-\frac{1}{2}}}\\ &\lesssim T^\theta\|u\|_{X_0^{-b'',s}}.\end{aligned}$$

This proves (48). The next Proposition contains the estimates needed to apply the contraction mapping principle.

Proposition 8. *For any $\phi \in Y^s(\mathbf{T}^2)$, $s \in \mathbf{R}$, there exists $\theta > 0$ such that the following inequalities hold*

$$\|\psi(t)S(t)\phi\|_{X_0^{\frac{1}{2}+,s}} \lesssim \|\phi\|_{Y^s} \tag{50}$$

$$\left\|\psi(t)\int_0^t S(t-t')\psi(t/T)\partial_x(u^2(t'))dt'\right\|_{X_0^{\frac{1}{2}+,s}} \lesssim T^\theta \|uu_x\|_{X_0^{-\frac{1}{2}+,s}}. \tag{51}$$

Proof of Proposition 8. The proof of (50) follows directly from the definition of the spaces $X_0^{b,s}(\mathbf{R}\times\mathbf{T}^2)$ (an observation similar to (13)). In order to prove (51) we first note that the following inequality holds

$$\left\|\psi(t)\int_0^t S(t-t')\psi(t/T)\partial_x(u^2(t'))dt'\right\|_{Y_0^{b,s}} \lesssim T^\theta\|uu_x\|_{Y_0^{b-1-,s}}, \tag{52}$$

provided $1/2 < b < 1$. In order to prove (52), we first apply the linear estimate in the framework of the Bourgain spaces (cf. (19) and (20) above with $T = 1$). Then we gain a small factor T^θ by the aid of (48). Now we write via (52)

$$\begin{aligned}\left\|\psi(t)\int_0^t S(t-t')\psi(t/T)\partial_x(u^2(t'))dt'\right\|_{X_0^{\frac{1}{2}+,s}} &\sim \|\cdots\|_{Y_0^{\frac{1}{2}+,s}} + \|\cdots\|_{Y_0^{\frac{5}{8}+,s-\frac{1}{2}}}\\ &\lesssim T^{\theta_1}\|uu_x\|_{Y_0^{-\frac{1}{2}+,s}} + T^{\theta_2}\|uu_x\|_{Y_0^{-\frac{3}{8}+,s-\frac{1}{2}}}\\ &\lesssim T^\theta\|uu_x\|_{X_0^{-\frac{1}{2}+,s}}.\end{aligned}$$

This completes the proof of Proposition 8.

Now Proposition 9 and Proposition 8 allow us to apply a fixed point argument in order to complete the proof of Theorem 4.

Remark. The exponent $-1/8$ which appears in the statement of Theorem 4 is not optimal. It is of technical nature and is chosen in order to make the computations more transparent. Actually a slight modification of the above argument carries out the bilinear estimate for $s > -1/4$.

6.2. Proof of the convolution lemma. We shall follow the idea of the proof of Lemma 4.1 of [14]. Note that Young inequality yields

$$\|u_1 \star u_2\|_{L^2} \leq \|u_1\|_{L^1}\|u_2\|_{L^2}.$$

Hence the expression in front of $\|u_1\|_{L^2}\|u_2\|_{L^2}$ in the right hand-side of (44) is a bound for the volume which appears when estimating the L^1 norm of the characteristic function of a bounded set in the terms of the L^2 norm. Clearly

$$\|u_1 \star u_2\|_{L^2}^2 = \sum_{m,n}\int d\tau \left|\sum_{m_1,n_1}\int d\tau_1\ u_1(\tau_1,m_1,n_1)u_2(\tau-\tau_1,m-m_1,n-n_1)\right|^2 .$$

The Cauchy-Schwarz inequality in the (τ_1, m_1, n_1) variables yields

$$\|u_1 \star u_2\|^2_{L^2(\tau,|m|\sim M,n)} \leq sup_{(\tau,m,n)}\mathrm{mes}\{A_{\tau mn}\}\|u_1\|^2_{L^2}\|u_2\|^2_{L^2},$$

where $A_{\tau mn} \subset \mathbf{R} \times \mathbf{Z}^2$ is the following set

$$A_{\tau mn} = \{(\tau_1, m_1, n_1) : \quad (\tau_1, m_1, n_1) \in \ \mathrm{supp}\ u_1, \quad (\tau-\tau_1, m-m_1, n-n_1) \in \ \mathrm{supp}\ u_2\}.$$

First we eliminate τ_1. Set $p(m,n) = -m^5 + \frac{n^2}{m}$. If $(\tau_1, m_1, n_1) \in$ supp u_1 and $(\tau-\tau_1, m-m_1, n-n_1) \in$ supp u_2 then

$$|\tau_1 + p(m_1,n_1)| \lesssim K_1, \quad |\tau - \tau_1 + p(m-m_1, n-n_1)| \lesssim K_2.$$

Hence

$$|\tau + p(m_1,n_1) + p(m-m_1,n-n_1)| \lesssim (K_1 \vee K_2).$$

Also for fixed (m_1, n_1) the maximal range for τ_1 is bounded by $c(K_1 \wedge K_2)$. Therefore

$$\mathrm{mes}\{A_{\tau mn}\} \lesssim (K_1 \wedge K_2)\mathrm{mes}\{B_{\tau mn}\},$$

where $B_{\tau mn} \subset \mathbf{Z}^2$ is defined as follows

$$\begin{aligned} B_{\tau mn} &= \{(m_1,n_1) \in \mathbf{Z}^2 : |m_1| \sim M_1, |m-m_1| \sim M_2, \\ &\qquad |\tau + p(m_1,n_1) + p(m-m_1,n-n_1)| \lesssim (K_1 \vee K_2)\}. \end{aligned}$$

Note that m_1 should range in an interval of size smaller than $M_1 \wedge M_2$. Fix now m_1. We shall estimate the size of the section with n_1. We have to estimate the number of n_1 such that

$$|\tau + p(m_1,n_1) + p(m-m_1,n-n_1)| \lesssim (K_1 \vee K_2),$$

for fixed (τ, m, n, m_1). Note that

$$\begin{aligned} p(m_1,n_1) + p(m-m_1,n-n_1) &= p(m,n) + 5m_1m(m-m_1)(m^2 - mm_1 + m_1^2) \\ &\quad + \frac{(m_1n - mn_1)^2}{m_1m(m-m_1)}. \end{aligned}$$

