OLIVIER GLASS

Sur la contrôlabilité des fluides parfaits incompressibles

Thèses d'Orsay, 2000

<http://www.numdam.org/item?id=BJHTUP11_2000__0577__P0_0>

ORSAY
n° d'ordre : 6032

UNIVERSITÉ DE PARIS-SUD

CENTRE D'ORSAY

THÈSE

présentée
pour obtenir

Le GRADE de DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS XI ORSAY

Spécialité : Mathématiques

PAR

Olivier Glass

Sujet : **Sur la contrôlabilité des fluides parfaits incompressibles**

Soutenue le: 13 Janvier 2000 devant la Commission d'examen

M.	Y. Brenier	
M.	J. M. Coron	
M.	G. Lebeau	
M.	J. P. Puel	Rapporteur
M.	J. C. Saut	
M.	E. Zuazua	Rapporteur

Remerciements

Au moment de terminer la rédaction de ce mémoire, je souhaiterais avant toute chose remercier MM. les Professeurs Jean-Pierre Puel et Enrique Zuazua de s'être intéressés à mon travail et d'avoir eu la patience de tenir la charge de rapporteur. Je suis également très reconnaissant à MM. les Professeurs Yann Brenier, Gilles Lebeau et Jean-Claude Saut d'avoir accepté de figurer dans le Jury de cette thèse. Je suis par ailleurs redevable à Jean-Claude Saut d'avoir répondu utilement à un certain nombre de questions que je lui ai posées au cours de ce travail.

Je voudrais aussi remercier les nombreuses personnes qui de plus ou moins près ont pu m'apporter leur aide. Certains m'ont fait partager leurs connaissances mathématiques (dans un bureau, mais aussi dans un couloir ou autour d'une tasse de café), d'autres m'auront apporté leur soutien en anglais ou encore en informatique (j'en passe). Qu'ils en soient tous remerciés. Dans cette liste un peu longue, nul doute que quelques personnes (d'importance) auront été au passage oubliées. J'espère simplement qu'elles n'en prendront pas trop ombrage.

Je tiens donc à remercier (par ordre alphabétique) : Stéphane Aicardi, Jacques Bailly, Pr. Claude Bardos, Jacques Beigbeder, Pr. Fabrice Béthuel, Gilles Blanchard, Sylvain Bruiltet, Yacine Chitour, Juliusz Chroboczek, Nicolas Chung Siong Fah, Charles Favre, Pr. Andrei Fursikov, mes parents, Vincent Guedj, Mickaël Gutnic, Danielle Le Meur, Vincent Maillot, Ivan Marin, Denis Petrequin, Marine Picon, Séverine Rigot, Sylvia Serfaty, Cédric Villani.

Mes derniers mots iront bien entendu en direction de Jean-Michel. Celuici s'est montré d'un soutien constant, et a fait preuve à mon égard d'attentions qui allaient largement au delà de ce que l'on peut attendre d'un directeur de thèse. Qu'il trouve ici un modeste témoignage de ma reconnaissance et de mon amitié.

Résumé

Nous étudions différents problèmes de contrôlabilité pour ce qui est de l'équation d'Euler pour les fluides parfaits incompressibles, étudiés initialement par Jean-Michel Coron dans le cas bidimensionnel. Dans le cas d'un fluide tridimensionnel, nous établissons un résultat de contrôlabilité exacte au bord pour des domaines ouverts bornés et réguliers de l'espace, et une zone de contrôle sur le bord qui satisfait la condition nécessaire, qu'elle rencontre toutes les composantes connexes du bord. Une attention particulière est apportée à la topologie du domaine. Dans le cas d'un fluide du plan, nous établissons plusieurs résultats de contrôlabilité approchée lorsque la zone de contrôle au bord ne satisfait pas la condition d'intersection non nulle avec chaque composante connexe du bord. J.-M. Coron a établi un résultat de contrôlabilité approchée dans ce cas ; la qualité de la contrôlabilité approchée, en termes de topologie de l'approximation, limitée par des invariants du processus, peut être améliorée à des espaces de Sobolev de dérivées supérieures, si on se restreint à des configurations qui précisément se correspondent du point de vue de ces invariants.

Abstract

We study different controllability problems concerning the Euler system for the incompressible inviscid fluids. The controllability of these equations was first studied by Jean-Michel Coron in the two-dimensional case. In the case of a thre e-dimensional fluid, we establish a result of boundary exact controllability, fo r open regular domains, and for a control zone which meets any connected compone nt of the boundary (this condition is necessary for the general result). The top ological nature of the domain is crucial for the kind of control applied on the boundary. In the case of a two-dimensional fluid, we establish some results of approximate controllability when the control zone does not satisfy the condition that it sh ould meet any connected component of the boundary. J.-M. Coron established a res ult of approximate controllability in this case ; but the quality of the control , in terms of the topology of the approximation, which is limited by certain inv ariants of the process, can be improved to some Sobolev spaces of higher derivat ives, if one considers only configurations for which the value of these invarian ts are equal.

Table des matières

Chapitre 1

Introduction

1.1 Introduction générale

1.1.1 Position du problème

Dans l'étude mathématique de systèmes physiques, en particulier pour ceux gouvernés par des équations aux dérivées partielles, il est naturel de se poser la question suivante : étant donné un système au sujet duquel on est suffisamment informé (pour lequel on connaît des résultats d'existence et éventuellement d'unicité des solutions), est-il possible de le contrôler, c'est à dire d'imposer à ses solutions de prendre une certaine forme souhaitée ? Ceci est l'objet de la théorie du contrôle des EDP, que l'on pourrait rattacher plus généralement à celle des «problèmes inverses».

Le but de ce travail est d'apporter quelques résultats en cette matière pour ce qui est de la mécanique des fluides. Celle-ci est un champ d'études mathématiques encore largement ouvert ; le problème de l'hydrodynamique tridimensionnelle laisse encore par exemple la place à de nombreuses interrogations. Cela dit, l'état actuel des connaissances autorise déjà la constitution d'un certain nombre de résultats sur le contrôle de ces équations.

Le contrôle actif en mécanique des fluides est une discipline qui concerne autant le champ mathématique que le champ physique voire industriel. Pour les physiciens, il s'agit de tenter de stabiliser un fluide par différents moyens (injection de fluide, contacts piézoélectriques, ondes sonores, etc.), avec comme objectif par exemple la réduction de la turbulence et du bruit. L'industrie aéronautique en particulier, qui pour l'instant se contente essentiellement de contrôles «passifs», y voit une ouverture vers la réduction active de traînée derrière les ailes d'avion, en vue de réduire ainsi le coefficient de résistance à l'avancement.

Cela étant, on est encore loin d'une solution mathématique susceptible de répondre aux besoins physiques en la matière. Il est évident que les problèmes considérés en mathématiques sont (encore) grossiers en regard des expériences physiques qui ont été menées. Il est cependant loisible d'espérer que l'étude mathématique de ces questions pourra contribuer d'un certain point de vue à une résolution du problème physique, même si la résolution complète d'un système adapté à la situation réelle reste encore largement hors de portée.

Mathématiquement et indépendamment des éventuelles applications, le contrôle en mécanique des fluides conduit à de nombreux problèmes ouverts. Il s'agit essentiellement de prouver la possibilité de «connecter» deux profils de vitesses différents en suivant la loi et les contraintes que l'on s'est fixées. Le présent travail répond à quelques questions concernant le système d'Euler (qui néglige les effets de viscosité) incompressible tant bidimensionnel que tridimensionnel.

Dans la suite de cette introduction, nous décrivons plus précisément les problèmes mathématiques considérés, nous évoquons d'autres résultats de la discipline, puis nous expliquons et justifions la méthode utilisée. Nous commentons également quelque peu les résultats établis dans les sections suivantes.

1.1.2 Le problème mathématique

Des différents systèmes considérés en mécanique des fluides incompressibles, le système de Navier-Stokes incompressible (qui concerne les fluides «newtonniens») :

$$\left\{ \begin{array}{l} \partial_t y + (y.\nabla)y - \nu\Delta y = \nabla p + f, \\ \operatorname{div} y = 0, \end{array} \right. \tag{1.1}$$

est le plus couramment utilisé. Dans (1.1), y désigne la vitesse locale du fluide, et p la pression (définie à constante près). Le paramètre ν est un réel strictement positif qui décrit la viscosité du fluide. La fonction f désigne une répartion locale de force sur le domaine occupé par le fluide. Enfin, ∂_t désigne la dérivation par rapport au temps t, Δ est le laplacien (portant sur les variables d'espace), $\operatorname{div} X = \sum_{i=1}^n \partial_i X^i$ et $(X.\nabla)Y : = \sum_{i=1}^n X^i \partial_i Y$, où n est la dimension de l'espace ambiant, X^i est la i-ème coordonnée du champ de vecteur X et ∂_i est la dérivée par rapport à cette i-ème coordonnée. Nous noterons enfin qu'il est simple de transformer ce système en un système où y est la seule inconnue, et où p n'apparaît plus ; aussi fréquemment oublierons-nous de citer p quand nous parlerons d'une solution de cette équation.

Lorsque l'on prend $\nu = 0$, c'est à dire que l'on néglige le terme de viscosité, le fluide est dit parfait et l'équation de Navier-Stokes correspondante est l'équation d'Euler :

$$\begin{cases} \partial_t y + (y.\nabla)y = \nabla p + f, \\ \operatorname{div} y = 0. \end{cases} \tag{1.2}$$

Cette équation constitue le premier système à avoir été proposé pour l'étude des fluides.

À ces équations, il faut bien entendu ajouter des conditions initiales (la vitesse locale à l'instant 0), et des conditions au bord du domaine, dont la nature dépend du système considéré.

On dispose d'un certain nombre de résultats d'existence et d'unicité de solutions fortes pour cette équation, pour tout temps en dimension 2, pour un temps minimal calculable en fonction des conditions initiales et des conditions au bord en dimension 3.

Dès lors, peut se poser le problème de la contrôlabilité de cette équation. Celui-ci est le suivant : on se fixe un profil de vitesse initial fixé, et un autre profil de vitesse «cible». Existe-t-il une solution de l'équation avec cette condition initiale, qui vérifie certaines contraintes (par exemple la condition au bord fixée sur une partie de celui-ci) et qui atteigne au bout d'un temps fixé à l'avance le profil cible, ou à défaut s'en approche arbitrairement près (au sens d'une topologie à préciser) ?

On peut distinguer deux cas de «contrôle» dans ce type de problème, selon le moyen que l'on possède pour exercer son influence le système :

- le contrôle intérieur, dans ce cas la condition au bord est fixée pour l'intégralité de celui-ci et la liberté dont on dispose pour trouver une solution au problème réside dans la possibilité d'imposer la répartition de force locale sur un sous-domaine fixé à l'avance ;

- le contrôle au bord, cas dans lequel la répartition de forces est fixée, mais on dispose en revanche de la condition au bord sur une partie de celui-ci.

Ces deux cas se recouvrent largement, et l'on s'est intéressé ici à la question exprimée sous sa forme de contrôlabilité au bord. Par ailleurs, le système qui nous a intéressé est celui où les forces extérieures sont nulles (ou dérivent d'un gradient). Le problème est alors formalisé de la façon suivante : soit Ω un domaine ouvert, non vide, borné et régulier du plan ou de l'espace ; on considère une partie ouverte non vide Σ de son bord $\partial\Omega$. On se fixe un réel strictement positif T, et deux champs de vecteurs y_0 et y_1 solénoïdaux, i.e. qui vérifient

$$\operatorname{div} y_0 = \operatorname{div} y_1 = 0 \text{ dans } \overline{\Omega}, \tag{1.3}$$

qui satisfont une contrainte précise sur $\partial\Omega\backslash\Sigma$ (les conditions au bord adaptées à l'équation d'Euler dépendent de la dimension, néanmoins ici la contrainte au bord se résume à

$$y_0.\nu = y_1.\nu = 0 \text{ sur } \partial\Omega\backslash\Sigma, \tag{1.4}$$

où ν est la normale unitaire extérieure). Existe-t-il une solution (y,p) du système (1.2) (le même problème se pose avec (1.1)) définie sur $\overline{\Omega} \times [0,T]$, dont la donnée initiale vérifie

$$y_{|t=0} = y_0 \text{ dans } \overline{\Omega}, \tag{1.5}$$

qui durant toute la période $[0,T]$ continue de subir la même contrainte au bord :

$$y.\nu = 0 \text{ sur } (\partial\Omega\backslash\Sigma) \times [O,T], \tag{1.6}$$

et telle que

$$y_{|t=T} = y_1 \text{ sur } \overline{\Omega}, \tag{1.7}$$

(on parle alors de «contrôlabilité exacte»), ou encore telle que

$$y_{|t=T} \text{ soit proche de } y_1 \text{ au sens d'une certaine topologie,} \tag{1.8}$$

(on parle alors de «contrôlabilité approchée») ?.

Autrement dit, peut-on passer d'un écoulement fixé à un autre, en un temps précis, et ce en injectant et aspirant du fluide sur une partie du bord ?

Si on obtient une réponse positive à l'un de ces problèmes, on appellera contrôle la condition au bord sur Σ qui réalise la solution du problème.

Ces questions ont été soulevées par J.-L. Lions dans [25]. Les premiers travaux sur le contrôle de l'équation d'Euler ont été réalisés par J.-M. Coron, dans le cadre bidimensionnel, dans [5] (pour les domaines simplement connexes) et [6] (dans le cas général). Citons également [8] où il est question de stabilisation par retour d'état stationnaire.

Dans la deuxième section de cette introduction, nous évoquerons les problèmes relatifs au contrôle de l'équation d'Euler de façon générale ; la troisième section constitue une introduction au deuxième chapitre de ce travail, qui concerne le cas tridimensionnel. Enfin, la quatrième section de cette introduction évoque le cas bidimensionnel et notre travail dans ce domaine, qui constitue le troisième chapitre de ce mémoire.

1.2 Équation d'Euler et contrôle

Dans cette section, nous présentons quelques généralités concernant le problème du contrôle de l'équation d'Euler. Nous reviendrons plus tard sur les spécificités des problèmes bidimensionnel et tridimensionnel.

1.2.1 Objections à la contrôlabilité

Dans un premier temps, on peut remarquer que, dans le cas d'un domaine dont le bord comporte plusieurs composantes connexes, des objections peuvent être soulevées à l'encontre de la contrôlabilité exacte si la zone de contrôle Σ ne rencontre pas chacune de ces composantes.

Deux objections notamment apparaissent naturellement : d'une part, comme remarqué dans [5] et [6], la loi de Kelvin, qui assure que la circulation de vitesse le long d'une courbe de Jordan est constante dans le temps lorsque cette courbe suit le flot, implique en particulier (en dimension 2) que les circulations de vitesse le long des courbes qui composent la frontière et qui ne sont pas rencontrées par Σ sont constantes. En dimension 3, on a la même propriété pour les courbes incluses dans les composantes connexes du bord qui ne rencontrent pas Σ, mais dans ce cas ces courbes ne sont plus fixes. D'où, dans le premier cas, la nécessité d'égalité des circulations le long de ces courbes pour y_0 et y_1 pour espérer un contrôle exact entre ces deux profils de vitesse. Dans le second cas, la contrôlabilité exacte ne peut pas être établie non plus quels que soient les y_0 et y_1 : prenons par exemple $y_0 = 0$, il devient nécessaire que la restriction de y_1 à n'importe quelle composante connexe du bord qui n'intersecte pas Σ soit un gradient sur cette surface.

L'autre objection est que la vorticité suit le flot de la vitesse (ou du moins son support en dimension 3), et qu'en conséquence la vorticité est «confinée» dans les courbes du bord non rencontrées par Σ.

En conséquence, lorsque l'on étudiera la contrôlabilité exacte du système, on se placera dans le cas où Σ rencontre toutes les composantes connexes du bord. Ce ne sera pas a priori le cas pour les problèmes de contrôlabilité approchée. Nous verrons cependant par la suite que ces objections peuvent être également valables dans une certaine mesure pour la contrôlabilité approchée, selon l'espace fonctionnel considéré.

Ces objections soulevées, étudions à présent le problème de plus près.

1.2.2 Simplifications du problème

Deux remarques peuvent apporter quelques simplifications au problème exact.

D'une part, l'équation d'Euler est réversible dans le temps. Autrement dit, considérons une solution y de (1.2) (qui part de $y_{|t=0}$ pour terminer au temps T à $y_{|t=T}$). Alors la fonction

$$(t,x) \mapsto -y(T-t,x),$$

est également solution de l'équation. Il s'ensuit immédiatement que l'on peut se limiter dans l'étude de la contrôlabilité exacte (bien entendu pas pour la contrôlabilité approchée), au cas où $y_1 = 0$. Il suffit alors de résoudre le problème restreint avec y_0 et $-y_1$ comme données de départ, puis de les «recombiner» pour obtenir une solution du problème général.

D'autre part, la seconde remarque est une observation d'homogénéité de l'équation. En effet, si y est de même une solution de l'équation d'Euler définie sur $\Omega \times [0,T]$ ($\Omega \subset \mathbb{R}^2$ ou $\mathbb{R}^3$), alors la fonction suivante

$$\hat{y} : \left\{ \begin{array}{l} \Omega \times [0,\epsilon T] \to \mathbb{R}^2 \text{ ou } \mathbb{R}^3, \\ (x,t) \mapsto \frac{1}{\epsilon} y(x, \frac{t}{\epsilon}), \end{array} \right.$$

est aussi solution de l'équation. Il s'ensuit que si l'on sait résoudre, pour n'importe quel temps, le problème de contrôlabilité exacte pour donnée initiale petite (au sens d'une topologie adéquate), et pour donnée finale nulle, on sait résoudre le problème général.

La contrôlabilité exacte se ramène donc à chercher une solution au système, qui passerait d'un point proche de 0 (dans une norme assez forte si on le souhaite) au point 0 exactement ; il est donc tout naturel d'essayer de résoudre le problème de contrôlabilité sur l'équation linéarisée autour de la fonction nulle (elle même solution du problème non linéaire).

1.2.3 Tentative de résolution

Or on s'aperçoit rapidement (la remarque émane de J.-L. Lions) que l'équation linéarisée autour de 0 n'est pas du tout contrôlable. En effet, celle-ci a la forme suivante :

$$\left\{ \begin{array}{l} \partial_t y = \nabla p, \\ \operatorname{div} y = 0. \end{array} \right.$$

Ainsi, partant d'un certain $y_{|t=0}$, on ne pourra atteindre que des champs de vecteurs du type $y_{|t=0} + \nabla q$ (avec $\Delta q = 0$), c'est à dire que l'on ne peut atteindre qu'une minuscule partie de l'ensemble des champs à divergence nulle et satisfaisant la contrainte au bord.