Hence the expression

$$\frac{(m_1 n - m n_1)^2}{m_1 m (m - m_1)}$$

has to value in an interval of size $c(K_1 \vee K_2)$. Therefore, for fixed m_1, n_1, m the integer n_1 has to range in an interval of size

$$\left[\frac{c(K_1 \vee K_2) m m_1 (m - m_1)}{m^2}\right]^{\frac{1}{2}} \lesssim (K_1 \vee K_2)^{\frac{1}{2}} \frac{(M_1 M_2)^{\frac{1}{2}}}{M^{\frac{1}{2}}}.$$

Hence we bound the measure of $B_{\tau m n}$ with the product of the projection on the m_1 line and the maximum of the sections with lines parallel to the n_1 axis. Thus

$$\text{mes}(B_{\tau m n}) \lesssim (K_1 \vee K_2)^{\frac{1}{2}} (M_1 \wedge M_2) \frac{(M_1 M_2)^{\frac{1}{2}}}{M^{\frac{1}{2}}}.$$

and furthermore

$$\text{mes}(A_{\tau m n}) \lesssim (K_1 \wedge K_2)(K_1 \vee K_2)^{\frac{1}{2}} (M_1 \wedge M_2) \frac{(M_1 M_2)^{\frac{1}{2}}}{M^{\frac{1}{2}}}.$$

This completes the proof of Proposition 6.

7. The case of periodic boundary conditions - 3D

As in 2D we can suppose that

$$\int_{\mathbf{T}^1} \phi(x, y, z) dx = 0. \tag{53}$$

Denote by $X_0^{b,\bar{b},s,\bar{s}}(\mathbf{R} \times \mathbf{T}^3)$ the completion of $C_0^\infty(\mathbf{R} \times \mathbf{T}^3)$ with zero x mean value with respect to the norm

$$\begin{aligned} \|u\|^2_{X_0^{b,\bar{b},s,\bar{s}}} &= \sum_{\substack{(m,n,l) \in \mathbf{Z}^3 \\ m \neq 0}} \int d\tau \left\langle \tau - m^5 + \frac{n^2}{m} + \frac{l^2}{m} \right\rangle^{2b} \langle m \rangle^{2s} |\widehat{u}(\tau, m, n, l)|^2 \\ &+ \sum_{\substack{(m,n,l) \in \mathbf{Z}^3 \\ m \neq 0}} \int d\tau \frac{\left\langle \tau - m^5 + \frac{n^2}{m} + \frac{l^2}{m} \right\rangle^{2\bar{b}}}{|m|^{2\bar{s}}} \langle m \rangle^{2s} |\widehat{u}(\tau, m, n, l)|^2. \end{aligned}$$

The next Proposition contains the bilinear estimate crucial for the proof of Theorem 5.

PROPOSITION 9. *Let $u \in X_0^{\frac{1}{2}+\varepsilon, \frac{1}{8}+2\varepsilon, s, \frac{1}{2}+4\varepsilon}(\mathbf{R} \times \mathbf{T}^3)$, $s \geq 8\varepsilon$ and $\varepsilon > 0$ be an arbitrary small number. Then the following inequality holds*

$$\|u u_x\|_{X_0^{-\frac{1}{2}+\varepsilon, \frac{1}{8}+2\varepsilon, s, \frac{1}{2}+4\varepsilon}} \lesssim \|u\|^2_{X_0^{\frac{1}{2}+\varepsilon, \frac{1}{8}+2\varepsilon, s, \frac{1}{2}+4\varepsilon}}. \tag{54}$$

Once we obtain (54) the rest of the proof of Theorem 5 follows the lines of the proof of Theorem 4 and hence will be omitted.

Proof of (54). Set $\zeta = (m, n, l) \in \mathbf{Z}^3$ and

$$\sigma = \sigma(\tau, \zeta) = \tau - m^5 + \frac{n^2}{m} + \frac{l^2}{m}, \quad \sigma_1 = \sigma(\tau_1, \zeta_1), \quad \sigma_2 = \sigma(\tau - \tau_1, \zeta - \zeta_1).$$

$$\theta = \frac{\langle \sigma \rangle^{\frac{1}{8}+2\varepsilon}}{|m|^{\frac{1}{2}+4\varepsilon}}, \quad \theta_1 = \theta(\tau_1, \zeta_1), \quad \theta_2 = \theta(\tau - \tau_1, \zeta - \zeta_1).$$

A duality argument writes (54) into the form

$$\sum_{\zeta_1 \in \mathbf{Z}^3} \sum_{\zeta \in \mathbf{Z}^3} \int d\tau_1 \int d\tau \frac{|m| \langle m \rangle^s \langle \theta \rangle \widehat{u}(\tau_1, \zeta_1) \widehat{v}(\tau - \tau_1, \zeta - \zeta_1) \widehat{w}(\tau, \zeta)}{\langle m_1 \rangle^s \langle m - m_1 \rangle^s \langle \theta_1 \rangle \langle \theta_2 \rangle \langle \sigma \rangle^{\frac{1}{2}-\varepsilon} \langle \sigma_1 \rangle^{\frac{1}{2}+\varepsilon} \langle \sigma_2 \rangle^{\frac{1}{2}+\varepsilon}}$$

$$\lesssim \|u\|_{L^2(\mathbf{R} \times \mathbf{T}^3)} \|v\|_{L^2(\mathbf{R} \times \mathbf{T}^3)} \|w\|_{L^2(\mathbf{R} \times \mathbf{T}^3)}, \tag{55}$$

where $\widehat{u}(\tau_1, \zeta_1), \widehat{v}(\tau - \tau_1, \zeta - \zeta_1)$ and $\widehat{w}(\tau, \zeta)$ are supposed to be positive. Similarly to 2D we consider the dyadic levels

$$D^{KK_1K_2}_{MM_1M_2} = \begin{cases} \langle \sigma \rangle \sim K, \quad \langle \sigma_1 \rangle \sim K_1, \quad \langle \sigma_2 \rangle \sim K_2, \\ |m| \sim M, \quad |m_1| \sim M_1, \quad |m - m_1| \sim M_2 \end{cases}$$

Denote by J the left-hand side of (55) and by $J^{KK_1K_2}_{MM_1M_2}$ the contribution of $D^{KK_1K_2}_{MM_1M_2}$ to J. Then

$$J \lesssim \sum_{K, K_1, K_2, M, M_1, M_2} J^{KK_1K_2}_{MM_1M_2}.$$

Let

$$\widehat{u}_{K_1M_1}(\tau_1, \zeta_1) = \begin{cases} \widehat{u}(\tau_1, \zeta_1), & \text{when } \langle \sigma_1 \rangle \sim K_1, \quad |m_1| \sim M_1 \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly one can define the localized $\widehat{v}$ and $\widehat{w}$ denoted respectively by:

$$\widehat{v}_{K_2M_2}(\tau - \tau_1, \zeta - \zeta_1) \quad \text{and} \quad \widehat{w}_{KM}(\tau, \zeta).$$

Hence we have

$$J^{KK_1K_2}_{MM_1M_2} \lesssim \frac{M^{1+s}\left(1 + \frac{K^{\frac{1}{8}+2\varepsilon}}{M^{\frac{1}{2}+4\varepsilon}}\right)}{K^{\frac{1}{2}-\varepsilon} K_1^{\frac{1}{2}+\varepsilon} K_2^{\frac{1}{2}+\varepsilon}} \int_{D^{KK_1K_2}_{MM_1M_2}} \frac{\widehat{u}_{K_1M_1}(\tau_1, \zeta_1) \widehat{v}_{K_2M_2}(\tau - \tau_1, \zeta - \zeta_1) \widehat{w}_{KM}(\tau, \zeta)}{M_1^s M_2^s \left(1 + \frac{K_1^{\frac{1}{8}+2\varepsilon}}{M_1^{\frac{1}{2}+4\varepsilon}}\right)\left(1 + \frac{K_2^{\frac{1}{8}+2\varepsilon}}{M_2^{\frac{1}{2}+4\varepsilon}}\right)},$$

where $\int_{D^{KK_1K_2}_{MM_1M_2}}$ means integration with respect to τ, τ_1 and summation with respect to ζ, ζ_1 on the range of $D^{KK_1K_2}_{MM_1M_2}$. In 3D the algebraic relation has the form

$$\sigma_1 + \sigma_2 - \sigma = 5m_1m(m-m_1)(m^2 - mm_1 + m_1^2) + \frac{(m_1n - mn_1)^2}{m_1m(m-m_1)} + \frac{(m_1l - ml_1)^2}{m_1m(m-m_1)}.$$

Hence on $D^{KK_1K_2}_{MM_1M_2}$ we have

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim \{M^3M_1M_2 + MM_1^2M_2^2 + M_2M^2M_1^2\}. \tag{56}$$

Now we state the convolution estimate in 3D.

PROPOSITION 10. *Let $u_1(\tau, m, n, l)$ and $u_2(\tau, m, n, l)$ be two functions defined on $\mathbf{R} \times \mathbf{Z}^3$ satisfying the support properties*

If $(\tau, m, n, l) \in supp\ u_j$ *then* $|m| \sim M_j, \quad \left|\tau - m^5 + \frac{n^2}{m} + \frac{l^2}{m}\right| \sim K_j, \quad j = 1, 2,$

where $M_j \geq 1$. Then the following inequality holds

$$\|(u_1 * u_2)(\tau, m, n)\|_{L^2(\tau, |m|\sim M, n, l)} \lesssim \tag{57}$$

$$(K_1 \wedge K_2)^{\frac{1}{2}}(K_1 \vee K_2)^{\frac{1}{2}}(M_1 \wedge M_2)^{\frac{1}{2}}\frac{(M_1M_2)^{\frac{1}{2}}}{M^{\frac{1}{2}}}\|u_1\|_{L^2}\|u_2\|_{L^2}$$

Proof of Proposition 10. The proof is similar to that of Proposition 6. Write

$$\|u_1 \star u_2\|^2_{L^2} = \sum_{m,n,l}\int d\tau \left|\sum_{m_1,n_1,l_1}\int d\tau_1 u_1(\tau_1, m_1, n_1, l_1)u_2(\tau - \tau_1, m - m_1, n - n_1, l - l_1)\right|^2.$$

Set $\zeta = (m, n, l)$. The Cauchy-Schwarz inequality in the (τ_1, m_1, n_1, l_1) variables yields

$$\|u_1 \star u_2\|^2_{L^2(\tau, |m|\sim M, n, l)} \leq sup_{(\tau,\zeta)}\text{mes}\{A_{\tau\zeta}\}\|u_1\|^2_{L^2}\|u_2\|^2_{L^2},$$

where $A_{\tau\zeta} \subset \mathbf{R} \times \mathbf{Z}^3$ is the following set

$$A_{\tau\zeta} = \{(\tau_1, \zeta_1): \quad (\tau_1, \zeta_1) \in \text{ supp } u_1, \quad (\tau - \tau_1, \zeta - \zeta_1) \in \text{ supp } u_2\}.$$

First we eliminate τ_1. Similarly to 2D we obtain

$$\text{mes}\{A_{\tau\zeta}\} \lesssim (K_1 \wedge K_2)\text{mes}\{B_{\tau\zeta}\},$$

where

$$B_{\tau\zeta} = \{\zeta_1 \in \mathbf{Z}^3 : |m_1| \sim M_1, |m - m_1| \sim M_2, |\tau + p(\zeta_1) + p(\zeta - \zeta_1)| \leq c(K_1 \vee K_2)\}$$

and $p(\zeta) = -m^5 + \frac{n^2}{m} + \frac{l^2}{m}$. Note that m_1 should range in an interval of size smaller than $M_1 \wedge M_2$. Fix now m_1. We shall estimate the surface of the section (n_1, l_1). We have to estimate the number of (n_1, l_1) such that

$$|\tau + p(\zeta_1) + p(\zeta - \zeta_1)| \lesssim (K_1 \vee K_2), \tag{58}$$

for fixed (τ, ζ, m_1). Note that

$$p(\zeta_1) + p(\zeta - \zeta_1) - p(\zeta) = 5m_1 m(m - m_1)(m^2 - mm_1 + m_1^2) + \frac{(m_1 n - mn_1)^2}{m_1 m(m - m_1)} + \frac{(m_1 l - ml_1)^2}{m_1 m(m - m_1)}.$$

Hence the expression

$$\frac{(m_1 n - mn_1)^2 + (m_1 l - ml_1)^2}{m_1 m(m - m_1)}$$

has to value in an interval of size $c(K_1 \vee K_2)$. Now we set

$$\tilde{m} = m_1 m(m - m_1), \quad \alpha = \tau + p(\zeta) + 5\tilde{m}, \quad \tilde{K} = c(K_1 \vee K_2).$$

Note that α, and $\tilde{m}$ do not depend on (n_1, l_1). Now it is easily seen that the measure of (n_1, l_1) such that (58) holds is

$$\text{mes}\left\{ (n_1, l_1) \ : \ \frac{-|\tilde{m}|\tilde{K} - \alpha\tilde{m}}{m^2} \le \left(n_1 - \frac{m_1 n}{m}\right)^2 + \left(l_1 - \frac{m_1 l}{m}\right)^2 \le \frac{|\tilde{m}|\tilde{K} - \alpha\tilde{m}}{m^2} \right\}$$

which is clearly equal to

$$\frac{c\tilde{K}|\tilde{m}|}{m^2} \sim (K_1 \vee K_2)\frac{M_1 M_2}{M}.$$

Hence we can bound the measure of $B_{\tau\zeta}$ with the product of the projection on the m_1 line and the maximum of the surfaces of the sections with planes parallel to the (n_1, l_1) plane. Thus

$$\text{mes}(B_{\tau\zeta}) \lesssim (K_1 \vee K_2)(M_1 \wedge M_2)\frac{M_1 M_2}{M}$$

and furthermore

$$\text{mes}(A_{\tau\zeta}) \lesssim (K_1 \wedge K_2)(K_1 \vee K_2)(M_1 \wedge M_2)\frac{M_1 M_2}{M}$$

This completes the proof of Proposition 10.