Cependant, si cela réduit les chances d'obtenir une solution au problème général directement, la non-contrôlabilité du problème linéarisé n'implique pas celle du problème non-linéaire. Au contraire, certains problèmes de contrôle en dimension finie donnent des exemples d'équations non-linéaires contrôlables, dont le linéarisé ne l'est pas.

Cependant, on peut considérer au lieu de l'équation linéarisée autour de 0, l'équation linéarisée autour d'une solution du problème non-linéaire, qui n'est plus nulle, mais qui fait une boucle en partant de 0 pour y revenir, avec l'espoir d'obtenir cette fois-ci un système linéarisé contrôlable. C'est là l'objet de la méthode du retour, introduite dans [3] par J.-M. Coron, initialement pour résoudre des problèmes de contrôle en dimension finie (voir aussi [4]).

Mais avant d'expliquer plus longuement comment pallier le manque de contrôlabilité de l'équation linéarisée autour de 0, nous souhaiterions mettre cette situation en regard de celle qui concerne l'équation de Navier-Stokes.

1.2.4 Une situation bien différente : les travaux de A.V. Fursikov et O.Yu. Imanuvilov concernant l'équation de Navier-Stokes

Revenons à l'équation (1.1). La question se pose de la même façon que pour l'équation d'Euler, de la contrôlabilité du système, soit par le biais d'un contrôle au bord (on «choisit» dans ce cas la vitesse entière sur une partie du bord, celle-ci étant imposée, ou fortement contrainte, sur le reste du bord), soit par le biais d'une répartition de force sur une partie ouverte préalablement imposée du domaine.

De nombreux problèmes de ce type ont été résolus dans les années 1990 par A.V. Fursikov et O.Yu. Imanuvilov en dimension 2 et en dimension 3 (voir par exemple les références de [14] à [17], ainsi que [22]).

Citons par exemple le résultat suivant :

Théorème 1 (Imanuvilov) *Soit Ω un domaine non vide, simplement connexe, borné et régulier de $\mathbb{R}^2$ ou $\mathbb{R}^3$. Soit ω un sous-domaine ouvert et non vide de Ω. Soit $T > 0$. Considérons $\hat{y}(x)$ une solution des équations de Navier-Stokes stationnaires*

$$\left\{ \begin{array}{l} -\Delta\hat{y} + (\hat{y}.\nabla)\hat{y} = \hat{f} + \nabla\hat{p}, \ \textit{dans } \Omega, \\ \operatorname{div}\hat{y} = 0 \ \textit{dans } \Omega, \end{array} \right. \tag{1.9}$$

de régularité $W^{1,\infty}$, où f est de régularité $L^2(\Omega)$, et supposons de plus qu'elle satisfasse

$$\textit{Supp } \hat{y} \subset\subset \Omega. \tag{1.10}$$

Soit $y_0(x)$ une fonction de $\overline{\Omega}$ dans $\mathbb{R}^2$ (respectivement $\mathbb{R}^3$) telle que

$$\operatorname{div} y_0 = 0 \ \textit{dans } \Omega,$$

$$y_0 = 0 \text{ sur } \partial\Omega, \tag{1.11}$$

et

$$\|\hat{y} - y_0\|_{H^1(\Omega)} < \epsilon,$$

où $\epsilon = \epsilon(\hat{y})$ est suffisament petit.

Alors il existe un contrôle intérieur $u \in L^2(\Omega)$ tel que la solution y de classe $H^1(\Omega)$ de l'équation de Navier-Stokes (1.1) avec y_0 comme condition initiale, avec comme condition au bord

$$y = 0 \text{ sur } \partial\Omega,$$

et avec $f := \hat{f} + \chi_\omega u$ comme répartition de forces (où χ_ω est la fonction caractérisitique de ω dans Ω) satisfasse

$$y(T, x) = \hat{y}(x) \text{ dans } \Omega.$$

Remarquons que du fait de l'effet régularisant de l'équation, on ne pouvait espérer une contrôlabilité exacte au sens indiqué plus haut. Notons de plus que le coefficient correspondant à la viscosité peut être choisi différent de 1 par homogénéité de l'équation.

Ajoutons qu'il existe un théorème équivalent où, au lieu d'une solution du système (1.9), on considère un profil de vitesse obtenu comme solution de l'équation de Navier-Stokes (1.1) au bout d'un temps T.

Ce théorème s'obtient, à la différence du cas des fluides parfaits, en prouvant la contrôlabilité du système linéarisé autour de la fonction $\hat{y}$ considérée (les remarques de la section 1.2.2 ne s'appliquant plus ici, on ne peut plus se limiter au linéarisé autour de 0).

Pour prouver cette contrôlabilité du linéarisé, ils établissent une inégalité d'observabilité pour le problème adjoint (dans un espace pondéré bien choisi). Il s'agit ici de l'application de la méthode HUM, introduite par J.-L. Lions pour les problèmes de contrôlabilité des équations linéaires (voir [27]). L'inégalité d'observabilité en question est une estimée de type Carleman, adaptée par les auteurs pour ce problème.

De là, par une méthode de point fixe particulière, ils parviennent ensuite à démontrer la contrôlabilité locale du système non linéaire.

Il est possible d'établir un résultat de contrôlabilité approchée (mais «globale») pour cette équation : pour cela, on utilise la méthode du retour en regardant d'autres équations linéarisées que celle autour de $\hat{y}$ (en particulier, autour de solutions potentielles de l'équation de Navier-Stokes, qui sont également des solutions de léquation d'Euler, et qui sont utilisées dans le traitement de celle-ci). Dans cette optique, l'étude du contrôle l'équation d'Euler donne une alternative pour le traitement des équations de Navier-Stokes (avec condition de «glissement de Navier» au bord). Pour plus de précisions, nous renvoyons le lecteur à l'article de Jean-Michel Coron référencé par [7], ainsi qu'à son article conjoint avec Andrei Fursikov, référencé par [10].

Pour ce qui est du contrôle de Navier-Stokes, nous citerons également l'article de Caroline Fabre référencé par [11], où il est question d'équation de Navier-Stokes avec terme non-linéaire tronqué. Dans cet article, l'auteur utilise une méthode introduite par C. Fabre J.-P. Puel et E. Zuazua dans [12] et [35], et par E. Fernandez-Cara et J. Real dans [13]. Enfin, citons l'article [29], où est étudié le contrôle des approximations de type Galerkin de l'équation de Navier-Stokes.

1.2.5 Le système d'Euler : utilisation de la méthode du retour

Le fait que le linéarisé du système d'Euler autour de 0 ne soit pas contrôlable n'empêchera pas le système non linéaire de l'être : le système linéarisé autour d'une solution $\overline{y}$ particulière du système, qui part de 0 pour y revenir, sera en revanche contrôlable.

De là, on essaiera de passer au système non-linéaire. Ainsi, dans cette construction, pour passer d'un profil de vitesse proche de 0 à un profil de vitesse nul, on ne tente plus de trouver une solution qui reste dans un voisinage de 0, mais au contraire, on fait sortir l'état du système loin, pour revenir et atteindre exactement la cible. Nous donnons dans la section suivante un exemple qui expose la nécessité de «sortir du voisinage de 0».

La solution $\overline{y}$ en question prend la forme d'un écoulement potentiel, i.e. $\overline{y} : = \nabla\theta(x,t)$ ou le symbole ∇ porte sur la partie spatiale. De cette manière, lorsqu'on aura finalement trouvé par cette méthode une solution à notre problème de contrôlabilité, on se sera «éloigné de 0» du point de vue de la vitesse locale, mais cependant du point de vue de la vorticité on sera resté proche de 0.

Comment établir la contrôlabilité de l'équation linéarisée autour de $\overline{y}$? Celle-ci peut être prouvée de manière constructive (une fois connu $\overline{y}$) ; elle dérive de la forme de $\overline{y}$ et de l'utilisation de la règle suivante : lorsque l'on considère l'equation linéarisée autour de $\overline{y}$, la vorticité (le rotationnel) de la solution est constante en suivant le flot de $\overline{y}$ (dans le cas bidimensionnel). Dans le cas tridimensionnel, la vorticité n'est pas conservée, mais à défaut, le support de cette vorticité suit le flot de $\overline{y}$, et sa modification est facile à mesurer, car elle satisfait une certaine équation linéaire.

On choisira alors (dans le cas d'un domaine simplement connexe) un $\overline{y}$ qui satisfasse à la propriété suivante : tout point du domaine $\overline{\Omega}$, en suivant le flot de $\overline{y}$, est à un moment ou un autre envoyé à l'extérieur de $\overline{\Omega}$, puis est remis en place (ce dernier point étant facultatif). On construit alors la solution du problème linéarisé autour de $\overline{y}$ en précisant le rotationnel de celle-ci : pour un point qui reste à l'intérieur du domaine en suivant le flot de $\overline{y}$, ce rotationnel est fixée par l'équation, mais lorsque le point sort, on peut modifier la vorticité qui lui est allouée de telle sorte que lorsqu'il reprend sa place, la valeur de vorticité affectée est celle requise.

1.2.6 De la «nécessité» de s'éloigner de 0

Expliquons en quoi la méthode du retour est en quelque sorte nécessaire pour obtenir le résultat final.

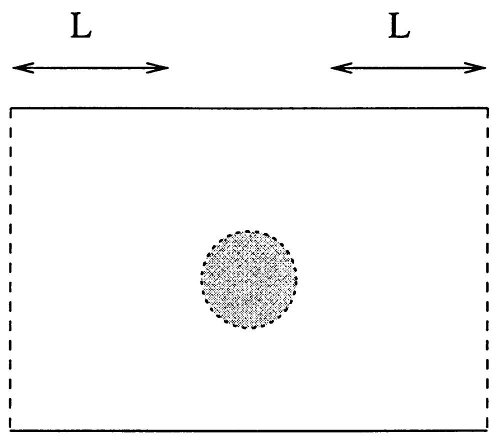

FIG. 1.1: *Un exemple de domaine de contrôle.*

On se positionne dans le cas où l'ouvert Ω où l'on cherche à contrôler le fluide est un rectangle (pour le cas de la dimesnion 2) ou un cylindre (pour le cas de la dimension 3). La zone de contrôle est donnée par les «faces latérales» (les disques dans le cas de la dimension 3), comme indiqué sur la figure 1.1.

Maintenant, on considère le profil de vitesse y_0 défini de la manière suivante : on définit une fonction w de classe $C_0^\infty(\Omega)$ (dans $\mathbb{R}$ en dimension 2, dans $\mathbb{R}^3$ en dimension 3), à support dans le petit sous-domaine grisé de la figure, non nulle, et de norme $C^{k,\alpha}$ inférieure à un réel strictement positif ϵ (k fixé dans $\mathbb{N}$). On demande que la distance de cette «zone de vorticité» soit éloignée d'une distance au moins L de la zone de contrôle. On définit alors y_0 comme l'unique fonction de $C^\infty(\overline{\Omega};\mathbb{R}^n)$ (n est la dimension de l'espace ambiant à savoir 2 ou 3) vérifiant

$$\begin{cases} \operatorname{div} y_0 = 0 \text{ dans } \Omega, \\ \operatorname{rot} y_0 = w \text{ dans } \Omega, \\ y_0.\nu = 0 \text{ sur } \partial\Omega. \end{cases}$$

On se pose le problème de la contrôlabilité exacte avec $y_1 := 0$, entre le temps 0 et le temps 1. Alors considérons ce qui se passe quand ϵ tend vers 0.

Bien entendu, selon l'estimation elliptique bien connue, y_0 a une norme $C^{k+1,\alpha}$ d'ordre ϵ. Pour autant, on ne peut guère espérer passer (en suivant le système d'Euler) de y_0 à y_1 «en restant dans un voisinage» de 0. En effet, la vorticité étant constante en suivant le flot de la vélocité du fluide, il s'ensuit que pour une solution y du système qui méne de y_0 à y_1 en un temps 1, on a l'inégalité suivante :

$$\|y\|_{L^\infty(\Omega\times[0,1])} \geq L.$$

Ainsi, il est nécessaire de quitter les alentours de 0 avant d'y retourner ; on ne peut pas espérer trouver une solution d'ordre ϵ. Notons que cela ne vaut que parce que le temps de contrôle est fixé ; si on laisse le temps devenir grand, cela devient possible de tendre vers 0 en restant dans un voisinage de 0, ce qui autorise la stabilisation asymptotique (voir [8]).

1.3 Le cas tridimensionnel

Dans cette partie, nous décrivons les spécificités du cas tridimensionnel en expliquant brièvement la méthode de démonstration du théorème suivant, dont la preuve précise fait l'objet du deuxième chapitre de ce travail.

Théorème 2 *Soit Ω un domaine ouvert, non vide, borné et régulier de $\mathbb{R}^3$. Soit Σ une partie ouverte non vide de son bord $\partial\Omega$, qui rencontre toutes les composantes connexes de $\partial\Omega$. Considérons $\alpha \in (0,1)$ et $T > 0$. Soient enfin y_0 et y_1 deux fonctions de $C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)$, satisfaisant (1.3) et (1.4). Alors il existe une fonction $y \in C^0([0,T],C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3)) \cap L^\infty([0,T],C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$, solution de (1.2) sur $\Omega\times[0,T]$ pour un certain $p \in \mathcal{D}'((0,T)\times\Omega)$, satisfaisant la contrainte (1.6) et de plus (1.5) et (1.7).*

A noter que dans cette situation, le contrôle peut être (par exemple) la donnée de la partie normale de la vitesse sur Σ et la partie tangente de la vorticité sur la partie de Σ où le fluide entre (i.e. où $y.n < 0$).

La démonstration diffère selon la topologie du domaine considéré : nous allons aborder les cas simplement connexe et multiplement connexe de façon séparée.

1.3.1 Domaines simplement connexes

Si le domaine est simplement connexe, «l'écoulement à suivre» $\overline{y}$ est celui décrit dans la section 1.2, c'est à dire un écoulement potentiel (qui vérifie la contrainte de glissement au bord sur $\partial\Omega \backslash \Sigma$), et qui successivement envoie des parties du domaine en dehors, puis les replace. La preuve de l'existence d'un tel $\overline{y}$, comme celle de [7], est faite par l'absurde ; en conséquence, la méthode de construction de $\overline{y}$ est non constructive.

Ceci constitue une différence avec le cas bidimensionnel (et simplement connexe), où la fonction $\overline{y}$ en question peut être décrite explicitement comme solution d'un problème de Dirichlet (voir [5]). Ajoutons qu'une autre différence de taille concerne cette fonction, à savoir que dans le cas bidimensionnel, le $\overline{y}$ peut être choisi stationnaire, ou plus précisément de la forme $\alpha(t)\nabla\theta(x)$ (dans ce cas, il ne fait «qu'expluser» les points sans les remettre à leur place, mais cette dernière propriété a peu d'importance) ; dans le cas tridimensionnel, la question reste à notre connaissance ouverte. Signalons qu'elle est d'importance pour la résolution du problème de stabilisation asymptotique.

Puis par la suite la résolution du problème linéarisé est opérée ainsi qu'il est décrit dans la section précédente ; cela se fait cependant de manière un peu plus complexe, du fait qu'en dimension 3, d'une part la vorticité est un champ de vecteurs solénoïdal (et non plus une fonction numérique), et que d'autre part, elle n'est plus simplement transportée par le flot de la vorticité.

Pour schématiser, pour atteindre 0, on vide le domaine de sa vorticité, puis on annule le contrôle au bord.

Notons de plus le phénomène suivant. On ne connaît l'existence d'une solution forte de l'équation tridimensionnelle (laissée sans contrôle) que pour un temps minimal calculable en fonction de y_0 (et d'éventuelles conditions au bord), et d'autant plus court que certaines normes de y_0 sont importantes. En particulier, pour certains y_0, sans utilisation de contrôle, on est tout à fait incapable d'assurer l'existence d'une solution jusqu'au temps T prescrit. On répond à cette difficulté en demandant donc de résoudre le problème de contrôlabilité suffisamment vite (d'aller suffisamment rapidement en 0), pour prévenir toute explosion.

1.3.2 Domaines non simplement connexes

Lorsque $\pi_1(\Omega)$ n'est pas trivial, le problème est plus complexe. En effet, dans ce cas,

$$\left.\begin{array}{l}\text{rot } \zeta(\cdot,T)=0 \text{ dans } \Omega,\\ \text{div } \zeta(\cdot,T)=0 \text{ dans } \Omega,\\ \zeta(\cdot,T).n=0 \text{ sur } \partial\Omega,\end{array}\right\} \not\Rightarrow \zeta(\cdot,T)=0,$$

mais $\zeta(\cdot,T)=\sum_{i=1}^{g}\lambda_i\mathcal{Q}_i$, pour des λ_i réels, où les $\mathcal{Q}_i$ sont des représentants des générateurs du premier espace de cohomologie de de Rham de l'ouvert Ω. Les $\mathcal{Q}_i$ peuvent être définis par exemple de la façon suivante (nous nous référons ici à [33]). On considère g hypersurfaces régulières Σ_i de $\overline{\Omega}$, reposant sur $\partial\Omega$, et telles que

$$\Omega\backslash(\bigcup_{i=1}^{g}\Sigma_i) \text{ soit simplement connexe,}$$

et que les intersections $\Sigma_i\cap\partial\Omega$ et les éventuelles intersections $\Sigma_i\cap\Sigma_j$ $(i\neq j)$ soient transversales.

Par exemple, dans le cas du tore, on peut choisir Σ_1 comme décrit dans la figure 1.2.

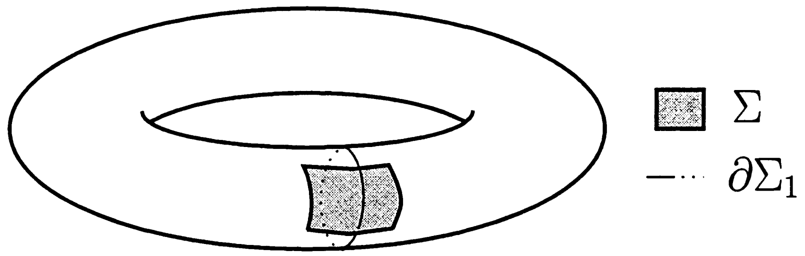

FIG. 1.2: *Exemple de surface* Σ_i.