As in 2D, in order to estimate $J_{MM_1M_2}^{KK_1K_2}$ we distinguish the case taking into account which symbol dominates in the left-hand side of (56). By symmetry we assume that $|\sigma_1| \ge |\sigma_2|$.

- Let $|\sigma|$ dominates and $K^{\frac{1}{8}+2\epsilon} \le M^{\frac{1}{2}+4\epsilon}$. Then due to Proposition 10 and (56) we obtain the following bound for the contribution of this region to $J_{MM_1M_2}^{KK_1K_2}$

$$\frac{M^s}{M_1^s M_2^s}\frac{cM}{K^{\frac{1}{2}-\epsilon}K_1^{\frac{1}{2}+\epsilon}K_2^{\frac{1}{2}+\epsilon}}\langle \widehat{u}_{K_1M_1} \star \widehat{v}_{K_2M_2}, \widehat{w}_{KM}\rangle_{L^2}$$

$$\lesssim \frac{M^{\frac{1}{2}}(M_1 \wedge M_2)^{\frac{1}{2}}(M_1 M_2)^{\frac{1}{2}}(K_1 \wedge K_2)^{\frac{1}{2}}(K_1 \vee K_2)^{\frac{1}{2}}}{M^{\frac{1}{2}-2\epsilon}(M_1 M_2)^{1-4\epsilon}K^{\epsilon}K_1^{\frac{1}{2}+\epsilon}K_2^{\frac{1}{2}+\epsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K^{\epsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K \gtrsim M^3 M_1 M_2$, $K \gtrsim K_1$ and $K \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $|\sigma|$ dominates and $K^{\frac{1}{8}+2\varepsilon} \geq M^{\frac{1}{2}+4\varepsilon}$. Then due to Proposition 10 and (56) we obtain the following bound for the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$

$$\frac{M^s}{M_1^s M_2^s}\frac{cM^{\frac{1}{2}-4\varepsilon}}{K^{\frac{3}{8}-3\varepsilon}K_1^{\frac{1}{2}+\varepsilon}K_2^{\frac{1}{2}+\varepsilon}}\langle \widehat{u}_{K_1M_1} \star \widehat{v}_{K_2M_2}, \widehat{w}_{KM}\rangle_{L^2}$$

$$\lesssim \frac{(M_1\wedge M_2)^{\frac{1}{2}}(M_1M_2)^{\frac{1}{2}-s}(K_1\wedge K_2)^{\frac{1}{2}}(K_1\vee K_2)^{\frac{1}{2}}}{M^{\frac{3}{8}-s+\varepsilon}(M_1M_2)^{\frac{3}{4}-8\varepsilon}K^{\varepsilon}K_1^{\frac{1}{2}+\varepsilon}K_2^{\frac{1}{2}+\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{(M_1\wedge M_2)^{\frac{1}{2}}}{M^{\frac{3}{8}-s+\varepsilon}(M_1M_2)^{\frac{1}{4}+s-8\varepsilon}K^{\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2} \qquad (\text{ since } s \geq 8\varepsilon)$$

$$\lesssim \frac{1}{K^{\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.$$

Since $K \gtrsim M^3 M_1 M_2$, $K \gtrsim K_1$ and $K \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case. Note that this is the only case when we use the assumption $s > 0$.

- Let $|\sigma_1|$ dominates and $K^{\frac{1}{8}+2\varepsilon} \leq M^{\frac{1}{2}+4\varepsilon}$. Note that in this case

$$\frac{1+\frac{K^{\frac{1}{8}+2\varepsilon}}{M^{\frac{1}{2}+4\varepsilon}}}{1+\frac{K_1^{\frac{1}{8}+2\varepsilon}}{M_1^{\frac{1}{2}+4\varepsilon}}} \lesssim \frac{M_1^{\frac{1}{2}+4\varepsilon}}{K_1^{\frac{1}{8}+2\varepsilon}}$$

Then due to Proposition 10 and (56) we obtain the following bound for the contribution of this region to $J^{KK_1K_2}_{MM_1M_2}$

$$\frac{M^s}{M_1^s M_2^s}\frac{cMM_1^{\frac{1}{2}+4\varepsilon}}{K^{\frac{1}{2}+\varepsilon}K_1^{\frac{5}{8}+\varepsilon}K_2^{\frac{1}{2}+\varepsilon}}\langle \widehat{w}_{KM} \star \widehat{v}_{K_2M_2}, \widehat{u}_{K_1M_1}\rangle_{L^2} \quad (\text{ since } K_1 \gtrsim K)$$

$$\lesssim \frac{MM_1^{\frac{1}{2}+4\varepsilon}(M\wedge M_2)^{\frac{1}{2}}(MM_2)^{\frac{1}{2}}M_1^{-\frac{1}{2}}(K\wedge K_2)^{\frac{1}{2}}(K\vee K_2)^{\frac{1}{2}}}{M_1^{\frac{5}{8}+\frac{\varepsilon}{2}}(MM_2)^{\frac{5}{4}+\varepsilon}K_1^{\frac{\varepsilon}{2}}K^{\frac{1}{2}+\varepsilon}K_2^{\frac{1}{2}+\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{M(M\wedge M_2)^{\frac{1}{2}}}{M_1^{\frac{5}{8}-\frac{7\varepsilon}{2}}(MM_2)^{\frac{3}{4}+\varepsilon}K_1^{\frac{\varepsilon}{2}}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K_1^{\frac{\varepsilon}{2}}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2} \quad (\text{ considering the cases } M_1 \leq M_2 \text{ and } M_1 \geq M_2).$$

Since $K_1 \gtrsim M^3 M_1 M_2$, $K_1 \gtrsim K$ and $K_1 \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

- Let $|\sigma_1|$ dominates and $K^{\frac{1}{8}+2\varepsilon} \geq M^{\frac{1}{2}+4\varepsilon}$. Note that in this case

$$\frac{1+\frac{K^{\frac{1}{8}+2\varepsilon}}{M^{\frac{1}{2}+4\varepsilon}}}{1+\frac{K_1^{\frac{1}{8}+2\varepsilon}}{M_1^{\frac{1}{2}+4\varepsilon}}} \lesssim \frac{M_1^{\frac{1}{2}+4\varepsilon}}{M^{\frac{1}{2}+4\varepsilon}}$$

Then due to Proposition 10 and (56) we obtain the following bound for the contribution of this region to $J_{MM_1M_2}^{KK_1K_2}$

$$\frac{M^s}{M_1^s M_2^s}\frac{cM^{\frac{1}{2}-4\varepsilon}M_1^{\frac{1}{2}+4\varepsilon}}{K^{\frac{1}{2}}K_1^{\frac{1}{2}}K_2^{\frac{1}{2}+\varepsilon}}\langle \widehat{w}_{KM} \star \widehat{v}_{K_2M_2}, \widehat{u}_{K_1M_1}\rangle_{L^2} \quad (\text{ since } K_1 \gtrsim K)$$

$$\lesssim \frac{M^{\frac{1}{2}-4\varepsilon}M_1^{\frac{1}{2}+4\varepsilon}(M\wedge M_2)^{\frac{1}{2}}(MM_2)^{\frac{1}{2}}M_1^{-\frac{1}{2}}(K\wedge K_2)^{\frac{1}{2}}(K\vee K_2)^{\frac{1}{2}}}{M_1^{\frac{1}{2}-\varepsilon}(MM_2)^{1-2\varepsilon}K_1^{\varepsilon}K^{\frac{1}{2}}K_2^{\frac{1}{2}+\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{M^{\frac{1}{2}-4\varepsilon}(M\wedge M_2)^{\frac{1}{2}}}{M_1^{\frac{1}{2}-\varepsilon}(MM_2)^{\frac{1}{2}-2\varepsilon}K_1^{\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}$$

$$\lesssim \frac{1}{K_1^{\varepsilon}}\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2} \quad (\text{ considering the cases } M_1 \leq M_2 \text{ and } M_1 \geq M_2).$$

Since $K_1 \gtrsim M^3 M_1 M_2$, $K_1 \gtrsim K$ and $K_1 \gtrsim K_2$ a summation over dyadic K, K_1, K_2, M, M_1, M_2 yields the needed bound in this case.

This completes the proof of (54).

CHAPTER 8

Remark on the local ill-posedness for KdV equation

This Chapter essentially contains the paper [59] *(Remark on the local ill-posedness for KdV equation, to appear in C.R. Acad. Sci. Paris).*

Abstract. We prove that the Cauchy problem for KdV equation is locally ill-posed in $\dot{H}^s$, $s < -3/4$ if one asks the flow map $u(0) \longmapsto u(t)$ to be C^2 (in [15] C^3 regularity is needed).

Remarque sur le problème de Cauchy pour l'équation de KdV.

Résumé. On remarque que l'on peut construire un exemple tel que le problème de Cauchy pour l'équation de KdV soit mal posé dans $\dot{H}^s(\mathbf{R})$, $s < -3/4$, si on suppose que l'application qui à une donnée initiale associe la solution à l'instant t soit de classe C^2 (dans [15], une régularité C^3 est supposée).

Version française abrégée

On considère le problème de Cauchy pour l'équation de KdV

$$\text{(1)} \qquad \begin{cases} u_t + u_{xxx} + uu_x = 0, \\ u(0,x) = \phi(x). \end{cases}$$

On a le résultat suivant dû à J. Bourgain (voir [15]).

THÉORÈME 1. *Supposons que $s < -3/4$. Alors il n'existe pas de $T > 0$ tel que l'application*

$$S_t : \phi \longmapsto u(t), \quad t \in [0,T]$$

pour (1) soit Fréchet différentiable de classe C^3 en zéro de $H^s(\mathbf{R})$ dans $H^s(\mathbf{R})$.

Dans [42] il est démontré que (1) est localement bien posé à données dans $H^s(\mathbf{R})$, $s > -3/4$. En fait, l'application S_t est analytique réelle de $H^s(\mathbf{R})$ dans $H^s(\mathbf{R})$. Le point essentiel de la preuve est une estimation bilinéaire dans le cadre des espaces "restriction de Fourier" introduits par J. Bourgain (voir [12, 14]). Dans[42] on trouve un exemple qui démontre que l'estimation bilinéaire cruciale n'est plus satisfaite pour $s < -3/4$. Dans [15], un exemple similaire est donné pour démontrer le théorème 1. En effet, on ne possède plus la régularité suivant les directions $\phi(x)$ telles que

$$\widehat{\phi}(\xi) \sim \mathbf{1}_{[-N-\gamma,-N+\gamma]}(\xi) + \mathbf{1}_{[N-\gamma,N+\gamma]}(\xi),$$

où $\mathbf{1}_A$ est la fonction caractéristique de l'ensemble A. La relation entre N et γ est $\gamma \sim N^{-\frac{1}{2}}$. Le but de cette note est de montrer qu'une régularité C^2 suffit pour démontrer que le problème de Cauchy (1) est mal posé dans les espaces de Sobolev $\dot{H}^s(\mathbf{R})$, $s < -3/4$. La relation entre N et γ est $\gamma \sim N^{-2}$. On a le théorème suivant.

THÉORÈME 2. *Le théorème 1 reste vrai si on demande que S_t soit de classe C^2 de $\dot{H}^s(\mathbf{R})$ dans $\dot{H}^s(\mathbf{R})$.*

Consider the initial value problem for KdV equation

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, \\ u(0,x) = \phi(x). \end{cases} \tag{1}$$

We have the following result due to J. Bourgain (cf. [**15**]).

THEOREM 1. *Suppose that $s < -3/4$. Then there is no $T > 0$ such that the map*

$$S_t : \phi \longmapsto u(t), \quad t \in [0, T]$$

for (1) is C^3 Fréchet differentiable at zero from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$.