On distingue les deux faces de chaque Σ_i, les dénotant Σ_i^+ et Σ_i^-. Pour une fonction f définie sur $\overline{\Omega}\backslash\cup_{i=1}^{i=g}\Sigma_i$, on par note par $f_{|\Sigma_i^+}$ et $f_{|\Sigma_i^-}$les limites de f de chaque côté de Σ_i lorsqu'elles existent. On note ensuite par $[f]$ la fonction $f_{|\Sigma_i^+}-f_{|\Sigma_i^-}$ sur Σ_i lorsqu'elle est définie.

On introduit ensuite des fonctions q_i de $\overline{\Omega}\backslash\cup_{i=1}^{g}\Sigma_i$ dans $\mathbb{R}$ telles que

$$\left\{\begin{array}{l}\Delta q_i=0 \text{ dans } \Omega\backslash\Sigma_i,\\ \partial_\nu q_i=0 \text{ sur } \partial\Omega,\\ q_{i|\Sigma_i^+}-q_{i|\Sigma_i^-}=1 \text{ sur } \Sigma_i,\end{array}\right.$$

comme solutions du problème variationnel suivant : soit X_i l'espace fonctionnel suivant :

$$X_i : = \{p \in H^1(\Omega) \ / \ [p]_i = 0, \ [p]_j \text{ est constant sur } \Sigma_i \text{ pour } j \neq i\}.$$

Puis on définit q_i' solution de

$$\int_\Omega \nabla q_i'.\nabla p = [p]_i, \quad \forall p \in X_i.$$

Alors on peut introduire les q_i en renormalisant les q_i', puis enfin définir $Q_i : = \nabla q_i \in C^\infty(\overline{\Omega})$.

En réalisant le même type de contrôle que dans la section précédente, on parvient dans le cas multiconnexe, non pas à annuler la solution au temps T, mais à la rendre irrotationnelle. Le nouveau problème est donc de savoir éliminer les termes résiduels en «Q_i».

On introduit pour cela un nouvel «$\overline{y}$» qui va compléter le précédent, et on utilise ensuite une méthode comparable. Contrairement au «$\overline{y}$» de la section précédente, ce nouvel «$\overline{y}$» sera déterminé non pas de telle sorte que certains points aient sous l'action de son flot la trajectoire voulue, mais de telle sorte que certaines courbes aient (approximativement) la bonne trajectoire sous l'action du flot.

On montre pour cela une proposition qui indique que étant donnée une courbe de Jordan dans Ω et un champ de vecteurs sur cette courbe de circulation nulle, on peut approcher ce dernier par la restriction sur la courbe du gradient d'une fonction harmonique sur Ω, satisfaisant la contrainte au bord $\partial_\nu \theta = 0$ sur $\partial\Omega \backslash \Sigma$. De la sorte, à l'aide d'un champ de vecteurs du type $\nabla\theta(t,x)$, on peut contrôler la trajectoire d'une courbe de manière approchée (tandis que par ailleurs on peut contrôler celle d'un point de façon exacte).

Grâce à ce résultat, on considérera des fonctions $\overline{y}^i$ pour éliminer particulièrement le terme en Q_i. On en déduira ensuite la «mise bout à bout» des $\overline{y}^i$ pour obtenir un nouvel $\overline{y}$ adapté aux problèmes posés par des champs de vitesse initiaux y_0 de la forme $\sum \lambda_i Q_i$.

Nous exposons à présent la forme de $\overline{y}^i$. Dans le cas d'un tore, la trajectoire requise est schématisée par la figure 1.3.

On pourrait se demander si la topologie de Σ relativement à $\partial\Omega$ n'est pas importante pour permettre ce processus. En fait, ce «passage» est possible y compris si Σ est une petite boule de $\partial\Omega$.

En effet, revenons au cas général. On peut supposer $\Sigma_i \cap \Gamma_0 \neq \emptyset$. Nous allons exposer cette trajectoire requise pour la courbe de Jordan dans le plan de coupe de Σ_i (voir la figure 1.2), les pointillés désignant Σ.

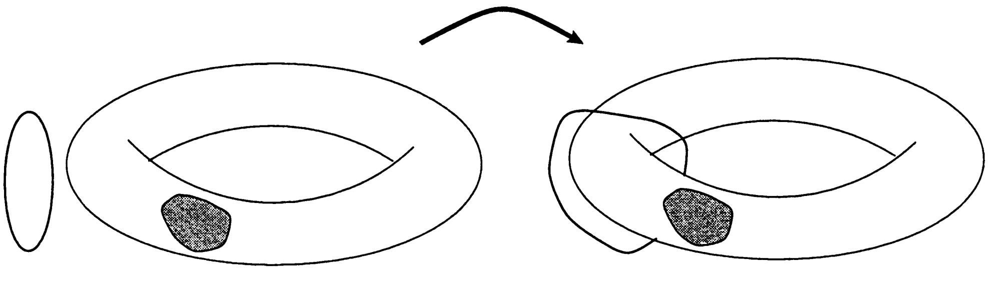

FIG. 1.3: *Trajectoire requise de la courbe.*

La trajectoire de la courbe se divise en trois phases. Dans la première, on exige de $\overline{y}^i$ que son flot fasse entrer la courbe à moitié dans Ω, selon la figure 1.4.

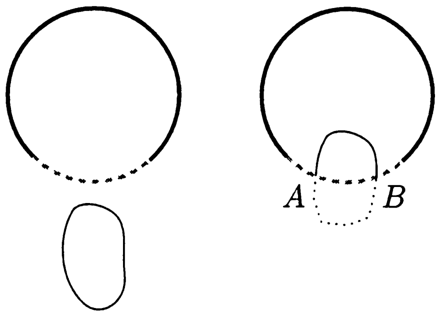

FIG. 1.4: *Entrée dans Ω.*

Dans la deuxième phase, on impose que la partie de la courbe intérieure à Ω «fasse le tour», i.e. décrive (approximativement) la surface Σ_i, jusqu'à ce que les points d'intersection de celle-ci et de $\partial\Omega$ se retrouvent dans Σ, selon la figure 1.5.

On fait dans la dernière phase sortir complèment la courbe de Ω (en restant proche de Σ_i), comme décrit dans la figure 1.6. Si durant cette phase la courbe rencontre des «obstacles» (un «tore intérieur», par exemple), ceux-ci peuvent être passés selon le même processus.

Ainsi, on choisit un $\overline{y}_i$ qui fait parcourir à une courbe de Jordan placée au départ à l'extérieur de Ω, une surface à l'intérieur de Ω qui est très proche de Σ_i (le choix de celui-ci étant libre, on pourrait même dire que la courbe parcourt exactement Σ_i). À la fin du processus, la courbe est enroulée autour de Ω comme dans la figure 1.3.

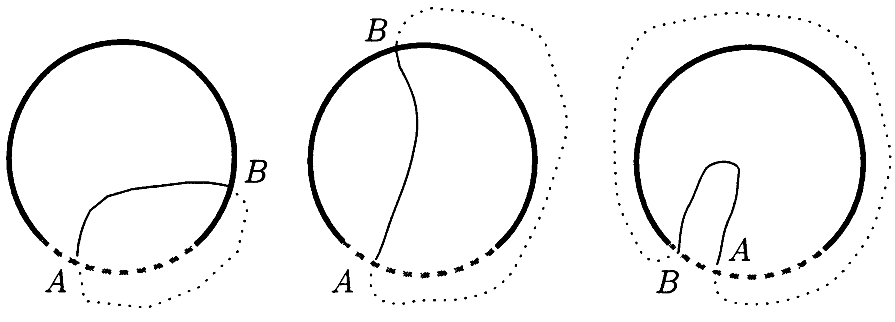

FIG. 1.5: *Description (approximative) de* Σ_i.

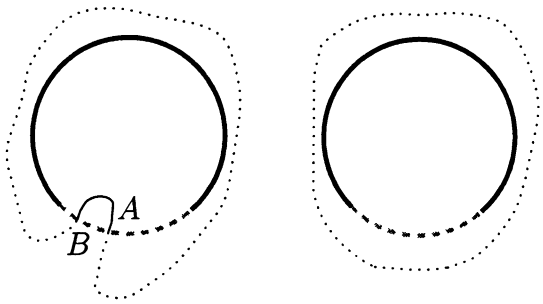

FIG. 1.6: *Sortie hors de* Ω.

On peut maintenant décrire le contrôle en vorticité que l'on utilise pour l'équation linéarisée autour de $\overline{y}^i$. Le principe est de placer au lieu de la courbe de Jordan J à l'instant 0, un filament de vortex (ou une régularisation par convolution de ce filament, dont le support est au voisinage de la courbe), d'intensité μ_i. Par cela, nous entendons la distribution suivante :

$$\omega_0 : = \mu_i \mathcal{M}_J \vec{\tau},$$

où $\mathcal{M}_J$ est la répartition linéique de Dirac sur J, et $\vec{\tau}$ une tangente unitaire sur J.

On montre alors que si l'on opère ainsi, on obtient comme solution de l'équation linéarisée autour de $\overline{y}_i$ un certain ζ qui vérifie

$$\zeta(T) = \sum_{j=1}^{g} \lambda_j \mathcal{Q}_j + \mu_i \mathcal{Q}_i. \tag{1.12}$$

On peut donc ainsi «régler» correctement les intensités μ_i, de sorte que l'on peut passer d'une répartition irrotationnelle de vitesse à zéro en utilisant la méthode du retour autour de $\overline{y}$.

Nous donnons à présent une explication imprécise de (1.12).

Considérons une solution ζ de l'equation d'Euler, proche de $\overline{y}_i$, qui «fait passer» les filaments de vortex comme indiqué, et qui part de y_0. Alors, on a la méthode suivante pour calculer les coefficients finaux de ζ selon $\mathcal{Q}_i$. On considère une courbe de Jordan γ_0^i à l'intérieur de Ω à l'instant 0, qui coupe Σ_i (une fois), et qui ne coupe pas d'autres hypersurfaces Σ_j.

Alors la circulation de vitesse le long de cette courbe, soit

$$\int_{\gamma_0^i} y_0.d\vec{\tau},$$

est égale à au coefficient de y_0 selon $\mathcal{Q}_i$, soit λ_i (ou $-\lambda_i$ selon nos choix de $\vec{\tau}$ et de Σ_i^+ et Σ_i^-).

Au temps T, le champ du vecteur vitesse est irrotationnel, et on peut calculer le coefficient de $\mathcal{Q}_i$ dans $\zeta(T)$ de la même façon, comme la circulation de $\zeta(T)$ le long d'une courbe $\tilde{\gamma}^i$ qui coupe simplement Σ_i et ne coupe pas les autres Σ_j. Alors on peut faire reposer sur les courbes γ_T^i (image de γ_0^i par le flot de ζ entre 0 et T) et $\tilde{\gamma}^i$ une surface (un cylindre). Celui-ci coupe simplement l'emplacement J_T du filament de vortex à la fin du processus. Il s'ensuit que le flux de vorticité à travers cette surface est de μ_i.

Prenons comme exemple celui du tore, vu selon un plan de coupe orthogonal à l'axe de révolution ; les courbes sont placées dans ce cas comme sur la figure 1.7.

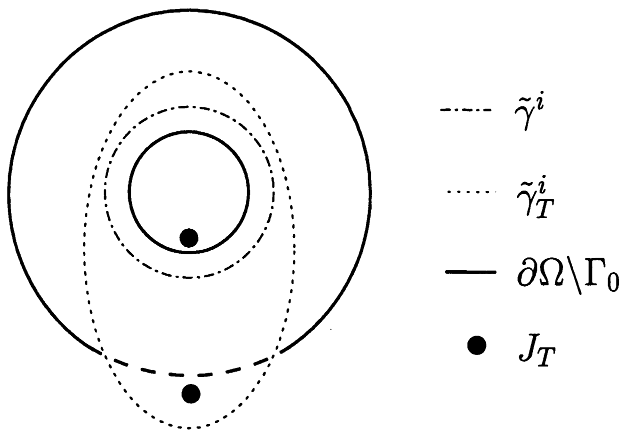

FIG. 1.7: *Emplacement des courbes γ_T^i et $\tilde{\gamma}^i$.*

Donc si on cherche le coefficient de $\mathcal{Q}_i$ dans $\zeta(T)$, on peut le chercher sous la forme

$$\int_{\tilde{\gamma}^i} \zeta(T).d\vec{\tau},$$

Or par la formule de Stokes celui-ci est égal à

$$\int_{\gamma_T^i} \zeta(T).d\vec{r} + \mu_i,$$

et l'intégrale précédente est égale à

$$\int_{\gamma_0^i} y_0.d\vec{r},$$

par le théorème de Kelvin. On arrive ainsi à la formulation de (1.12).

1.4 Le cas bidimensionnel

Le cas bidimensionnel est le cadre de l'article de J.-M. Coron référencé par [6]. Dans cet article, l'auteur démontre un théorème de contrôlabilité exacte dans le cas où la zone de contrôle rencontre toutes les composantes connexes du bord. Contrairement au cas tridimensionnel, si cette condition n'est pas remplie, on peut obtenir néanmoins le théorème suivant ([6, Theorem 1.3]), où l'on note par Γ^b la réunion des composantes connexes de $\partial\Omega$ qui ne coupent pas Σ :

Théorème 3 (J.-M.Coron) *Il existe une constante c_0 dépendant de Ω tel que pour tout $T > 0$, pour tout $\epsilon > 0$, et quels que soient y_0 et y_1 de $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ vérifiant (1.3) et (1.4), il existe $y \in C^\infty(\overline{\Omega} \times [0, T]; \mathbb{R}^2)$ satisfaisant (1.2) sur $\Omega \times [0, T]$ tel que*

$$y(x, T) = y_1(x) \quad \forall x \in \overline{\Omega} \text{ tel que } dist(x, \Gamma^b) \geq \epsilon, \tag{1.13}$$

$$\|y(\cdot, T)\|_{L^\infty} \leq c_0(\|y_0\|_{L^2} + \|y_1\|_{L^2} + \|\operatorname{rot} y_0\|_{L^\infty} + \|\operatorname{rot} y_1\|_{L^\infty}). \tag{1.14}$$

On obtient donc de la contrôlabilité approchée $\bigcap_{1 \leq p < +\infty} L^p(\Omega)$; mais comme remarqué dans [6], on ne peut guère espérer mieux, en raison de la loi de Kelvin, qui assure des invariants pendant l'évolution du système d'Euler, à savoir les intégrales suivantes

$$\int_\Gamma y(t, \cdot).d\vec{r}, \quad t \in [0, T],$$

calculées pour chaque composante connexe Γ du bord qui ne rencontre pas la zone de contrôle Σ.

Par conséquent, si ces circulations calculées pour y_0 et y_1 diffèrent, on ne peut pas espérer résoudre le problème de contrôlabilité approchée L^∞.

Il est en revanche assez naturel de se demander si ces invariants constituent l'unique objection à la contrôlabilité approchée L^∞. C'est l'objet du premier théorème démontré en partie 3 :

Théorème 4 *Pour tout $T > 0$, pour tout $\epsilon > 0$, et quels que soient y_0 et y_1 de $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ vérifiant (1.3) , (1.4) et*

$$\int_\Gamma y_0.d\vec{\tau} = \int_\Gamma y_1.d\vec{\tau}, \ \forall \Gamma \text{ connexe de } \Gamma^b, \tag{1.15}$$

il existe une suite (y^n) de fonctions de $C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2)$ satisfaisant (1.2) sur $\Omega \times [0,T]$, et telle que

$$y^n(\cdot, T) \longrightarrow y_1 \text{ au sens } W^{1,p}(\Omega), \quad \forall p \in [1, +\infty). \tag{1.16}$$

On obtient ainsi mieux que la contrôlabilité approchée L^∞. Cependant une fois encore ce résultat ne peut guère être amélioré, car d'autres invariants interviennent et empêchent la contrôlabilité $W^{1,\infty}$. Cette fois-ci c'est la répartition de vorticité sur chaque composante connexe du bord non atteinte par la zone de contrôle, à difféomorphisme direct (qui conserve l'orientation) près qui constitue en effet une donnée permanente du système. La possibilité d'un contrôle $W^{1,\infty}$ est ainsi limitée à des données y_0 et y_1 pour lesquelles, sur chaque composante connexe de $\partial\Omega$ qui n'intersecte pas Σ, $\operatorname{rot} y_1$ peut s'écrire comme la composition de $\operatorname{rot} y_0$ par un difféomorphisme direct de la composante.

Il est encore naturel de se demander si, dans le cas où précisément on se limite à de telles données, la contrôlabilité approchée $W^{1,\infty}$ a lieu ou non. Le second théorème démontré en partie 3 répond positivement à cette question. On a en effet le résultat suivant :

Théorème 5 *Pour tout $T > 0$, pour tout $\epsilon > 0$, et quels que soient y_0 et y_1 de $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ vérifiant (1.3) , (1.4), (1.15), et*

$$(\operatorname{rot} y_0)_{|\Gamma} \equiv (\operatorname{rot} y_1)_{|\Gamma} \text{ mod } Diff^+(\Gamma), \quad \forall \Gamma \text{ connexe de } \Gamma^b, \tag{1.17}$$

il existe une suite (y^n) de fonctions de $C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2)$ satisfaisant (1.2) sur $\Omega \times [0,T]$, (1.13), et telle que

$$y^n(\cdot, T) \longrightarrow y_1 \text{ au sens } W^{2,p}(\Omega), \quad \forall p \in [1, +\infty). \tag{1.18}$$

On obtient à plus forte raison la contrôlabilité dans $W^{1,\infty}$.

Notons à présent que même sous ces hypothèses supplémentaires, on n'a pas de contrôlabilité approchée $W^{2,\infty}$. En effet, partons d'un profil de vitesse nul. Maintenant, on peut considérer un autre profil de vitesse au voisinage du bord tel que la vorticité correspondante soit nulle au bord, mais telle que la dérivée de cette vorticité dans le sens de la normale au bord soit non nulle.

«Complétons» ce profil de vitesse sur le reste du domaine de sorte que la condition (1.15) soit vérifiée. Alors on n'a pas contrôlabilité approchée $W^{2,\infty}$ entre ces deux points, car on aurait alors convergence $W^{1,\infty}$ de la vorticité. Or, partant de 0, on aura forcément à la fin du processus une vorticité nulle au voisinage des composantes du bord non contrôlées.

Nous allons à présent exposer les méthodes utilisées pour parvenir à ces résultats.

1.4.1 Contrôlabilité $W^{1,p}$

Supposons donc disposer de l'hypothèse supplémentaire que pour toute composante connexe Γ de $\partial\Omega$ qui ne rencontre pas Σ, on a (1.15).