It is known that (1) is locally well-posed for data $\phi \in H^s$, $s > -3/4$ (existence, uniqueness, continuous dependence with respect to the data) (cf. [**42**]). In fact the map data-solution turns out to be real analytic. The essential ingredient of the proof is a bilinear estimate in the Fourier transform restriction spaces introduced by J. Bourgain (cf. [**12, 13**]). In [**42**] is given an example which shows that the crucial bilinear estimate fails for $s < -3/4$. In [**15**] a similar construction is performed in order to prove Theorem 1. Actually the C^3 differentiability fails along the directions $\phi(x)$ such that

$$\widehat{\phi}(\xi) \sim \mathbf{1}_{[-N-\gamma,-N+\gamma]}(\xi) + \mathbf{1}_{[N-\gamma,N+\gamma]}(\xi),$$

where $\mathbf{1}_A$ is the characteristic function of the set A. The relation between the frequency N and the parameter γ is $\gamma \sim N^{-\frac{1}{2}}$. Our aim is to show that C^2 regularity of the map data solution suffices to construct the needed example. In our example the relation between N and γ will be $\gamma \sim N^{-2}$. We have the following Theorem.

THEOREM 2. *Theorem 1 is valid if one asks S_t to be of class C^2 from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$.*

Proof. Similarly to [**15**] we consider the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, \\ u(0,x) = \delta\phi(x), \quad \delta \in \mathbf{R}, \end{cases} \tag{2}$$

where $\phi \in \dot{H}^s(\mathbf{R})$, $s < -3/4$, will be chosen later. Suppose that $u(\delta, t, x)$ solves (2) and S_t to be C^2. We have that

$$u(\delta,t,x) = \delta U(t)\phi(x) + \int_0^t U(t-t')u(\delta,t',x)u_x(\delta,t',x)dt',$$

where $U(t) = \exp(-t\partial_x^3)$. Note that

$$\begin{aligned} \frac{\partial u}{\partial \delta}(0,t,x) &= U(t)\phi(x) := u_1(t,x), \\ \frac{\partial^2 u}{\partial \delta^2}(0,t,x) &= 2\int_0^t U(t-t')u_1(t',x)\partial_x u_1(t',x)dt' := u_2(t,x). \end{aligned}$$

We can define in a similar fashion $u_k(t,x)$ as $\frac{\partial^k u}{\partial \delta^k}(0,t,x)$, $k = 3, 4, \cdots$. Taking into account that $u(0,t,x) = 0$ we can formally write a Taylor expansion

$$u(\delta,t,x) = \delta u_1(t,x) + \delta^2 u_2(t,x) + \delta^3 u_3(t,x) + \cdots \tag{3}$$

Our aim here is to show that in some sense the first two terms of (3) are the main contribution to the solution of (2). The assumption of C^2 regularity of S_t yields

$$\|u_2(t,.)\|_{\dot H^s} \le c\|\phi\|^2_{\dot H^s}. \tag{4}$$

We seek for data ϕ such that (4) fails. Consider initial data of type

$$\phi(x) = \gamma^{-\frac{1}{2}} N^{-s} \left\{ \exp(-iNx) \left(\int_0^\gamma \exp(ix\xi) d\xi \right) + \exp(iNx) \left(\int_\gamma^{2\gamma} \exp(ix\xi) d\xi \right) \right\}.$$

Here $N \gg 1$, $\gamma \ll 1$ and the relation between N and γ is to be fixed later. We have

$$\widehat{\phi}(\xi) = \gamma^{-\frac{1}{2}} N^{-s} \left\{ \mathbf{1}_{[-N,-N+\gamma]}(\xi) + \mathbf{1}_{[N+\gamma,N+2\gamma]}(\xi) \right\},$$

Clearly $\|\phi\|_{\dot H^s} \sim 1$. Let $I_1 = [-N, -N+\gamma]$ and $I_2 = [N+\gamma, N+2\gamma]$. We have that

$$\mathcal{F}_{x\mapsto\xi}(u_1)(t,\xi) = \exp(it\xi^3)\widehat{\phi}(\xi)$$

and hence

$$u_1(t,x) \sim \gamma^{-\frac{1}{2}} N^{-s} \int_{\xi\in I_1\cup I_2} \exp(ix\xi + it\xi^3) d\xi.$$

A straightforward computation gives

$$u_2(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi \exp(ix\xi + it\xi^3) \frac{e^{it(\tau-\xi^3)} - 1}{\tau - \xi^3} \left(\widehat{u_1} \star \widehat{u_1}\right)(\tau,\xi) d\tau d\xi.$$

Since $\widehat{u_1}(\tau,\xi) = \delta(\tau - \xi^3)\widehat{\phi}(\xi)$ (δ stays for Dirac delta function) we arrive at

$$\left(\widehat{u_1} \star \widehat{u_1}\right)(\tau,\xi) = \int_{-\infty}^{\infty} \delta(\tau - \xi_1^3 - (\xi-\xi_1)^3)\widehat{\phi}(\xi_1)\widehat{\phi}(\xi-\xi_1) d\xi_1.$$

Hence

$$u_2(t,x) \sim \gamma^{-1} N^{-2s} \int_{\xi_1\in I_1\cup I_2,\, \xi-\xi_1\in I_1\cup I_2} \xi \exp(ix\xi + it\xi^3) \frac{e^{-it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi-\xi_1)} d\xi d\xi_1$$

Further we have

$$\begin{aligned}
\mathcal{F}_{x\mapsto\xi}(u_2)(t,\xi) &\sim \gamma^{-1} N^{-2s} \exp(it\xi^3)\xi \int_{\xi_1\in I_1\cup I_2,\, \xi-\xi_1\in I_1\cup I_2} \frac{e^{-it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi-\xi_1)} d\xi_1 \\
&\sim \gamma^{-1} N^{-2s} \exp(it\xi^3)\xi \left\{ \int_{A_1(\xi)} \cdots + \int_{A_2(\xi)} \cdots + \int_{A_3(\xi)} \cdots \right\} \\
&\sim g_1(t,\xi) + g_2(t,\xi) + g_3(t,\xi),
\end{aligned}$$

where

$$A_1(\xi) = \{\xi_1 : \xi_1 \in I_1, \xi - \xi_1 \in I_1\}, \quad A_2(\xi) = \{\xi_1 : \xi_1 \in I_2, \xi - \xi_1 \in I_2\},$$

$$A_3(\xi) = \{\xi_1 : \xi_1 \in I_1, \xi - \xi_1 \in I_2 \text{ or } \xi_1 \in I_2, \xi - \xi_1 \in I_1\}.$$

Let $f_j = \mathcal{F}^{-1}_{\xi\mapsto x}(g_j)$, $j = 1,2,3$. Then clearly

$$u_2(t,x) \sim f_1(t,x) + f_2(t,x) + f_3(t,x).$$

We shall first give an upper bound for the $\dot{H}^s$ norm of f_1. If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_1$, then $|\xi_1| \sim |\xi - \xi_1| \sim |\xi| \sim N$ and hence

$$\|f_1(t,.)\|_{\dot{H}^s} \leq \gamma^{-1}N^{-2s}N^{-2}\gamma N^s\gamma^{\frac{1}{2}} \sim \gamma^{\frac{1}{2}}N^{-s-2}.$$