On peut alors tenter d'utiliser la même stratégie que l'on utilisait pour le cas de la contrôlabilité L^p, à savoir que, dans un premier temps, on «remplace» la vorticité de y_0 par celle de y_1 (ou par une répartition bien choisie), en «soufflant suffisament fort» pour bien placer celle-ci. Puis dans un deuxième temps, reste à régler le problème suivant : même si l'on a parfaitement remplacé la vorticité et que l'on impose la bonne vitesse normale au bord au temps T, cela n'assure rien du fait de la non-trivialité du premier espace de cohomologie de de Rham pour les domaines qui nous occupent dans cette partie. Concrètement, sur de tels domaines (non simplement connexes) :

$$\left.\begin{array}{l} \operatorname{rot} y_0 = \operatorname{rot} y_1 \text{ dans } \Omega, \\ \operatorname{div} y_0 = \operatorname{div} y_1 = 0 \text{ dans } \Omega, \\ y_0.\nu = y_1.\nu \text{ sur } \partial\Omega, \end{array}\right\} \not\Rightarrow y_0 = y_1 \text{ dans } \Omega.$$

En revanche, sous ces hypothèses, $y_0 - y_1$ est une combinaison linéaire des fonctions $\nabla^\perp \tau_i$, pour i variant entre 2 et k, où k est le nombre de composantes connexes de $\partial\Omega$, les fonctions τ_i étant définies de la façon suivante :

$$\left\{\begin{array}{l} \tau_i = 1 \text{ sur } \Gamma_i, \\ \tau_i = 0 \text{ sur } \partial\Omega \backslash \Gamma_i, \end{array}\right.$$

où l'on a numéroté les composantes connexes de $\partial\Omega$ par $\Gamma_1, \ldots, \Gamma_k$ en commençant par exemple par les courbes qui rencontrent Σ.

Il s'agit donc dans un deuxième temps de savoir «éliminer» ces courants irrotationnels. Pour les indices i correspondant à des composantes connexes qui coupent Σ, il n'y a pas de problème pour régler le $\nabla^\perp \tau_i$. Il suffit en effet, comme l'a démontré J.-M. Coron dans [6], de faire passer de la vorticité dans le domaine, en entrant par $\Gamma_1 \cap \Sigma$ et en sortant par $\Gamma_i \cap \Sigma$. En réglant la quantité de vorticité en question, on dispose parfaitement de la composante de $y_{|t=T}$ selon $\nabla^\perp \tau_i$.

Pour les autres indices i, on ne dispose pas de ce moyen. Dans [6], la contrôlabilité approchée est obtenue en plaçant de la vorticité supplémentaire près de Γ^b, ce qui règle le problème des $\nabla^\perp \tau_i$, mais qui a évidemment un prix en termes de qualité du contrôle approché (en particulier en ce qui concerne la norme $W^{1,p}$).

Ajoutons qu' un problème se superpose à celui-ci, à savoir que l'on souhaite, en plus de la contrôlabilité $W^{1,p}$, à obtenir au final un champ des vitesses qui coïncide avec y_1 sur le domaine privé des points les plus proches des composantes du bord «incontrôlées». Les deux problèmes sont réglés simultanément dans une sorte de couche limite près de Γ^b.

Dans [6], la valeur finale de la fonction solution pour le problème linéarisé autour de $\overline{y}$ s'exprime de façon relativement simple en fonction des fonctions de courant de y_0 et y_1 et du flot de $\overline{y}$. Ici, celle-ci est obtenue par un procédé un peu plus complexe (valable grâce à l'hypothèse supplémentaire), qui permet à la fois d'obtenir la condition de coïncidence et de résoudre le problème des $\nabla^\perp \tau_i$, et ce d'une manière pas trop coûteuse en norme $W^{1,p}$. Cette résolution fait appel à une problème de type contrôlabilité mentionné dans [26, p. 86].

1.4.2 Contrôlabilité $W^{2,p}$

Le problème de la contrôlabilité $W^{2,p}$ lorsque l'on a l'hypothèse supplémentaire (1.17), est de nature un peu différente à celui considéré au paragraphe précédent.

Pour ce résultat en effet, on doit modifier le «$\overline{y}$» pour obtenir le résultat, c'est à dire que l'on doit ajouter une «préparation» supplémentaire avant de revenir à une action semblable à celle utilisée précédemment. En effet, grâce à l'hypothèse précédente, on peut passer dans un premier temps de y_0 à une répartition de vitesse qui, sur les composantes de Γ^b a un rotationnel proche (en norme C^1) de celui de y_1. Par la suite, un procédé similaire au précédent (mais encore légèrement adapté) permet d'obtenir le résultat de contrôle approché voulu (après réexamen de la partie du domaine proche de Γ^b).

Pour ce qui est de «bien répartir» la vorticité initiale sur chaque composante de Γ^b, on fait encore appel à la méthode du retour. Si on trouve un $\overline{y}$ (solution potentielle de l'équation d'Euler partant et finissant en 0) dont le flot sur chaque composante de Γ^b donne la bonne transformation, on pourra espérer trouver une solution du système non-linéaire qui l'effectue également.

L'argument central pour trouver ce nouveau «$\overline{y}$» utilise l'outil complexe : il s'agit d'approcher une fonction continue sur une courbe, à valeurs complexes, par une fonction holomorphe sur un ensemble fixé à l'avance (à savoir

$\overline{\Omega}$), et satisfaisant la contrainte au bord. Précisément, la proposition suivante est établie :

Proposition 1 *Soit Ω un ouvert non vide borné et régulier du plan. Soit Σ une partie ouverte non vide de son bord $\partial\Omega$. Soit $\vec{v}$ un champ de vecteurs de classe $C^k(\partial\Omega\backslash\Sigma;\mathbb{R}^2)$ ($k \in \mathbb{N}$) tel que :*

$$\int_\Gamma \vec{v}.d\vec{\tau} = 0, \tag{1.19}$$

pour toute composante connexe Γ de $\partial\Omega$ qui ne rencontre pas Σ.

Alors pour tout $\varepsilon > 0$, il existe $\theta \in C^\infty(\overline{\Omega};\mathbb{R})$ tel que

$$\Delta\theta = 0 \textit{ dans } \Omega, \tag{1.20}$$

$$\|\nabla\theta - \vec{v}\|_{C^k(\partial\Omega\backslash\Sigma;\mathbb{R}^2)} \leq \varepsilon. \tag{1.21}$$

Dans notre cas, cette proposition peut se démontrer de manière élémentaire, en utilisant le théorème de Runge (notons que, ainsi, l'élaboration de cet écoulement potentiel est encore une fois non constructive).

1.4.3 Une remarque supplémentaire

Dans ce paragraphe, nous donnons une petite conséquence supplémentaire du lemme que nous venons d'évoquer.

Nous pouvons remarquer que la Proposition 1 permet de mettre en évidence de nouveaux types d'écoulement potentiel tels que ceux utilisés pour le contrôle de l'équation 2-D. En effet, on peut dans ce cas contrôler non seulement la trajectoire d'un point par le flot, imposer une vitesse normale nulle sur une partie précise du bord, mais également imposer (approximativement, dans une bonne norme) la vitesse *tangentielle* sur la même partie du bord. On peut par exemple obtenir le résultat suivant :

Proposition 2 *Soit Ω un ouvert borné régulier non vide de $\mathbb{R}^2$. Soit Σ une partie ouverte non vide de son bord. Soit $\Omega^\#$ une partie ouverte non vide de Ω. Considérons $k \in \mathbb{N}$. Soit enfin $\epsilon > 0$.*

Alors pour tout x de Ω, il existe $r(x) > 0$ et une fonction θ de classe $C^\infty(\overline{\Omega} \times [0,T];\mathbb{R})$ qui satisfait les conditions suivantes :

$$\Delta\theta = 0 \textit{ dans } \Omega, \tag{1.22}$$

$$\partial_n\theta = 0 \textit{ sur } \partial\Omega, \tag{1.23}$$

$$\|\nabla\theta\|_{C^0([0,T],C^k(\partial\Omega\backslash\Sigma))} < \epsilon, \tag{1.24}$$

$$\phi^{\nabla\theta}(B(x,r(x)),1,0) \subset \Omega^{\#}, \tag{1.25}$$

où $B(x,r(x))$ est la boule fermée de centre x et de rayon $r(x)$, et $\phi^{\nabla\theta}$ désigne le flot du champ de vecteurs $\nabla\theta$.

Ce résultat généralise un résultat de J.-M. Coron, à savoir le Lemme A.1 de [7], qui au lieu de (1.24), donne

$$\|\frac{\partial\theta}{\partial\tau}\|_{C^0(\partial\Omega\backslash\Sigma)} + \|\frac{\partial^2\theta}{\partial\tau^2}\|_{C^0(\partial\Omega\backslash\Sigma)} < \epsilon, \tag{1.26}$$

où τ est la tangente unitaire à $\partial\Omega$ orientée directement.

Cette dernière proposition est utilisée pour établir la contrôlabilité approchée de l'équation de Navier-Stokes avec condition de glissement de Navier sur le bord. Aussi une ouverture vers une prochaine recherche peut être de voir si le résultat de contrôle de Navier-Stokes peut être ainsi amélioré. En particulier, il serait intéressant de chercher à renforcer la qualité du contrôle approché ou de traiter la condition de non-glissement de Stokes.

De manière générale, on peut espérer trouver par ce biais d'autres «$\overline{y}$» qui satisfont des contraintes supplémentaires par rapport à ceux considérés jusqu'ici.

Chapitre 2

Contrôle des fluides tridimensionnels

Résumé

Nous prouvons la contrôlabilité de l'équation d'Euler tridimensionnelle pour les fluides parfaits incompressibles sur un domaine borné et régulier, lorsque le contrôle opère sur une partie ouverte du bord qui en rencontre les différentes composantes connexes.

Abstract

We prove the exact boundary controllability of the 3-D Euler equation of incompressible inviscid fluids on a regular connected bounded open set when the control operates on an open part of the boundary that meets any of the connected components of the boundary.

2.1 Introduction

Let Ω be a non-empty, open, connected, bounded and regular (say C^∞-regular) subset of $\mathbb{R}^3$. Let Γ_0 be an open and non-empty subset of its boundary $\partial\Omega$, which meets any connected component of $\partial\Omega$. We are interested in the exact boundary controllability of the 3-D Euler equation of inviscid incompressible fluids for (Ω, Γ_0), that is, the following question: given $T > 0$, given y_0 and y_1 two solenoidal vector fields, i.e. satisfying

$$\text{div } y_0 = \text{ div } y_1 = 0, \tag{2.1}$$

regular (in this paper, $C^{2,\alpha}$ for some Hölder coefficient $\alpha \in (0,1)$) and which satisfy

$$y_0.n = y_1.n = 0 \text{ on } \partial\Omega\backslash\Gamma_0, \tag{2.2}$$

where n is the outward unit normal vector field on $\partial\Omega$, does there exist a solution y of the Euler system

$$\partial_t y + (y.\nabla)y = \nabla p \text{ in } \Omega, \tag{2.3}$$

for some $p \in \mathfrak{D}'(\Omega \times (0,T))$ and

$$\operatorname{div} y = 0 \text{ in } \Omega, \tag{2.4}$$

with

$$y(x,t).n = 0, \quad \forall t \in [0,T], \ \forall x \in \partial\Omega\backslash\Gamma_0, \tag{2.5}$$

and such that

$$y_{|t=0} = y_0 \text{ in } \overline{\Omega}, \tag{2.6}$$

$$y_{|t=T} = y_1 \text{ in } \overline{\Omega}? \tag{2.7}$$

This problem, raised by J.-L. Lions in [25], was solved by J.-M. Coron in [5] and [6] in the two-dimensional case. In a previous paper [18], we have sketched a proof of a solution to this problem in dimension 3 when Ω is simply connected. Here we give the details of the demonstration and prove that, as announced in [19], the result still holds when Ω is not necessarily simply connected. Actually, we prove the following result:

Theorem 2.1 *Given $\alpha \in (0,1)$, two functions y_0 and y_1 in $C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)$ satisfying (2.1) and (2.2) and $T > 0$, then there exists a function y in the space $C([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3)) \cap L^\infty([0,T];C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$ such that (2.3) to (2.7) hold for some $p \in \mathfrak{D}'(\Omega \times (0,T))$.*

Remark 2.1 *As noticed in [6], the condition that Γ_0 meets any connected component of the boundary is necessary for the exact controllability as a consequence of the Kelvin law.*

Indeed, suppose that we choose $y_1 = 0$ on some connected component Γ^* of the boundary, which does not meet Γ_0. Then the existence of y and the Kelvin law for any loop γ in this connected component of the boundary imply that

$$\int_\gamma y_0 d\tau = \int_{\tilde{\gamma}} y_1 d\tau = 0,$$

where $\tilde{\gamma}$ is the loop obtained when transporting γ by the flow of y. This necessarily implies that $y_{0|\Gamma^*}$ is a gradient, which is not generally the case.

Now we briefly describe the method. As in [5] and [6], the steps of the proof of Theorem 2.1 are the following: first, we prove that this question can be reduced to the problem of zero-controllability with small initial data (that is $y_1 = 0$ and $\|y_0\|_{C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)} < \epsilon$) and small time T.

To be more precise, we prove in section 5 that Theorem 2.1 is a consequence of the following proposition:

Proposition 2.1 *There exists $\nu > 0$ such that if $y_0 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)$ satisfies (2.1), (2.2) and $\|y_0\|_{C^{2,\alpha}(\overline{\Omega},\mathbb{R}^3)} < \nu$, then there exists a function y in the space $C([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3)) \cap L^\infty([0,1];C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$ and $p \in \mathfrak{D}'(\Omega \times (0,1))$ satisfying (2.3) to (2.7) for $y_1 = 0$, and $T = 1$, and also*

$$y = 0 \textit{ and } p = 0, \quad \forall t \in [\frac{1}{2}, 1]. \tag{2.8}$$

In order to prove the last proposition, we use a method called the "return method", used in [5] and [6] and introduced in [3] for a stabilization problem. Precisely, – since the linearized Euler equation around $y \equiv 0$ is not controllable – we consider the linearized system around other solutions of the Euler control system $\overline{y}$ satisfying $\overline{y}_{|t=0} = \overline{y}_{|t=1} = 0$ (a kind of "loop"). If this linearized control system is controllable, then for y_0 small enough, one can hope to find y close to $\overline{y}$ answering to the general problem. In order to prove the existence of y, we use a construction of solutions of the Euler system due to C. Bardos and U. Frisch (see [2]).

In the previous presentation of the problem, the control itself was not explicit. As a control, we can take for example $y.n$ on $\Gamma_0 \times [0,T]$ and the tangent part of the vorticity where the fluid enters, that is $\omega \wedge n$ where $y.n < 0$ in Γ_0 (see for that [24]).

In the next section, we will present the different tools we need to introduce the particular solution $\overline{y}$. This function will be found in the particular potential form "$\nabla\theta$", in order for its flow to satisfy precise properties. In the simply connected case, as in dimension 2, $\overline{y}$ has the property that any particle in $\overline{\Omega}$ following the flow of $\overline{y}$ must go out of $\overline{\Omega}$. The major difference is that in dimension 3, as in the 2-D case for the Navier-Stokes equation [7], this "$\nabla\theta$" can no longer be chosen stationary. In the multi-connected case, we will have to introduce an other type of "$\nabla\theta$" (which we need to append to the previous one), whose flow moves certain Jordan curves properly.

In section 3, we define a function F on a certain functional set, of which y will be found as a fixed point. Given y near $\overline{y}$, F associates the solution of a linear control problem relied to (2.3)-(2.7).

In section 4, we prove Proposition 2.1, by showing that F admits a fixed point which gives a solution to the non-linear problem.

Section 5 deduces Theorem 2.1 from Proposition 2.1.

Section 6 is devoted to the proof of Lemma 2.1, which corresponds to the first type of "$\nabla\theta$".

Section 7 corresponds to the second type of "$\nabla\theta$" presented in Lemma 2.2.

In sections 8 and 9, we give the details of the proofs of technical lemmas needed in sections 6 and 7 respectively.

2.2 The particular solution of Euler system : $\overline{y}$

We first set up the following lemma, which stands for any regular bounded open set $\tilde{\Omega}$ such that $\tilde{\Omega}$ contains $\overline{\Omega}$. This $\overline{y}$ is useful to get rid of vorticity in Ω.

Lemma 2.1 *For all a in $\overline{\Omega}$, there exists $\theta \in C^\infty(\overline{\tilde{\Omega}} \times [0,1];\mathbb{R})$ satisfying:*

$$Supp\ \theta \subset \tilde{\Omega} \times (0,1), \tag{2.9}$$

$$\theta = 0 \ in\ \overline{\tilde{\Omega}} \times ([0,\frac{1}{4}] \cup [\frac{3}{4},1]), \tag{2.10}$$

$$\Delta\theta = 0 \ in\ \overline{\Omega} \times [0,1], \tag{2.11}$$

$$\frac{\partial\theta}{\partial n} = 0 \ on\ (\partial\Omega\backslash\Gamma_0) \times [0,1], \tag{2.12}$$

$$\phi^{\nabla\theta}(a,0,1) \in \tilde{\Omega}\backslash\overline{\Omega}, \tag{2.13}$$

where we denote by $\phi^{\nabla\theta} : \overline{\tilde{\Omega}} \times [0,1] \times [0,1] \longrightarrow \overline{\tilde{\Omega}}$, $(x,t_1,t_2) \mapsto \phi^{\nabla\theta}(x,t_1,t_2)$ the flow of $\nabla\theta$, i.e. the function which satisfies

$$\frac{\partial\phi}{\partial t_2} = \nabla\theta(\phi,t_2), \tag{2.14}$$

$$\phi(x,t_1,t_1) = x. \tag{2.15}$$

With the help of that lemma, we will be able to single out a solution of the Euler system, which makes each part of the fluid go out $\overline{\Omega}$ (far enough), and then go back the same way.

⋆

In the multi-connected case, we will also need another type of "$\nabla\theta$", in order to control flows without vorticity, which class in de Rham's first cohomology space is not trivial. Let us describe these flows (we use the description of [33, Appendix I]).

We introduce, in the multi-connected case, precisely when $H_1(\Omega) = \mathbb{Z}^s$ with $s \geq 1$, s smooth hypersurfaces $\Sigma_1, \dots, \Sigma_s$ included in $\overline{\Omega}$, with boundaries in $\partial\Omega$, not tangent to $\partial\Omega$, such that:

$$\Omega \backslash (\bigcup_{i=1}^{s} \Sigma_i) \text{ is simply connected,} \tag{2.16}$$

and such that if the Σ_i have mutual intersections (which, in some cases, cannot be avoided), these have to be transversal.

For $i \in \{1, ..., s\}$, we distinguish the two sides Σ_i, that we denote by Σ_i^+ and Σ_i^-. For a function f defined in $\Omega \backslash \cup_{i=1}^{s} \Sigma_i$, which trace on Σ_i^+ may differ from the one on Σ_i^-, one defines $[f]_i : = f_{|\Sigma_i^+} - f_{|\Sigma_i^-}$ considered as a function on Σ_i.

We introduce the functional space

$$X_i : = \left\{ p \in H^1(\Omega \backslash \bigcup_{i=1}^{s} \Sigma_i) \ / \ [p]_i = \text{constant}, \ [p]_j = 0 \text{ for } j \neq i \right\}.$$

(The transversality allows the proper definition of the X_i, and particularly the fact that $X_i \neq X_j$ for $i \neq j$.)

Then using the Lax-Milgram theorem on $X_i/\mathbb{R}$ ($\mathbb{R}$ representing the constant functions), one easily deduces the existence of a function q_i' in X_i such that:

$$\int_{\Omega} \nabla q_i' . \nabla p = [p]_i, \ \forall p \in X_i. \tag{2.17}$$

This leads to the existence of a function $q_i \in X_i$ such that:

$$\Delta q_i = 0 \text{ in } \Omega \backslash \bigcup_{j=1}^{s} \Sigma_j, \tag{2.18}$$

$$\partial_n q_i = 0 \text{ on } \partial\Omega, \tag{2.19}$$

$$[q_i]_i = 1, \tag{2.20}$$

$$[q_i]_j = 0 \text{ for } j \neq i, \tag{2.21}$$

$$[\partial_n q_i]_i = 0. \tag{2.22}$$

By (2.20), (2.21) and (2.22), the $Q_i : = \nabla q_i$ are in $C^0(\Omega)$. In fact they are in $C^\infty(\overline{\Omega})$ (see [33, Appendix I, Remark 1.3.ii]). This can be easily obtained by considering in a neighbourhood V of a point in Σ_i the function $\tilde{q}_i$ equal to q_i in the part of V situated on the "Σ_i^-" side of Σ_i and equal to $q_i - [q_i]_i$ on the "Σ_i^+" one. This function appears to be H^1 and harmonic.

Remark 2.2 *One deduces from the previous construction (see [33, Appendix I, Proposition 1.1]) that any (regular) vector field X satisfying*

$$\operatorname{curl} X = 0 \textit{ in } \Omega, \tag{2.23}$$

can then be written as

$$X = \nabla\chi + \sum_{i=1}^{s} \alpha_i \mathcal{Q}_i,$$

for some function χ and some real numbers α_i, $i \in \{1, ..., s\}$. If we add to (2.23) the conditions:

$$\operatorname{div} X = 0 \text{ in } \Omega, \tag{2.24}$$
$$X.n = 0 \text{ on } \partial\Omega, \tag{2.25}$$

then we obtain

$$X = \sum_{i=1}^{s} \alpha_i \mathcal{Q}_i.$$

In the multiconnected case, it is hence not sufficient to find a control which makes the vorticity vanish, to reduce the flow to zero.

This fact will oblige us to set up a second lemma to define our particular solution $\overline{y}$ and to get rid of the terms "$\mathcal{Q}_i$". It is in particular necessary to treat the problem with (for instance) $y_0 = \mathcal{Q}_i$ and $y_1 = 0$. Roughly speaking, the following lemma solves this precise case.

Lemma 2.2 *There exists $\overline{\nu} > 0$, such that for i in $\{1, ..., s\}$, there exists $\theta^i \in C^\infty(\overline{\tilde{\Omega}} \times [0,1];\mathbb{R})$ and $\aleph^i \in C^\infty(\overline{\tilde{\Omega}};\mathbb{R}^3)$ satisfying:*

$$Supp\ \theta^i \subset \tilde{\Omega} \times (0,1), \tag{2.26}$$
$$\theta^i = 0 \text{ in } \overline{\tilde{\Omega}} \times ([0, \frac{1}{4}] \cup [\frac{3}{4}, 1]), \tag{2.27}$$
$$\Delta\theta^i = 0 \text{ in } \overline{\Omega} \times [0,1], \tag{2.28}$$
$$\frac{\partial\theta^i}{\partial n} = 0 \text{ on } \partial\Omega\backslash\Gamma_0 \times [0,1], \tag{2.29}$$
$$Supp\ \aleph^i \subset \tilde{\Omega}\backslash\overline{\Omega}, \tag{2.30}$$

and such that for any $f \in C([0,1], C^{2,\alpha}(\overline{\tilde{\Omega}};\mathbb{R}^3))$ with

$$\|f - \nabla\theta^i\|_{C([0,1]\times\overline{\tilde{\Omega}})} < \overline{\nu}, \tag{2.31}$$

if we define $w^i \in C^\infty(\overline{\tilde{\Omega}} \times [0,1];\mathbb{R}^3)$ by

$$w^i(\cdot,0) = \operatorname{curl}(\aleph^i) \text{ on } \overline{\tilde{\Omega}}, \tag{2.32}$$
$$\partial_t w^i + (f.\nabla)w^i = (w^i.\nabla)f - w^i \operatorname{div} f \text{ on } \tilde{\Omega} \times [0,1], \tag{2.33}$$

and if we define the function ζ^i *in* $C^\infty(\overline{\tilde{\Omega}} \times [0,1]; \mathbb{R}^3)$ *by*

$$\operatorname{curl} \zeta^i = w^i \text{ in } \overline{\Omega} \times [0,1], \tag{2.34}$$

$$\operatorname{div} \zeta^i = 0 \text{ in } \overline{\Omega} \times [0,1], \tag{2.35}$$

$$\zeta^i.n = \partial_n \theta^i \text{ on } \partial\Omega \times [0,1], \tag{2.36}$$

$$\int_\Omega \zeta^i(0).Q_j dx = 0, \tag{2.37}$$

$$\int_\Omega (\partial_t \zeta^i + f \wedge \operatorname{curl}(\zeta^i)).Q_j dx = 0, \quad \forall j \in \{1, ..., s\}, \tag{2.38}$$

then we have

$$Supp\ w^i(\cdot, 1) \subset \tilde{\Omega} \backslash \overline{\Omega}, \tag{2.39}$$

and

$$\zeta^i(1) = Q_i. \tag{2.40}$$

As we will see in section 6, $\overline{y} :\ = \nabla\theta$ in this lemma will be chosen, not in terms of the flow of points, but in terms of the flow of certain Jordan curves.

We can now present what our particular solution to Euler system $\overline{y}$ will be.

We denote by $B(x_i, r_i)$ the open ball of center x_i and of radius r_i, and by $\overline{B}(x_i, r_i)$ its closure. By Lemma 2.1, one can find by compactness of $\overline{\Omega}$ a positive integer k, k points x_i in $\overline{\Omega}$, k real numbers $r_i > 0$ and k smooth functions $\theta_i \in C^\infty(\overline{\tilde{\Omega}} \times [0,1], \mathbb{R})$, $i \in \{1, ..., k\}$, satisfying (2.9)-(2.12), and an open bounded regular set Ω_2 with $\overline{\Omega} \subset \Omega_2$ and also $\overline{\Omega_2}$ such that

$$\overline{B}(x_i, r_i) \subset \tilde{\Omega}, \tag{2.41}$$

$$\overline{\Omega} \subset \bigcup_{i=1}^{i=k} B(x_i, r_i), \tag{2.42}$$

$$\phi^{\nabla\theta_i}(\overline{B}(x_i, r_i), 0, 1) \cap \overline{\Omega}_2 = \emptyset. \tag{2.43}$$

Let us split the time-segments $[1/4, 1/2]$ and $[1/2, 3/4]$ as follows:

$$t_i = \frac{1}{4} + i\frac{1}{4k}, \quad \forall i \in \{0, ..., k\}, \tag{2.44}$$

$$t_{i+\frac{1}{2}} = \frac{1}{4} + (i + \frac{1}{2})\frac{1}{4k}, \quad \forall i \in \{0, ..., k-1\}, \tag{2.45}$$

$$t_i = \frac{1}{2} + (i-k)\frac{1}{4s}, \quad \forall i \in \{k, ..., k+s\}. \tag{2.46}$$

We can now define θ in $C^\infty(\tilde{\Omega} \times [0,1/2], \mathbb{R})$ by

$$\theta(x,t) = 0, \ \forall (x,t) \in \overline{\tilde{\Omega}} \times [0, \frac{1}{4}], \tag{2.47}$$

$$\theta(x,t) = 8k\theta_j(x, 8k(t - t_{j-1})), \quad \forall j \in \{1, ..., k\}, \\ \text{and } \forall (x,t) \in \tilde{\Omega} \times [t_{j-1}, t_{j-\frac{1}{2}}], \tag{2.48}$$

$$\theta(x,t) = -8k\theta_j(x, 8k(t_j - t)), \quad \forall j \in \{1, ..., k\}, \\ \text{and } \forall (x,t) \in \tilde{\Omega} \times [t_{j-\frac{1}{2}}, t_j]. \tag{2.49}$$

During the interval of time $[\frac{1}{2}, 1]$, we define θ by

$$\theta(x,t) = 4s\theta^{j-k+1}(x, 4s(t - t_j)), \quad \forall j \in \{k, ..., k+s-1\}, \\ \text{and } \forall (x,t) \in \tilde{\Omega} \times [t_j, t_{j+1}]. \tag{2.50}$$

Let $\overline{y} : = \nabla\theta$. We remark that $\overline{y}$ restricted to $\overline{\Omega} \times [0,1]$ is a C^∞ solution of (2.3)-(2.7) with $T = 1$, $y_0 = y_1 = 0$ and with $p(x,t) = \partial\theta/\partial t + |\nabla\theta|^2/2$.

2.3 The application F

2.3.1 Introduction

In this section,we use this particular solution to single out the application F, the fixed point of which give a solution for Proposition 2.1. For that purpose we first introduce a certain functional set X_ν.

Let $\mu : [0,1] \to [0,1]$ C^∞-regular, such that

$$\begin{cases} 0 \le \mu \le 1 \text{ in } [0,1], \\ \mu = 1 \text{ in } [0, 1/8], \\ \mu = 0 \text{ in } [1/4, 1]. \end{cases} \tag{2.51}$$

Then the set X_ν for $\nu > 0$ small enough, is defined as:

$$X_\nu = \Big\{ u \in C^0([0,1], C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3)) / \ \operatorname{div} u = 0, \ \|u - \overline{y}\|_{C^0(\overline{\Omega}\times[0,1])} < \nu, \\ u(x,t).n(x) = \mu(t) y_0(x).n(x) + \overline{y}.n \text{ on } \partial\Omega \times [0,1]. \ \Big\}. \tag{2.52}$$

The value of F will be a solution to a certain linear controllability problem in $C([0,1], C^{2,\alpha}(\overline{\Omega}))$.

We introduce a linear operator π which extends functions defined on $\overline{\Omega}$ to functions defined on $\overline{\tilde{\Omega}}$, and with support in Ω_2. We will require also for it to send continuously $C^{[\lambda],\lambda-[\lambda]}(\overline{\Omega};\mathbb{R}^3)$ into $C^{[\lambda],\lambda-[\lambda]}(\overline{\tilde{\Omega}};\mathbb{R}^3)$, for all $\lambda \in [0,3)\backslash\mathbb{N}$.

Now we define the application F. For $u \in X_\nu$, we set

$$\tilde{u} = \overline{y} + \pi(u - \overline{y}). \tag{2.53}$$

Then $F(u)$ will be a solution of the following problem:

$$F(u)(\cdot,0) = y_0 \text{ in } \overline{\Omega}, \tag{2.54}$$

$$F(u)(\cdot,\cdot).n = 0 \text{ on } [0,1] \times (\partial\Omega\backslash\Gamma_0), \tag{2.55}$$

$$\operatorname{div} F(u) = 0 \text{ in } [0,1] \times \Omega, \tag{2.56}$$

and if we set $\omega := \operatorname{curl}(F(u))$, then it should satisfy

$$\int_\Omega (\partial_t F(u) + \tilde{u} \wedge \omega).Q_i = 0 \text{ in } [0,1],\ \forall i \in \{1,\dots,s\}, \tag{2.57}$$

$$\partial_t \omega + (\tilde{u}.\nabla)\omega = (\omega.\nabla)\tilde{u} \text{ in } [0,1] \times \Omega. \tag{2.58}$$

The controllability problem is to find a $F(u)$ such that

$$F(u)(1,\cdot) = 0 \text{ in } \overline{\Omega}. \tag{2.59}$$

Of course, this linear problem becomes "close" to the Euler problem as ω approaches $\operatorname{curl} u$.

2.3.2 Preliminaries

Before making F explicit, we introduce some notations.

For a regular open bounded subset E of $\mathbb{R}^3$, we denote by $\|\cdot\|_{i,\alpha,E}$ for $i \in \mathbb{N}$ and $\alpha \in (0,1)$, the usual norm for $C^{i,\alpha}(\overline{E})$ and by $\|\cdot\|_{i,E}$ the usual norm for $C^i(\overline{E})$.

We introduce a partition of unity adapted to the open covering of $\overline{\Omega}$ by the open sets $B(x_i,r_i)$ (described in (2.41)-(2.43)), that is some functions $\kappa_i \in C_0^\infty(\tilde{\Omega};[0,1])$ such that

$$\operatorname{Supp} \kappa_i \subset B(x_i,r_i), \tag{2.60}$$

$$\sum_{i=1}^{i=k} \kappa_i \equiv 1 \text{ in } \overline{\Omega}. \tag{2.61}$$

In this section, we will frequently use the following lemma, of which we postpone the demonstration to section 2.3.4.

Lemma 2.3 *Let U be a function in $C^0([0,T], C^{2,\alpha}(\overline{\tilde{\Omega}}, \mathbb{R}^3))$, and W_0 be a function in $C^{1,\alpha}(\overline{\tilde{\Omega}}, \mathbb{R}^3)$. Let W be a function in $C^0([0,T], C^{1,\alpha}(\overline{\tilde{\Omega}}, \mathbb{R}^3))$ defined by the following system*

$$\left\{ \begin{array}{l} W(\cdot,0) = W_0 \ \textit{in} \ \overline{\tilde{\Omega}}, \\ \partial_t W + (U.\nabla)W = (W.\nabla)U - (\operatorname{div} U)W \ \textit{in} \ \tilde{\Omega} \times [0,T]. \end{array} \right. \tag{2.62}$$

Then for all $t \in [0,T]$, one has

$$\operatorname{div} W(\cdot,t) = 0. \tag{2.63}$$

Moreover, if $\operatorname{div} U = 0$ in Ω and $W_0 = \operatorname{curl} V_0$ in Ω, then there exists $V \in C^0([0,T], C^{2,\alpha}(\overline{\tilde{\Omega}}, \mathbb{R}^3))$ such that for all $t \in [0,T]$

$$W(t) = \operatorname{curl} V(t) \ \textit{in} \ \Omega. \tag{2.64}$$

2.3.3 Construction of F

We now give an explicit formulation of $F(u)$. Let $u \in X_\nu$ for ν small enough (say $\nu < \nu_0$ with $\nu_0 < \overline{\nu}$). We associate $\tilde{u}$ defined by (2.53).

We define $F(u)$ by its curl ω in Ω and by its "coordinates" λ_i with respect to the functions Q_i. We define the functions ω and λ_i in a first step, during the times $[0,1/2]$, and then we define them in the interval $[1/2,1]$.

Along the construction of F, we will allow ourselves to reduce ν_0 in order to make F correctly defined.

We introduce a first function ω^* in $C^0([0,1], C^{1,\alpha}(\overline{\tilde{\Omega}}))$. We define ω^* by the relations

$$\left\{ \begin{array}{l} \omega^*(\cdot,0) = \operatorname{curl}\left(\sum_{i=1}^k (\kappa_i \pi(y_0))\right) \ \text{in} \ \overline{\tilde{\Omega}}, \\ \partial_t \omega^* + (\tilde{u}.\nabla)\omega^* = (\omega^*.\nabla)\tilde{u} - (\operatorname{div}\tilde{u})\omega^* \ \text{in} \ \tilde{\Omega} \times (0,1). \end{array} \right. \tag{2.65}$$

By Lemma 2.3, $\omega^*(\cdot,1/4)$ is a curl in Ω: let us say

$$\omega^*(\cdot,1/4) = \operatorname{curl} \mathcal{W} \ \text{in} \ \Omega, \tag{2.66}$$

with $\mathcal{W} \in C^{2,\alpha}(\overline{\tilde{\Omega}})$.

We define then the functions w^l in $C^0([1/4,1/2], C^{1,\alpha}(\overline{\tilde{\Omega}}))$ by the equations:

$$\left\{ \begin{array}{l} w^l(\cdot,1/4) = \operatorname{curl}(\kappa_l \pi(\mathcal{W})) \ \text{in} \ \overline{\tilde{\Omega}}, \\ \partial_t w^l + (\tilde{u}.\nabla)w^l = (w^l.\nabla)\tilde{u} - (\operatorname{div}\tilde{u})w^l \ \text{in} \ \tilde{\Omega} \times [1/4,1/2]. \end{array} \right. \tag{2.67}$$

Of course, we have the relation for $t \in [1/4, 1/2]$

$$\omega^* = \sum_{l=1}^{l=k} w^l. \tag{2.68}$$

Let us now build $\omega : \bar{\tilde{\Omega}} \times [0,1] \longrightarrow \mathbb{R}^3$, continuous in the variable t from $[0,1]\backslash\{t_{i-\frac{1}{2}},\ i \in \{1,\ldots,k\}\}$ into $C^{1,\alpha}(\bar{\tilde{\Omega}})$, continuous at the right of each $t_{i-\frac{1}{2}}$ and with a limit in $C^{1,\alpha}(\bar{\tilde{\Omega}})$ at the left of each $t_{i-\frac{1}{2}}$ (for $i \in \{1,...,k\}$). We will extract $F(u)$ from this ω.