Similarly

$$\|f_2(t,.)\|_{\dot{H}^s} \leq c\gamma^{\frac{1}{2}}N^{-s-2}.$$

Now we shall show that with a proper choice of γ and N the main contribution of the $\dot{H}^s$ norm of $u_2(t,.)$ is that of f_3. If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$ then $|\xi_1| \sim |\xi - \xi_1| \sim N$ and $|\xi| \sim \gamma$. Therefore $|\xi\xi_1(\xi - \xi_1)| \sim N^2\gamma$. Choose N and γ such that $N^2\gamma = o(1)$. Hence for $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$ one has

$$\left| \frac{e^{-it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi - \xi_1)} \right| \geq \text{const}$$

and therefore

$$\|f_3(t,.)\|_{\dot{H}^s} \geq cN^{-2s}\gamma^{\frac{3}{2}+s} \sim N^{-4s-3}.$$

Hence

$$\begin{aligned} 1 \sim \|\phi\|^2_{\dot{H}^s} &\geq \|u_2(t,.)\|_{\dot{H}^s} \\ &\geq \|f_3(t,.)\|_{\dot{H}^s} - \|f_1(t,.)\|_{\dot{H}^s} - \|f_2(t,.)\|_{\dot{H}^s} \\ &\geq cN^{-4s-3} - cN^{-s-3}. \end{aligned}$$

Hence $N^{-4s-3} \leq c(1 + N^{-s-3})$. Let $-3 \leq s < -3/4$. Then $N^{-4s-3} \leq c$. Contradiction for $N \gg 1$. Let $s < -3$. Then $N^{-4s-3} \leq cN^{-s-3}$ and hence $N^{-3s} \leq c$. Again contradiction for $N \gg 1$. This completes the proof of Theorem 2.

The proof of Theorem 2 uses the arithmetic fact involving the symbol of the linearized KdV equation

$$(\tau_1 - \xi_1^3) + (\tau - \tau_1 - (\xi - \xi_1)^3) - (\tau - \xi^3) = 3\xi\xi_1(\xi - \xi_1). \tag{5}$$

The relation (5) was used in **[13, 15, 42]** to prove local existence results for KdV equation. We can regard (5) as a smoothing effect which helps to recuperate the loss of one derivative in the nonlinear term. We note that in the example performed here the frequencies ξ_1, $\xi - \xi_1$ and ξ are chosen so that the quantity $\xi\xi_1(\xi - \xi_1)$ be small, which somehow means the loss of the smoothing relation (5).

The fact that $s = -3/4$ turns out to be the critical exponent for the local well-posedness in $H^s(\mathbf{R})$ for the KdV equation is surprising, since it differs from the scaling exponent. The scaling exponent for KdV equation is $s = -3/2$, since the norm $H^{-3/2}(\mathbf{R})$ is invariant under the scale transformations

$$u(t,x) \longmapsto u_\lambda(t,x) = \lambda^2 u(\lambda^3 t, \lambda x),$$

which preserves the KdV equations.

In general, one can try to prove local well-posedness for the Cauchy problem of any evolution equation by considering some exact solutions. KdV equation possesses solitary wave solutions of type $u_c(t,x) = \phi(x-ct)$, where $\phi(x) = \{3/2\text{sech}^2(x/2)\}^{1/2}$. One check that $u_c(0,.)$ converges weakly in H^s, $s < -3/2$ to $\sqrt{2}\pi\partial_x\delta(x)$ as c tends to infinity. A similar consideration shows that $u_c(t,.)$, $t > 0$ converges weakly to zero as c tends to infinity. We also have that $\|u_c(t,.)\|_{H^s} = c^{1/2}\|\phi\|_{H^s}$. Since $\partial_x\delta(x) \in H^s$, $s < -3/2$ (the scaling exponent) the last arguments strongly suggest local ill-posedness for KdV equation assuming that S_t is continuous from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$, $s < -3/2$.

Bibliography

[1] M. Abramowitz, I.A. Stegun, *Handbook of mathematical functions*, Dover (1965).

[2] L.A. Abramyan, Y.A. Stepanyants, *The structure of two-dimensional solitons in media with anomalously small dispersion* , Sov. Phys. JETP, 61 (1985), 963-966.

[3] J. Albert, J. Bona, J.C. Saut, *Model equations for waves in stratified fluids*, Proc. Royal Soc. London A, 453 (1997), 1213-1260.

[4] J.M. Ash, J. Cohen, G. Wang, *On strongly interacting internal solitary waves*, J. Fourier Anal. and Appl. 5 (1996), 507-517.

[5] M. Ben-Artzi, J.-C. Saut, *Uniform decay estimates for a class of oscillatory integrals and applications*, Diff. Int. Eq. 12, (1999), 137-145.

[6] Besov, Ilin, Nikolski, *Integral representation of functions and embedding theorems*, J. Wiley, 1978.

[7] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*, Annales I.H.P., Analyse non linéaire 13 (1996), 529-535.

[8] B. Birnir, G. Ponce, N. Svanstedt, *The local ill-posedness of the modified KdV equation*, J. London Math. Soc. 53 (1996), 551-559.

[9] J. Bona, G. Ponce, J.C. Saut, M. Tom, *A model system for strong interaction between internal solitary waves*, Comm. Math. Phys. 143 (1992), 287-313.

[10] A. de Bouard, J.C. Saut, *Solitary waves of generalized Kadomtsev-Petviashvili equations*, Annales I.H.P., Analyse non linéaire 14 (1997), 211-236.

[11] A. de Bouard, J.C. Saut, *Symmetries and decay of the generalized KP solitary waves*, SIAM J. Math. Anal. 28 (1997), 1064-1085.

[12] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I. Schrödinger equations*, GAFA 3 (1993), 107-156.

[13] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations II. The KdV equation*, GAFA 3 (1993), 209-262.

[14] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA 3 (1993), 315-341.

[15] J. Bourgain, *Periodic Korteweg de Vries equation with measures as initial data*, Sel. Math. New. Ser. 3 (1997), 115-159.

[16] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, IMRN No.5 (1998), 253-283.

[17] J. Bourgain, *Global well-posedness of defocusing critical nonlinear Scrödinger equation in the radial case*, J. AMS (1999), 145-271.

[18] R. Coifman, Y. Meyer, *Au delà des operateurs pseudodifférentiels*, Asterisque 57, Societe Mathematique de France (1978).