Let us define ω this way:

$$\omega(x,t) = \omega^*(x,t) \text{ in } \bar{\tilde{\Omega}} \times [0, \frac{1}{4}], \tag{2.69}$$

then for $t \in [1/4, 1/2]$:

$$\partial_t \omega + (\tilde{u}.\nabla)\omega = (\omega.\nabla)\tilde{u} - (\operatorname{div}\tilde{u})\omega,$$
$$\text{in } \{[\frac{1}{4}, t_{\frac{1}{2}}) \bigcup_{i=0}^{i=k-1} (t_{i-\frac{1}{2}}, t_{i+\frac{1}{2}}) \cup [t_{k-\frac{1}{2}}, \frac{1}{2}]\} \times \tilde{\Omega}. \tag{2.70}$$

Thus to define ω properly, we have yet to define it at times $t_{i-\frac{1}{2}}$. We do it in order that, at time t_i, only $\sum_{l=i+1}^{k} w^l(x,t_i)$ stays on Ω, instead of $\omega^*(x,t_i)$. For that, we simply have to consider ω at time $t^-_{i-\frac{1}{2}}$. Let us suppose by induction that, one has

$$\omega(\cdot, t^-_{i-\frac{1}{2}}) = \sum_{l=i}^{k} w^l(\cdot, t_{i-\frac{1}{2}}). \tag{2.71}$$

Then, we just have to define

$$\omega(x, t^+_{i-\frac{1}{2}}) = \sum_{l=i+1}^{k} w^l(\cdot, t_{i-\frac{1}{2}}), \tag{2.72}$$

with the convention

$$\omega(x, t^+_{k-\frac{1}{2}}) = 0. \tag{2.73}$$

So (2.70) and (2.72) do completely define ω for t in $[1/4, 1/2]$. Note that by (2.67), (2.70) and (2.72), one gets

$$\omega(\cdot, t) = \sum_{l=i+1}^{k} w^l(\cdot, t), \ \forall t \in [t_{i-\frac{1}{2}}, t_{i+\frac{1}{2}}). \tag{2.74}$$

Relations (2.43) and (2.53) imply that for $\|u-\overline{y}\|_{C^0(\overline{\Omega}\times[0,1])}$ small enough (that is, for a suitable choice of ν_0), one has

$$\phi^{\tilde{u}}(\overline{B}(x_i,r_i),1/4,t_{i-\frac{1}{2}})\cap\overline{\Omega}=\emptyset.$$

But by (2.60) and (2.67), at time 1/4, the support of w^i is included in $\overline{B}(x_i,r_i)$. It follows from the form of (2.67) that the support of w^i follows the flow of $\tilde{u}$. We deduce that

$$\text{Supp } w^i(t_{i-\frac{1}{2}},\cdot)\cap\overline{\Omega}=\emptyset.$$

This way, we get that the restriction of ω to $\overline{\Omega}\times[0,\frac{1}{2}]$ is $C([0,\frac{1}{2}],C^{1,\alpha}(\overline{\Omega}))$-regular and that we have in $\Omega\times[0,\frac{1}{2}]$ the relation

$$\partial_t\omega+(\tilde{u}.\nabla)\omega=(\omega.\nabla)\tilde{u}. \tag{2.75}$$

Furthermore, by Lemma 2.3, ω stays divergence-free in $\tilde{\Omega}\times[0,\frac{1}{2}]$.

We want to define v in $C([0,\frac{1}{2}],C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$ by

$$\operatorname{curl} v=\omega \text{ in } \Omega\times[0,\frac{1}{2}], \tag{2.76}$$

$$\operatorname{div} v=0 \text{ in } \Omega\times[0,\frac{1}{2}], \tag{2.77}$$

$$v.n=\mu(t)y_0.n+\overline{y}.n \text{ in } \partial\Omega\times[0,\frac{1}{2}], \tag{2.78}$$

$$\int_\Omega v.Q_i dx=0,\ \forall t\in[0,\frac{1}{2}]. \tag{2.79}$$

But to prove that it is possible, let us point out that, for the existence of such a v, we need, in addition to $\operatorname{div}\omega=0$, the fact that ω is a curl in Ω. This is proved also by Lemma 2.3.

By the way, we remark that the relation (2.79) is necessary to obtain the unicity of v.

Now, we can see that any

$$v' :=v+\sum_{j=1}^{s}\lambda_j(t)Q_j(x), \tag{2.80}$$

still satisfies (2.76), (2.77) and (2.78), for any choice of λ_i. We choose λ_i, and hence v' such that

$$\int_\Omega v'(0).Q_i dx=\int_\Omega y_0.Q_i dx, \tag{2.81}$$

$$\int_{\Omega} (\partial_t v' + \tilde{u} \wedge \omega).\mathcal{Q}_i dx = 0, \quad \forall i \in \{1, ..., s\}, \forall t \in [0, \frac{1}{2}]. \tag{2.82}$$

Note that this is made possible because the matrix

$$\left(\int_{\Omega} \mathcal{Q}_i.\mathcal{Q}_j dx \right)_{1 \leq i \leq s,\ 1 \leq j \leq s},$$

is invertible, as the $(\mathcal{Q}_i)$ is a free family.

We are now able to define ω in the time-interval $(\frac{1}{2}, \frac{3}{4})$. We consider $i \in \{k, \dots, k+s-1\}$. Let $\tilde{w^i} \in C([t_i, t_{i+1}], C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$ be defined by

$$\tilde{w^i}(t_i) = \operatorname{curl} \aleph^i, \tag{2.83}$$

$$\partial_t \tilde{w}^i + (\tilde{u}.\nabla)\tilde{w}^i = (\tilde{w}^i.\nabla)\nabla\tilde{u} - \tilde{u}(\operatorname{div} \tilde{w}^i) \text{ in } \tilde{\Omega} \times [t_i, t_{i+1}], \tag{2.84}$$

where $\aleph^i$ is defined in Lemma 2.2 (see (2.32)). Here precisely, will be needed the fact that $\nu_0 < \overline{\nu}$, in such a way that

$$\operatorname{Supp} \tilde{w}^i(\cdot, t_{i+1}) \subset \tilde{\Omega} \backslash \overline{\Omega}. \tag{2.85}$$

Then, we define the applications ζ^i (which differ from those in Lemma 2.2 by the time intervals only), for $i = 1, \dots, s$, respectively on the intervals $[t_{i+k-1}, t_{i+k}]$ by the relations:

$$\operatorname{curl} \zeta^i = \tilde{w}^i \text{ in } \overline{\Omega} \times [t_{i+k-1}, t_{i+k}], \tag{2.86}$$

$$\operatorname{div} \zeta^i = 0 \text{ in } \overline{\Omega} \times [t_{i+k-1}, t_{i+k}], \tag{2.87}$$

$$\zeta^i.n = \partial_n \theta^i \text{ on } \partial\Omega \times [t_{i+k-1}, t_{i+k}], \tag{2.88}$$

$$\int_{\Omega} \zeta^i(t_{k+i-1}).\mathcal{Q}_j dx = 0, \tag{2.89}$$

$$\int_{\Omega} (\partial_t \zeta^i + u \wedge \operatorname{curl}(\zeta^i)).\mathcal{Q}_j dx = 0, \quad \forall j \in \{1, ..., s\}, \ \forall t \in [t_{i+k-1}, t_{i+k}]. \tag{2.90}$$

We can then define:

$$\omega(x, t) = -\lambda_i(\frac{1}{2})\tilde{w}^i(x, t) \quad \text{in } [t_i, t_{i+1}) \times \overline{\tilde{\Omega}}, \quad \forall i \in \{1, ..., s\}. \tag{2.91}$$

As in $[\frac{1}{4}, \frac{1}{2}]$, ω is continuous in variable t from $\bigcup_{i=k}^{k+s-1}(t_i, t_{i+1})$ to $C^{1,\alpha}(\overline{\tilde{\Omega}})$, and with a limit in $(C^{1,\alpha}(\overline{\tilde{\Omega}}))$ at the left and at the right of each t_i (for $i \in \{k+1, ..., s+k-1\}$). Moreover, also as in $[\frac{1}{4}, \frac{1}{2}]$, it is continuous from $[1/2, 3/4]$ into $C^{1,\alpha}(\overline{\Omega})$.

Now we extend formula (2.76)-(2.79) to the whole interval $[0,1]$:

$$\begin{cases} \operatorname{curl} v = \omega \text{ in } \Omega \times [0,1], \\ \operatorname{div} v = 0 \text{ in } \Omega \times [0,1], \\ v.n = \mu(t) y_0.n + \overline{y}.n \text{ in } \partial\Omega \times [0,1], \\ \int_\Omega v.Q_i dx = 0, \ \forall t \in [0,1], \end{cases} \tag{2.92}$$

and also extend formula (2.82) (and hence extend the functions λ_i), in addition to (2.81):

$$\int_\Omega (\partial_t v + \sum_{j=1}^{j=s} \lambda_j'(t) Q_j + \tilde{u} \wedge \omega).Q_i dx = 0, \quad \forall i \in \{1, ..., s\}, \ \forall t \in [0,1].(2.93)$$

We can now define $F(u)$ from $\overline{\Omega} \times [0,1]$ into $\mathbb{R}^3$:

$$F(u) : = \begin{cases} v + \sum_{i=1}^s \lambda_i(t) Q_i(x) \ \text{ in } \overline{\Omega} \times [0, \frac{3}{4}], \\ 0 \ \text{ in } \overline{\Omega} \times [\frac{3}{4}, 1], \end{cases} \tag{2.94}$$

By this way, the function F is correctly defined.

By (2.74) and (2.85) we get that

$$\operatorname{curl} F(u) \in C^0([0,1]; C^{1,\alpha}(\overline{\Omega})). \tag{2.95}$$

By (2.86)-(2.90), (2.91) and (2.93), we get that

$$\lambda_i(\frac{3}{4}) = 0.$$

Together with (2.95), this proves that

$$F(u) \in C([0,1], C^{2,\alpha}(\overline{\Omega})) \tag{2.96}$$

and that the relation (2.75) holds in $\Omega \times [0,1]$. Moreover, $F(u)$ is obviously a solution to the controllability problem (2.59).

2.3.4 Proof of Lemma 2.3

As it can be seen from (2.62), $\operatorname{div} W$ satisfies the equation

$$\partial_t(\operatorname{div} W) + (U.\nabla)(\operatorname{div} W) = -(\operatorname{div} U)(\operatorname{div} W). \tag{2.97}$$

The point (2.63) is hence clear.

To get the second point (2.64), we need more that (2.63). Let us indeed introduce a family of special functions of Ω.

Let us consider, when $\partial\Omega$ has many connected components (that is in the case where $H_2(\Omega) \neq 0$), the set of functions $\mathcal{P}^j$ constructed as follows. We note the connected components of $\partial\Omega$: $\gamma_0, \dots, \gamma_{\overline{s}}$, and we define $\mathcal{P}^j := \nabla\mathfrak{p}^j$ for any $j \in \{1,..,\overline{s}\}$ where $\mathfrak{p}^j$ is defined by the relations

$$\begin{cases} \Delta\mathfrak{p}^j = 0 \text{ in } \Omega, \\ \mathfrak{p}_{|\partial\Omega} = 0 \text{ on } (\partial\Omega\backslash\gamma_j), \\ \mathfrak{p}_{|\partial\Omega} = 1 \text{ on } \gamma_j, \end{cases} \tag{2.98}$$

It is well known that a solenoidal vector field on Ω can be written as the sum of the curl of a vector field and of a linear combination of the $\mathcal{P}^j$.

Consequently, if Ω is an open set such that $H_2(\Omega) \neq 0$, in order that (2.64) occurs, we need, besides the divergence free condition, the following relations to hold:

$$\int_\Omega W(\cdot,t).\mathcal{P}^j dx = 0, \quad \forall t \in [0,T], \quad \forall j \in \{1,..,\overline{s}\}, \tag{2.99}$$

where we defined $\overline{s}$ and $\mathcal{P}^j$ in (2.98). These relations (2.99) are true for $t = 0$, since $W(\cdot,0) = \operatorname{curl} V_0$. We want to show this property stays true after $t = 0$. To prove it, we compute (indices j for $\mathcal{P}^j$ are dropped),

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_\Omega W.\nabla\mathfrak{p}dx &= \int_\Omega \{(U.\nabla)W\}.\nabla\mathfrak{p}dx - \int_\Omega \{(W.\nabla)U\}.\nabla\mathfrak{p}dx, \\ &= \int_\Omega U^i W_i^j \mathfrak{p}_j dx - \int_\Omega W^i.U_i^j \mathfrak{p}_j dx, \end{aligned}$$

where we denote derivations by lower indices and vector coordinates by upper exponents. Then, integrating by parts, we obtain, since $\operatorname{div} U = \operatorname{div} W = 0$,

$$\tfrac{d}{dt}\textstyle\int_\Omega W.\nabla\mathfrak{p}dx = \int_{\partial\Omega} (W^i U^j \mathfrak{p}_j n^i - W^j \mathfrak{p}_j U^i n^i) d\sigma. \tag{2.100}$$

As $\mathfrak{p}$ is constant on each connected component of $\partial\Omega$, $\nabla\mathfrak{p}$ is normal to the boundary everywhere on the boundary. We can deduce from this fact, that $(W.n)(U.\nabla\mathfrak{p}) = (W.\nabla\mathfrak{p})(U.n)$ on the boundary. The term on the right hand side of (2.100) is thus 0, so (2.99) stays true for all times. From that, we deduce that $W(\cdot,t)$ is a curl in Ω. □

2.4 Proof of Proposition 2.1

2.4.1 Introduction

The goal of this section is to prove that F admits a fixed point, and then to prove that it gives a proper solution to Proposition 2.1.

The first part of this proof is thus to find a set invariant by F.

We denote by $\mathcal{B}(B)$ the ball in $C^0([0,1], C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$ with radius B and center 0. Then this invariant set will be found as a certain $X_\nu \cap \mathcal{B}(B)$, for proper B and ν.

In a first step, we prove the following proposition:

Proposition 2.2 *For any $B > 0$, there exists ν_0 and ν_1, such that if one has $\|y_0\|_{1,\alpha,\Omega} < \nu_1$, then for all $\nu < \nu_0$, for all $u \in X_\nu \cap \mathcal{B}(B)$, one has $F(u) \in X_\nu$.*

In a second step we prove this proposition:

Proposition 2.3 *There exists $\nu_2 > 0$, such that if $\|y_0\|_{C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)} \leq \nu_2$, and if we define the sequences of functions $(y^m)_{m\geq 0} \in (C^0([0,1], C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)))^{\mathbb{N}}$ and $(\omega^m)_{m\geq 0} \in (C^0([0,1], C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3)))^{\mathbb{N}}$ as follows:*

$$\begin{cases} y^0(x,t) = \mu(t)y_0(x) + \overline{y}(x,t), \\ y^{m+1} = F(y^m), \\ \omega^{m+1} \text{ defined as previously on } \tilde{\Omega}\times[0,1], \end{cases} \tag{2.101}$$

then the sequence $(y^m)_{m\geq 0}$ is bounded in $C^0([0,1], C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$, the bound depending only on Ω and ν_2.

A fortiori, we will be able to find B and ν such that

$$F(X_\nu \cap \mathcal{B}(B)) \subset X_\nu \cap \mathcal{B}(B).$$

The last step of the proof of Proposition 2.1 is then to establish that F has a fixed point solution to the non-linear controllability problem.

The proofs of these propositions will require a technique introduced by C. Bardos and U. Frisch in [2]. Particularly, we will use the following lemma:

Lemma 2.4 *[2, Lemma 1] Let u, v and g be three functions of regularity $C^0([0,T], C^{1,\alpha}(\overline{\tilde{\Omega}},\mathbb{R}^3))$, satisfying the relations*

$$\frac{\partial u}{\partial t} + (v.\nabla)u = g,\ v.n_{|\partial\tilde{\Omega}\times[0,T]} = 0. \tag{2.102}$$

Then we have on $[0,T]$

$$\frac{d}{dt^+}\|u\|_{0,\alpha,\tilde{\Omega}} \leq \|\partial_t u\|_{0,\alpha,\tilde{\Omega}} \leq \|g\|_{0,\alpha,\tilde{\Omega}} + \alpha\|\nabla v\|_{0,\alpha,\tilde{\Omega}}.\|u\|_{0,\alpha,\tilde{\Omega}}. \tag{2.103}$$

2.4.2 Proof of Proposition 2.2

In the sequel, we will denote by C, C', C_1 and C_2 different positive constants depending only on Ω.

In this section, we will mark each object introduced in the previous section and corresponding to the m-th iteration of the operator F in the construction of the sequence $(y^m)_{m\geq 0}$ by a lower index m.

As previously, we will first consider t in the set $[0,1/2]$, and then t in the interval $[1/2,1]$.

When considering the w^m introduced in section 3, we will no longer use the upper index l (corresponding to the l-th ball $B(x_l,r_l)$) in order not to confuse with the index m corresponding to this m-th iteration. All the assertions about w will be valid for any upper index. We will do the same with the index 'i' in λ_i. Also, when considering a time-dependent function $f\ :\ = f(x,t)$, we will make no difference between $\|f(\cdot,t)\|$ and $\|f\|(t)$, whatever spatial norm we use.

We first get an estimate on ω^*_{m+1}. By (2.65) and Lemma 2.4, one easily gets for $t\in[0,1]$

$$\frac{d}{dt^+}\|\omega^*_{m+1}\|_{0,\alpha,\Omega}(t)\leq(2+\alpha)\|\tilde{y}^m\|_{1,\alpha,\tilde{\Omega}}(t)\|\omega^*_{m+1}\|_{0,\alpha,\Omega}(t). \tag{2.104}$$

With Gronwall's lemma, we deduce from (2.104) that for $t\in[0,1]$

$$\|\omega^*_{m+1}\|_{0,\alpha,\Omega}(t)\leq\|\omega^*_{m+1}\|_{0,\alpha,\Omega}(0)e^{(2+\alpha)t\|\tilde{y}^m\|_{C^0([0,1],C^{1,\alpha}(\bar{\Omega}))}}, \tag{2.105}$$

We do the same with the equation (2.67), and get by Lemma 2.4 the estimate for $t\in[1/4,1/2]$

$$\frac{d}{dt^+}\|w_{m+1}\|_{0,\alpha,\Omega}(t)\leq(2+\alpha)\|\tilde{y}^m\|_{1,\alpha,\tilde{\Omega}}\|w_{m+1}\|_{0,\alpha,\Omega}(t). \tag{2.106}$$

We easily deduce from that and from Gronwall's lemma that for $t\in[1/4,1/2]$

$$\|w_{m+1}\|_{0,\alpha,\Omega}(t)\leq\|w_{m+1}\|_{0,\alpha,\Omega}(\frac{1}{4})e^{(2+\alpha)(t-\frac{1}{4})\|\tilde{y}^m\|_{C^0([0,1],C^{1,\alpha}(\bar{\Omega}))}}, \tag{2.107}$$

from which we get, with (2.74), that for $t\in[1/4,1/2]$

$$\|\omega^{m+1}\|_{0,\alpha,\Omega}(t)\leq k\|\omega^m\|_{0,\alpha,\Omega}(\frac{1}{4})e^{3(t-\frac{1}{4})\|\tilde{y}^m\|_{C^0([0,1],C^{1,\alpha}(\bar{\Omega}))}}, \tag{2.108}$$

from what we deduce with (2.65) and (2.105) that for $t\in[0,1/2]$

$$\|\omega^{m+1}\|_{0,\alpha,\Omega}(t)\leq k\|y_0\|_{1,\alpha,\Omega}e^{3t\|\tilde{y}^m\|_{C^0([0,1],C^{1,\alpha}(\bar{\Omega}))}}. \tag{2.109}$$

We now want to get (2.109) for the rest of the time $[1/2, 1]$, and by the way obtain an estimate on the λ_i^m. By (2.92) and (2.93) we have on $[0, 1]$

$$\frac{d}{dt^+}|\lambda_i^m| \leq C\|\omega^{m+1}\|_{0,\Omega}\|y^m\|_{0,\Omega}. \tag{2.110}$$

We deduce that for $t \in [0, 1/2]$

$$|\lambda_i^{m+1}(t)| \leq C_1\|y_0\|_{1,\alpha,\Omega}\|y^m\|_{C^0([0,1],C^1(\overline{\Omega}))}e^{3\|\tilde{y}^m\|_{C^0([0,1],C^1(\overline{\Omega}))}}. \tag{2.111}$$

As for (2.105), one can deduce that for all $i \in \{k, \dots, k+s-1\}$ and for all $t \in [t_i, t_{i+1})$, one has

$$\|\omega^{m+1}\|_{0,\alpha,\Omega}(t) \leq \|\omega^{m+1}\|_{0,\alpha,\Omega}(t_i)e^{(2+\alpha)(t-t_i)\|\tilde{y^m}\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}}.$$

With (2.91) and (2.111), we get that for $t \in [1/2, 3/4]$

$$\|\omega^{m+1}\|_{0,\alpha}(t) \leq C_2\|y_0\|_{1,\alpha,\Omega}\|y^m\|_{C^0([0,1],C^1(\overline{\Omega}))}e^{3\|\tilde{y}^m\|_{C^0([0,1],C^1(\overline{\Omega}))}}. \tag{2.112}$$

We can deduce from it, with (2.92)-(2.94), (2.109), (2.111) and that for $t \in [0, 1]$

$$\|F(u) - \overline{y}\|_{0,\Omega}(t) \leq C(\|y^m\|_{1,\alpha,\Omega})\|y_0\|_{1,\alpha,\Omega}, \tag{2.113}$$

where $C(\cdot)$ is an increasing, positive real-valued, numerical function.

So we have proved that F is well defined on X_ν and that for any $B > 0$, there exists $\nu_1 = \nu_1(B) > 0$ such that for any y_0 satisfying $\|y_0\|_{2,\alpha,\Omega} < \nu < \nu_1$, one has

$$F(X_\nu \cap \mathcal{B}^{C([0,1],C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))}(B)) \subset X_\nu,$$

where we have denoted by $\mathcal{B}^{C([0,1],C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))}(B)$ the 0-centered open ball in $C([0,1], C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$ with radius B, at least if ν_0 and ν_1 (depending on B) are small small enough. □

2.4.3 Proof of Proposition 2.3

Let us consider the sequence $(y^m)_{m\geq 0} \in (C^0([0,1], C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3)))^{\mathbb{N}}$ by (2.101).

In a first step, we just deal with the boundedness of the sequence (y^m) in the space $C^0([0,1], C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$. We will come back to the boundedness in $C^0([0,1], C^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$ at the end of this section.

Let us denote by C_i, $i \geq 1$, various constants which do not depend on m. Combining (2.109) and (2.112), one can get for any $t \in [0, 1]$ that

$$\|\omega^{m+1}\|_{0,\alpha,\Omega} \leq \\ C_3\|y_0\|_{1,\alpha,\Omega}e^{3\|y^m\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}}(1 + \|y^m\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}). \tag{2.114}$$

By (2.110) and (2.112), one as also in $[0,1]$

$$|\lambda_i^{m+1}| \leq C_4\|y_0\|_{1,\alpha,\Omega}e^{3\|y^m\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}}(1+\|y^m\|^2_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}). \quad (2.115)$$

On another side, by (2.92) and (2.94), one can find some constants such that, for any $t \in [0,1]$,

$$\|y^{m+1}\|_{1,\alpha,\Omega}(t) \leq C_7\|y_0\|_{1,\alpha,\Omega} + C_8\|\omega^{m+1}\|_{0,\alpha}(t) + C_9\sum_i \|\lambda_i\|_{C^0([0,1])}. \quad (2.116)$$

We deduce from (2.114), (2.115) and (2.116) that for every $t \in [0,1]$

$$\begin{aligned}&\|y^{m+1}\|_{1,\alpha,\Omega}(t)\\ &\leq C_{10}\|y_0\|_{1,\alpha,\Omega}(1+e^{3\|y^m\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}})(1+\|y^m\|^2_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}).\end{aligned} \quad (2.117)$$

Note that this is made valid in $[3/4,1]$ because of the trivial form of y^m in this time segment.

We want to deduce from (2.117) that, reducing ν_0 if necessary, one can get

$$\|y^m\|_{C^0([0,1],C^{1,\alpha}(\Omega))} \leq 2\|\overline{y}\|_{C^0([0,1],C^{1,\alpha}(\Omega))}, \ \forall m \in \mathbb{N}. \quad (2.118)$$

The proof of (2.118) is done by induction.

We check (2.118) for $m=0$. As $y^0 = \mu(t)y_0 + \overline{y}$, (2.118) is satisfied if $\nu_0 < \|\overline{y}\|_{1,\alpha}$.

We now suppose that (2.118) is satisfied for a fixed m, and show it is still valid at rank $m+1$.

We impose ν_0 in order that

$$\begin{aligned}C_{10}\nu_0(1+e^{6\|\overline{y}\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}})(1+4\|\overline{y}\|^2_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))})\\ < 2\|\overline{y}\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))}.\end{aligned} \quad (2.119)$$

Using (2.117) and the induction hypothesis, we get (2.118) at rank $m+1$. Consequently, we have a bound on the sequence $(\|y^m\|_{1,\alpha})_{m\geq 0}$. Hence, we can choose a "universal" ν_1 so that actually y^m belongs to X_ν at each step.

★

Now we prove the boundedness of the sequence (y^m) in $C^0([0,1],C^{2,\alpha}(\overline{\Omega}))$.

The proof is quite the same as for the $C^{1,\alpha}$ bound. It consists in the majoration of $\|y^{m+1}\|_{C^{2,\alpha}}$ by a factor of the form $\|y_0\|_{2,\alpha}C(\|y^m\|_{2,\alpha})$.

We already have this type of bound on the λ_i. It is hence sufficient to have such a bound for $\|\omega^m\|_{C^{2,\alpha}}$.

As previously, it is consequently sufficient to get a $C^0([0,1], C^{2,\alpha}(\tilde{\Omega}))$ bound for the (w^m). But considering the derivatives of the relation (2.67) and using Lemma 2.4, we get terms in the right hand side all majored either by a certain $\|\tilde{y}^m\|_{2,\alpha}\|w^{m+1}\|_{1,\alpha}$, or by a constant because we know that (y^m) is bounded in $C^0([0,1], C^{1,\alpha}(\tilde{\Omega}))$.

Hence, the same demonstration works again, if $\|y_0\|_{2,\alpha}$ is chosen small enough.

2.4.4 Back to the proof of Proposition 2.1

Now we prove the convergence of the sequences (y^m) and (ω^m). We first show the convergence on the interval $[0,1/2]$. We still follow [2], and aim at proving that the sequence (ω^m, λ^m) satisfies the Cauchy criterion in $\cap_{i=0}^{i=k} C^0([t_{i-\frac{1}{2}}, t_{i+\frac{1}{2}}], C^{0,\alpha}(\bar{\tilde{\Omega}}) \times \mathbb{R}^s)$.

For all times except $t_{i+\frac{1}{2}}$, $i \in \{0, \dots, k-1\}$, we have

$$\begin{aligned}\partial_t(\omega^p - \omega^m) + (\tilde{y}^{m-1}.\nabla)(\omega^p - \omega^m) = \; & [(\tilde{y}^{p-1} - \tilde{y}^{m-1}).\nabla]\omega^p \\ & +[(\omega^p - \omega^m).\nabla]\tilde{y}^{p-1} \\ & +(\omega^m.\nabla)(\tilde{y}^{p-1} - \tilde{y}^{m-1}) \\ & -(\operatorname{div}\tilde{y}^{p-1})(\omega^p - \omega^m) \\ & -(\operatorname{div}\tilde{y}^{m-1} - \operatorname{div}\tilde{y}^{p-1})\omega^m.\end{aligned} \tag{2.120}$$

Any term at the right of this equality can be bounded in norm $\|\cdot\|_{0,\alpha,\tilde{\Omega}}$ (within a multiplicative constant) either by the norm $\|\omega^p - \omega^m\|_{0,\alpha,\tilde{\Omega}}(t)$, or by the norm $\|\tilde{y}^{p-1} - \tilde{y}^{m-1}\|_{1,\alpha,\tilde{\Omega}}(t)$.

With the help of Lemma 2.4, we deduce that for $t \in [\frac{1}{4}, \frac{1}{2}]$, except at times $t_{i+\frac{1}{2}}$, one has

$$\|\partial_t(\omega^p - \omega^m)\|_{0,\alpha,\tilde{\Omega}}(t) \leq C(\|\tilde{y}^{p-1} - \tilde{y}^{m-1}\|_{0,\alpha,\tilde{\Omega}}(t) + \|\omega^p - \omega^m\|_{0,\alpha,\tilde{\Omega}}(t)),$$

and consequently

$$\|\partial_t(\omega^p - \omega^m)\|_{0,\alpha,\tilde{\Omega}}(t) \leq C(\|y^{p-1} - y^{m-1}\|_{1,\alpha,\Omega}(t) + \|\omega^p - \omega^m\|_{0,\alpha,\tilde{\Omega}}(t)),$$

which gives finally

$$\begin{aligned}\|\partial_t(\omega^p - \omega^m)\|_{0,\alpha,\tilde{\Omega}}(t) \leq C(&\|\omega^p - \omega^m\|_{0,\alpha,\tilde{\Omega}}(t) \\ &+ \|\omega^{p-1} - \omega^{m-1}\|_{0,\alpha,\tilde{\Omega}}(t) + |\lambda^{m-1} - \lambda^{p-1}|(t)),\end{aligned} \tag{2.121}$$

As the expression (2.120) is also valid when replacing w by ω^*, one can get that (2.121) is valid in fact in $[0, t_{\frac{1}{2}}]$.

We do the same for $\lambda_i^m - \lambda_i^p$ and get the same way, using (2.93), that

$$|\partial_t(\lambda^p - \lambda^m)|(t) \leq C(\|\omega^p - \omega^m\|_{1,\alpha,\tilde{\Omega}}(t) + \|y^{p-1} - y^{m-1}\|_{1,\alpha,\Omega})(t),$$

and hence that

$$\begin{aligned} |\partial_t(\lambda^p - \lambda^m)|(t) \leq C(&\|\omega^p - \omega^m\|_{1,\alpha,\tilde{\Omega}}(t) \\ &+ \|\omega^{p-1} - \omega^{m-1}\|_{1,\alpha,\Omega}(t) + |\lambda^{p-1} - \lambda^{m-1}|(t)). \quad (2.122) \end{aligned}$$

Let us denote by $\mathcal{K}^{m,p}(t) : = \|\omega^p - \omega^m\|_{1,\alpha,\tilde{\Omega}}(t) + |\lambda^p - \lambda^m|(t)$.
Putting together (2.121) and (2.122), one gets

$$\frac{d}{dt^+}\mathcal{K}^{m,p}(t) \leq C(\mathcal{K}^{m,p}(t) + \mathcal{K}^{m-1,p-1}(t)). \tag{2.123}$$

By Gronwall's lemma, one gets, for $t \in [0, t_{\frac{1}{2}}]$

$$\mathcal{K}^{m+1,p+1}(t) \leq C \int_0^t \mathcal{K}^{m,p}(t_1) e^{Ct_1} dt_1, \tag{2.124}$$

and then by induction

$$\mathcal{K}^{m+k,p+k}(t) \leq C^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \mathcal{K}^{m,p}(t_k) e^{Ct_1 + \cdots + Ct_k} dt_1 \dots dt_k.$$

Finally, one gets on $[0, t_{\frac{1}{2}}]$:

$$\mathcal{K}^{m+k,p+k}(t) \leq K \frac{e^{kCt}}{k!} \max_{s \in [0, t_{\frac{1}{2}}]} \mathcal{K}^{m,p}(s), \tag{2.125}$$

from what we get the convergences of ω^m and λ^m to some $\mathbf{w}$ and some $\overline{\lambda}$ for times in $[0, t_{\frac{1}{2}}]$ (to be more precise, the convergence are to be understood respectively in $C^0([0, t_{\frac{1}{2}}], C^{0,\alpha}(\overline{\tilde{\Omega}}; \mathbb{R}^3))$ and in $C^0([0, t_{\frac{1}{2}}], \mathbb{R}^s))$. Furthermore, these convergences determine those of y^m and w_i^m respectively in $C^0([0, t_{\frac{1}{2}}], C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$ and $C^0([0, t_{\frac{1}{2}}], C^{0,\alpha}(\overline{\tilde{\Omega}}; \mathbb{R}^3))$.

Consequently, we get the convergence of $\omega^m(t_{\frac{1}{2}}^+)$ also, and we can repeat the same method during the interval $[t_{\frac{1}{2}}, t_{\frac{3}{2}}]$: instead of (2.124), we get

$$\mathcal{K}^{m+1,p+1}(t) \leq C \int_{t_{\frac{1}{2}}}^t \mathcal{K}^{m,p}(t_1) e^{Ct_1} dt_1 + \mathcal{K}^{m+1,p+1}(t_{\frac{1}{2}}^+) - \mathcal{K}^{m,p}(t_{\frac{1}{2}}^+),$$

which leads, for $t \in [t_{\frac{1}{2}}^+, t_{\frac{3}{2}}]$, to

$$\mathcal{K}^{m+k,p+k}(t) \leq K \frac{e^{kCt}}{k!} \max_{s \in [t_{\frac{1}{2}}^+, t_{\frac{3}{2}}]} \mathcal{K}^{m,p}(s) + \mathcal{K}^{m+k,p+k}(t_{\frac{1}{2}}^+) - \mathcal{K}^{m,p}(t_{\frac{1}{2}}^+).$$

We get the convergence on the interval $[t_{\frac{1}{2}}+, t_{\frac{3}{2}}]$, and then step by step we get the convergences in $C^0([t_{\frac{3}{2}}, t_{\frac{5}{2}}], C^{0,\alpha}(\overline{\tilde{\Omega}}; \mathbb{R}^3))$ and in $C^0([t_{\frac{3}{2}}, t_{\frac{5}{2}}], \mathbb{R}^s)$, etc.

Remains the problem of convergence in $[1/2, 3/4]$.

Clearly, the convergence of (ω^m) and the one of (y^m) on the interval $[0, 1/2]$ determine the one of the $\lambda_i^m(1/2)$ to a certain s-uplet $\overline{\lambda}(1/2) \in \mathbb{R}^s$.

As ω and λ are governed by the same equations as during $[0, \frac{1}{2}]$, the result (2.123) is also valid for $t \in [\frac{1}{2}, 1]$, except at times t_i, with $i \in \{k+1, k+s+1\}$. One consequently deduces the same convergence result in intervals $[t_i, t_{i+1}]$ for i in $\{k+1, k+s+1\}$. Finally, y^m and ω^m do converge to some y and $\mathbf{w}$ in $C^0([0,1], C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3))$ and $C^0([0,1], C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^3))$.

⋆

Let us now prove the fixed point y is a solution to our controllability problem.

From (2.75), (2.92) and (2.94), we get that

$$\left\{\begin{array}{l} \operatorname{div} y = 0 \text{ in } \Omega \times [0,1] \\ y.n = 0 \text{ on } (\partial\Omega \backslash \Gamma_0) \times [0,1] \\ \operatorname{curl} y = \mathbf{w} \text{ in } \ \Omega \times [0,1] \\ \partial_t \mathbf{w} + (y.\nabla)\mathbf{w} = (\mathbf{w}.\nabla) y \text{ in } \ \Omega \times [0,1]. \end{array}\right. \tag{2.126}$$

With (2.94), (2.82) and (2.126), we get (2.3) for some $p \in \mathfrak{D}'(\Omega \times (0,1))$. From (2.72) (with (2.73)), we get

$$\mathbf{w}(\cdot, 1) = 0.$$

Together with (2.91), this leads to

$$y(\cdot, 1) = 0.$$

As (2.6) is obviously satisfied by y, we get a solution for Proposition 2.1.

2.5 End of the proof of Theorem 2.1

Here we deduce Theorem 2.1 from Proposition 2.1. Let us consider y_0 and y_1 two divergence-free elements of $C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^3)$, . We use Proposition 2.1 and we obtain a certain ν by this proposition. For y_0, we choose ϵ in $]0, T/2[$ small enough so that $|\epsilon y_0|_{2,\alpha} < \nu$. Proposition 2.1 for ϵy_0 give us a couple

(y,p). Then $(\tilde{y},\tilde{p})$ defined by

$$\begin{aligned}
\tilde{y}(x,t) &= \epsilon^{-1}y(x,\epsilon^{-1}t) \ \ \forall t\in[0,\epsilon],\\
\tilde{p}(x,t) &= \epsilon^{-2}p(x,\epsilon^{-1}t) \ \ \forall t\in[0,\epsilon],\\
\tilde{y}(x,t) &= 0 \ \ \forall t\in[\epsilon,T/2],\\
\tilde{p}(x,t) &= 0 \ \ \forall t\in[\epsilon,T/2],
\end{aligned}$$

is still solution of the Euler system with $\tilde{y}_{|t=0}=y_0$. We operate similarly for $-y_1$, so that we obtain $(\tilde{y}',\tilde{p}')$. Then $(-\tilde{y}'(T-t),\tilde{p}'(T-t))$ is again solution of Euler equation. The function chosen equal to $\tilde{y}'$ on $[0,T/2]$ and to $\tilde{y}'(T-t)$ on $[T/2,T]$, associated to the pressure function equal to $\tilde{p}$ on $[0,T/2]$ and to $\tilde{p}'(T-t)$ on $[T/2,T]$, gives an answer to the problem.