[19] J. Colliander, G. Staffilani, H. Takaoka *Global well posedness of KdV below L^2*, Preprint (1998).

[20] A.V. Faminski, *The Cauchy problem for the Kadomtsev-Petviashvili equation*, Russian Math. Sur. 5 (1990), 203-204.

[21] G. Fonseca, F. Lineares, G. Ponce *Global well-posedness for the Modified KdV equation* , Comm. PDE, 24 (1999), 683-705.

[22] J.A. Gear, R. Grimshaw, *Weak and strong interactions between internal solitary waves* , Stud. Appl. Math. 65 (1984), 235-258.

[23] J. Ginibre, *Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain)*, Séminaire Bourbaki n. 796, Astérisque 237 (1996), 163-187.

[24] J. Ginibre, Y. Tsutsumi, G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal., 151 (1997), 384-436.

[25] R. Grimshaw, Y. Zhu, *Oblique interactions between internal solitary waves,* Stud. Appl. Math. 92 (1994) 249-270.

[26] M. Haragus, *Model equations for water waves in the presence of surface tension*, Eur. J. Mech. B/Fluids, 15, 4 (1996), 471-492.

[27] M. Haragus-Courcelle, A. Ill'ichev, *Three dimensional solitary waves in the presence of additional surface effects*, Eur. J. Mech. B/Fluids, 17 (1998).

[28] J.K. Hunter, J. Scheurle, *Existence of perturbed solitary waves to a model equation for water waves*, Physica D, 32 (1988), 253-268.

[29] A. Ill'ichev, *Self-channeling of surface water waves in the presence of an additional surface pressure*, Preprint (1998).

[30] A.T. Il'ichev, A.Yu. Semenov, *Stability of solitary waves in dispersive medias described by a fifth-order evolution equation*, Theoret. Comput. Fluid Dynamics, 3 (1992), 307-326.

[31] R.J. Iório Jr., W.V.L. Nunes, *On equations of KP-type*, Proc. Roy. Soc. Edinburgh A, 128 (1998), 725-743.

[32] P. Isaza, J. Mejia, V. Stallbohm, *El problema de Cauchy para la ecuacion de Kadomtsev-Petviashvili (KP-II) en espacios de Sobolev $H^s, s > 0$*, Preprint (1998).

[33] B.B. Kadomtsev, V.I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Soviet Phys. Dokl. 15 (1970), 539-541.

[34] V.I. Karpman, *Transverse stability of Kawahara solitons*, Phys. Rev. E, 47, 1 (1993), 674-676.

[35] V.I. Karpman, V. Yu. Belashov, *Dynamics of two dimensional solitons in weakly dispersive media*, Phys. Lett. A 154 (1991), 131-139.

[36] V.I. Karpman, V. Yu. Belashov, *Evolution of three-dimensional nonlinear pulses in weakly dispersive media*, Phys. Lett. A 154 (1991), 140-144.

[37] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equations*, Advances in Mathematics, Suplementary Studies, Stud. Appl. Math., 8 (1983), 93-128.

[38] T. Kawahara, *Oscillatory solitary waves in dispersive media*, J. Phys. Soc. Japan, 33, 1, (1972), 260-264.

[39] C. Kenig, G. Ponce, L. Vega, *On the (generalized) Korteweg-de Vries equation*, Duke Math. J., 59 (1989), 585-610.

[40] C. E. Kenig, G. Ponce, L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. AMS, 4 (1991), 323-347.

[41] C. E. Kenig, G. Ponce, L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J., 71 (1993), 1-21.

[42] C. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equations*, J. AMS, 9 (1996), 573-603.

[43] C. Kenig, G. Ponce, L. Vega, *Quadratic forms for $1 - D$ semilinear Schrödinger equation*, Trans. Amer. Math. Soc., 348 (1996), 3323-3353.

[44] S.N. Kruzhkov, A.V. Faminski, *Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation*, Math. U.S.S.R. Sbornik 48 (1984), 93-138.

[45] F. Linares, *L^2 global well-posedness of the initial value problem associated to the Benjamin equation*, J. Diff. Equations, 152 (1999), 377-393.

[46] A.V. Marchenko, *Long waves in shallow liquid under ice cover*, PMM USSR, 52, 2 (1998), 180-183.

[47] L. Molinet, *On the asymptotic behaviour of solutions to the (generalized) Kadomtsev-Petviashvili-Burgers equations*, J. Diff. Eq. 152 (1999), 30-74.

[48] J.C. Saut, *Remarks on the generalized Kadomtsev-Petviashvili equations*, Indiana Univ. Math. J. 42 (1993), 1017-1029.

[49] J.C. Saut, N. Tzvetkov, *The Cauchy problem for higher order KP equations*, J. Diff. Eq. 153 (1999), 196-222.

[50] J.C. Saut, N. Tzvetkov, *The Cauchy problem for the fifith order KP equations*, Preprint (1999).

[51] R. Strichartz, *Restriction of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J., 44 (1977), 705-714.

[52] H. Takaoka, *Well-posedness for the Kadomtsev-Petviashvili II equation*, Preprint (1998).

[53] H. Takaoka, *Global well-posedness for the Kadomtsev-Petviashvili II equation*, Discrete Contin. Dynam. Systems (to appear).

[54] H. Takaoka, N. Tzvetkov, *On the local regularity of Kadomtsev-Petviashvili II equation*, Preprint (1999).

[55] M.M. Tom, *On a generalized Kadomtsev-Petviashvili equation*, Contemporary Mathematics A.M.S. 200 (1996), 193-210.

[56] N. Tzvetkov, *Remarque sur la régularité locale de l'équation de Kadomtsev-Petviashvili*, C.R. Acad. Sci. Paris, t. 326, Série I, 709-712, 1998.

[57] N. Tzvetkov, *On the Cauchy problem for Kadomtsev-Petviashvili equations*, Comm. PDE, 24 (1999), 1367-1397.

[58] N. Tzvetkov, *Global low regularity solutions for Kadomtsev-Petviashvili equation*, Diff. Int. Eq. (to appear).

[59] N. Tzvetkov, *Remark on the local ill-posedness for KdV equation* , C.R. Acad. Sci. Paris (to appear).

[60] S. Ukai, *Local solutions of the Kadomtsev-Petviashvili equation*, J. Fac. Soc. Univ. Tokyo Sect. I A Math., 36 (1989), 193-209.

[61] M.V. Wickerhauser, *Inverse scattering for the heat operator and evolution in* $2+1$ *variables*, Comm. Math. Phys., 108 (1987), 67-87.