2.6 Proof of Lemma 2.1

The proof is quite the same as the one of [7, Lemma A.1], which proves the result for interior points.

Let us recall the lemma in [7]:

Lemma 2.5 *[7, Lemma A.2] For any $\overline{x}$ in Ω,*

$$\left\{\nabla\theta(\overline{x});\ \theta\in C^\infty(\overline{\Omega};\mathbb{R}),\ \Delta\theta=0 \ in\ \overline{\Omega}\ and\ \frac{\partial\theta}{\partial n}=0\ on\ \partial\Omega\backslash\Gamma_0\right\}=\mathbb{R}^3 \tag{2.127}$$

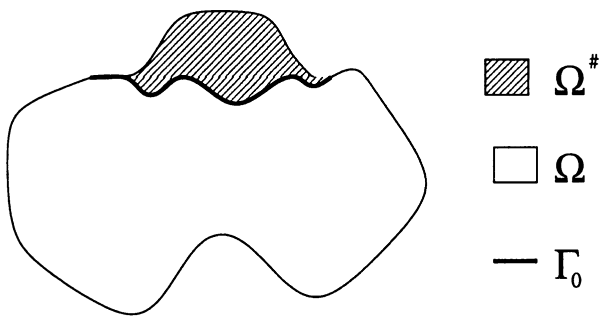

FIG. 2.1: *The enhanced domain.*

(The proof of J.-M. Coron in [7] still holds for dimension 3 with $\Omega^\#$ of [7] defined as in figure 2.1.)

In this paper, we prove this result holds for a point $\overline{x}$ of the boundary as well:

Lemma 2.6 *For any $\overline{x}$ in $\partial\Omega$, the set*

$$\left\{\nabla\theta(\overline{x});\ \theta \in C^\infty(\overline{\Omega};\mathbb{R})\ s.\ t.\ \Delta\theta = 0\ in\ \overline{\Omega}\ and\ \frac{\partial\theta}{\partial n} = 0\ on\ \partial\Omega\backslash\Gamma_0\right\} \tag{2.128}$$

is equal to the tangent plane to $\partial\Omega$ at the point $\overline{x}$, which we will denote by $T_{\overline{x}}(\partial\Omega)$.

We will prove this lemma in section 8.

We then follow [7]. We fix $\overline{x}$ in $\partial\Omega$ (the case when $\overline{x} \in \Omega$ has already been treated in [7]) and $\overline{y} \in \tilde{\Omega}\backslash\overline{\Omega}$. We choose some F in $C^\infty([0,1],\tilde{\Omega})$, such that

$$F(t) = \overline{x},\ \forall t \in [0, \frac{1}{4}],$$

$$F(t) = \overline{y},\ \forall t \in [\frac{3}{4}, 1],$$

$$\frac{dF(t)}{dt} \in T_{F(t)}\partial\Omega \text{ for } F(t) \in \partial\Omega\backslash\Gamma_0.$$

It follows from the previous lemma that one can find $h_1, ..., h_l$ and $\xi^1, ..., \xi^l$, $2l$ functions respectively in $C^\infty([0,1],\mathbb{R})$ and in $C_0^\infty(\tilde{\Omega})$ such that

$$\begin{aligned} \text{Supp } h_i \subset [\frac{1}{4}, \frac{3}{4}],\ \forall i \in \{1, ..., l\}, \\ \Delta\xi^i = 0 \text{ in } \Omega,\ \forall i \in \{1, ..., l\}, \\ \partial_n\xi^i = 0, \text{ on } \partial\Omega\backslash\Gamma_0,\ \forall i \in \{1, ..., l\}, \\ \text{and } \phi^{\nabla\xi}(x, 0, t) = F(t),\ \forall t \in [0,1], \end{aligned}$$

where

$$\xi(x', t) = \sum_{i=1}^{l} h_i(t)\xi^i(x'),\ \forall (x', t) \in \overline{\tilde{\Omega}} \times [0,1].$$

So we found the desired function.

2.7 Proof of Lemma 2.2

In order to prove Lemma 2.2, we will use vortex located on loops which we move around in the domain.

Note that, reducing Γ_0 if necessary, we can suppose it is a ball drawed in $\partial\Omega$ and which does not contain a loop which is non trivial in $\partial\Omega$.

We fix a certain $\mathbf{i}$ in $\{1, \dots, g\}$. For that $\mathbf{i}$, we will construct a certain $\overline{y}^{\mathbf{i}}$, and a certain $\omega_0^{\mathbf{i}}$ such that finally the matrix

$$\mathfrak{M} = \left(\int_\Omega \zeta^i(1).Q_j dx \right)_{i=1\dots g, j=1\dots g} \tag{2.129}$$

satisfies (2.40) where the functions ζ^i are defined by (2.34)-(2.38).

We introduce a hypersurface $\Sigma'_{\mathbf{i}}$ in Ω, equivalent to $\Sigma_{\mathbf{i}}$ (in the sense that there exists a part $\Sigma''_{\mathbf{i}}$ of $\partial\Omega$ such that $\Sigma_{\mathbf{i}} \cup \Sigma'_{\mathbf{i}} \cup \Sigma''_{\mathbf{i}}$ is topologically a sphere) and such that any connected component of its boundary meets Γ_0. This is made possible because Γ_0 meets all the connected components of $\partial\Omega$. In fact, we could have required directly from $\Sigma_{\mathbf{i}}$ to cut Γ_0.

In a first step, let us define a certain smooth vector field on $\mathbb{R}^3$, with compact support in time, which we will denote by $y^{\mathbf{i}}(x,t)$. Then we will define actually $\overline{y}^{\mathbf{i}}$ of the required form, so that it will be close enough to $y^{\mathbf{i}}$.

We consider a Jordan curve J_0 in $\Omega^\#\backslash\overline{\Omega}$, where $\Omega^\#$ is defined as previously (in section 6). More precisely, we set J_0 in a part of $\Omega^\#$ which is in contact with Σ'_i. The vector field $y^{\mathbf{i}}$ will be chosen according to the trajectory of J_0 (or more precisely, of a part of J_0) inside $\overline{\Omega}$.

Along the construction of $y^{\mathbf{i}}$, we will denote by $J(t)$ the Jordan curve obtained as the image of J_0 by the flow of $y^{\mathbf{i}}$ between times 0 and t.

To clarify the required motion of J_0 by the flow of $y^{\mathbf{i}}$, we will represent it in the cross-section $\Sigma'_{\mathbf{i}}$ of Ω described in figure 2.2 (in the case of a simple torus).

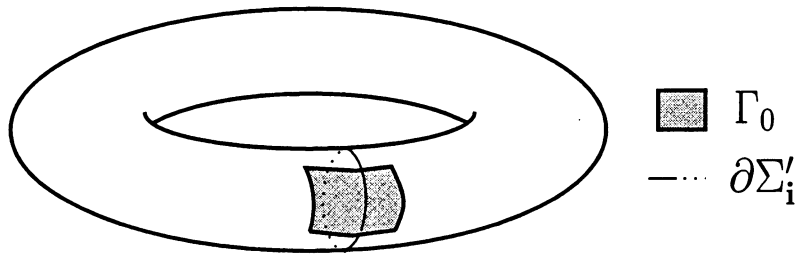

FIG. 2.2: *The cross-section from which the next figures are viewed.*

We divide the time interval $[\frac{1}{4}, \frac{3}{4}]$ in three stages.

In a first step $y^{\mathbf{i}}$ makes J_0 partially enter inside Ω by the "hole" Γ_0 as described in the figure 2.3 and such that $J \cap \Omega$ stays close to $\Sigma'_{\mathbf{i}}$.

At the end of this first stage, $J(t)$ cuts Γ_0 at two points a and b, and we will denote $\tilde{J}$ the part of $J(t)$ inside Ω at this moment.

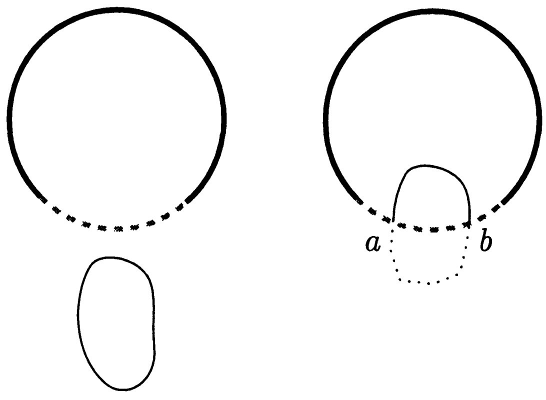

FIG. 2.3: *Entering* Ω.

In the second step, we demand that a stays fixed, and make b describe $\partial\Sigma_{\mathrm{i}}'$ and also that $\tilde{J}$ "nearly" describes Σ_{i}'. We make this process continue until b belongs again to Γ_0 such as described in figure 2.4.

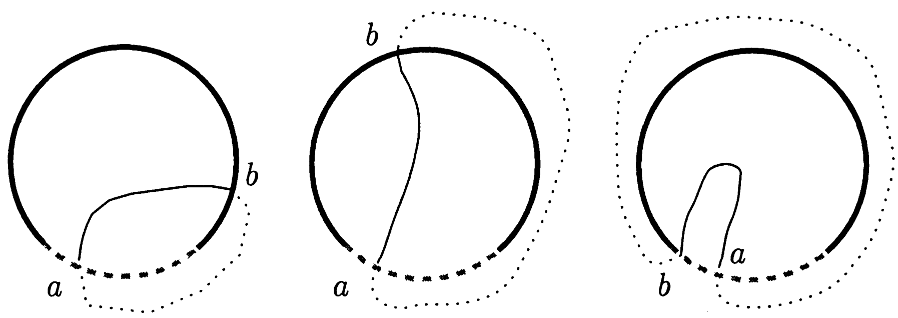

FIG. 2.4: *Describing* Σ_{i}' *(or nearly it).*

The last step of the movement consists in making $\tilde{J}$ leave $\overline{\Omega}$, staying close to Σ_{i}', and taking care that $J(1)$ does no longer cut $\partial\Omega\backslash\Gamma_0$ (so that at the end of the processes, $J(1)$ "describes a loop around" Ω). This step is described in figure 2.5.

In conclusion, we could say that we have chosen y^{i} so that inside Ω, $J(t)$ describes a surface $\tilde{\Sigma}_{\mathrm{i}}$ equivalent to Σ_{i}.

In fact, the process described in figures 2.3-2.5 may be more complicated than it appears in the previous presentation. Indeed, there can be some obstacles "on the way" back of the vortex filament, during the step described by figure 2.5. For example, one can think of a plain torus, inside which one has cut out another plain torus which winds around the "hole" of the first one (that is precisely, such that there is a generator – or a non-trivial element

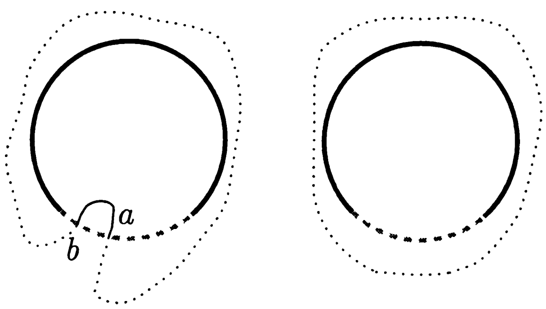

FIG. 2.5: *Leaving* Ω.

– of the fundamental group of the first torus represented in the second one). Then the filament must meet the internal torus during the process described by figures 2.3-2.5. One can also think of the same domain, where one has "glued" the two tori by means of a cylinder. This domain raises the same problem, and moreover has a connected boundary. Let us denote it by $\mathcal{T}$.

It appears that this kind of obstacle can be passed during the process, the same way as the filament passes the torus in figures 2.3-2.5. This is made possible because the control zone encounters any connected component of the boundary.

We give in the following figures 2.6-2.8 the example of a torus $\mathfrak{T}$, in which one has cut out an other plain torus T_1 and a domain of type $\mathcal{T}$, which both wind around the hole of $\mathfrak{T}$ (that is there exists in T_1 and in $\mathcal{T}$ some loops, non-trivial in $\mathfrak{T}$). The obtained domain is represented in color according to the section Σ_i'.

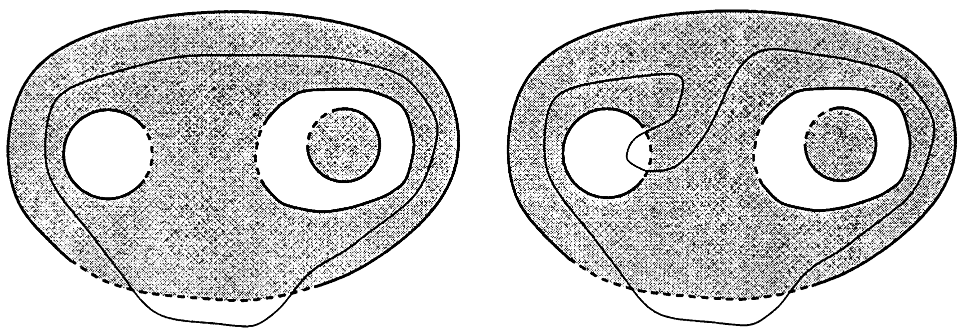

FIG. 2.6: *After the preceedings steps.*

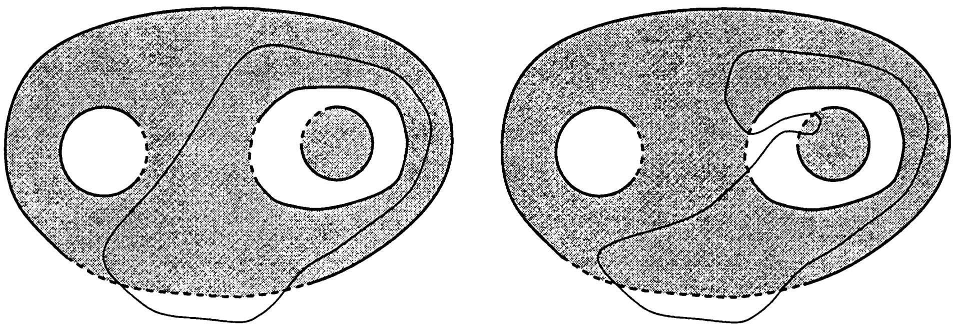

FIG. 2.7: *Once eliminated the first obstacle.*

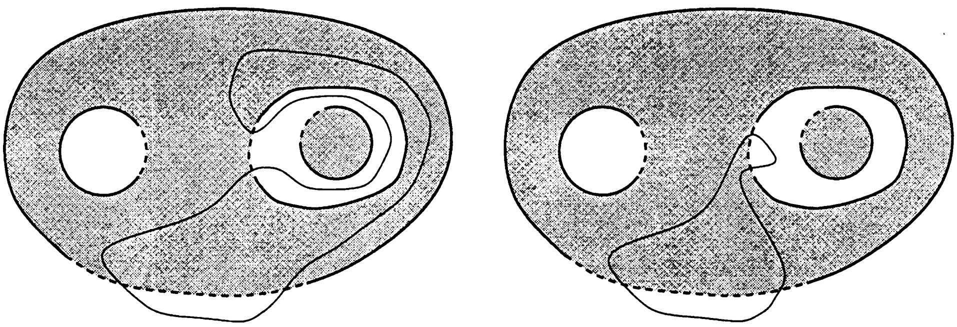

FIG. 2.8: *Eliminating the second obstacle.*

The same way as described in these figures, the vortex can "cross" any obstacle in its way in the general case (at each step, one repeats the process described by figures 2.3-2.5).

Our goal is now to find $\overline{y}^{\mathbf{i}}$ in the form $\nabla\theta(x,t)$ with θ satisfying (2.26)-(2.29) and such that the flow of J_0 along $\overline{y}^{\mathbf{i}}$ is approximately the same as the one along $y^{\mathbf{i}}$.

This is a consequence of the following lemma:

Lemma 2.7 *Given $\tilde{J}$ a curve such as described above, and $\mathbf{v} \in C^\infty(\mathbb{R}^3 \times [0,1], \mathbb{R}^3)$, for all $\epsilon > 0$, there exists $\theta \in C_0^\infty([0,1] \times \mathbb{R}^3, \mathbb{R}^3)$ satisfying (2.26)-(2.29), and such that for any $t \in [0,1]$, one has*

$$|\phi^{\mathbf{v}}(x,0,t) - \phi^{\nabla\theta}(x,0,t)| < \epsilon, \ \forall x \in \tilde{J}. \tag{2.130}$$

Moreover, we have the same result on the whole curve J_0, under the additional assumption that

$$\int_{\phi^{\mathbf{v}}(J_0,0,t)} \mathbf{v}.d\tau = 0, \ \forall t \in [0,1]. \tag{2.131}$$

We delay the proof of this lemma till the end of the paper. For $\mathbf{v} = y^i$, it directly gives us the wanted $\overline{y}^{\mathrm{i}}$, because it suffices then to define $\overline{y}^{\mathrm{i}} : = \nabla\overline{\theta}^{\mathrm{i}}$, where $\overline{\theta}^{\mathrm{i}}$ is given to us by Lemma 2.7. Precisely, we apply the first statement of this lemma for the second stage, and the second one for stages 1 and 3.

Before presenting the exact form of $\aleph^{\mathrm{i}}$, let us make the computation of $\mathrm{curl}\,\zeta^{\mathrm{i}}(1)$ when for $w^{\mathrm{i}}(0)$ we take a vortex filament along the position of J at the time 0. That is, we take a linear repartition of Dirac measure on J_0, say $\mathcal{M}$, and then we set

$$\overline{w}_{\mathrm{i}}(\cdot, 0) = \mathcal{M}\tau, \tag{2.132}$$

where τ is the unit tangent vector on J_0 (which sense does not matter much for the moment). Then we will consider a more regular vortex repartition, which will work all the same.

We consider some f as in the statement of Lemma 2.2. Let us prove that, with that choice of $\overline{w}_{\mathrm{i}}(\cdot, 0)$, a solution of (2.33) (in the distribution sense) is given by

$$\overline{w}_{\mathrm{i}}(\cdot, t) = \mathcal{M}_{\gamma(t)}\tau_{\gamma(t)}, \tag{2.133}$$

where $\gamma(t)$ is defined as the curve obtained by the flow of f on J_0 at time t, $\mathcal{M}_{\gamma(t)}$ and $\tau_{\gamma(t)}$ are respectively the linear Dirac repartition and the unit tangent vector on $\gamma(t)$ (also following the flow of f).

Let us fix a row-vector valued test function $\phi \in C_0^\infty(\tilde{\Omega} \times (0,1), \mathbb{R}^3)$. Then the function $\overline{w}_{\mathrm{i}}$ described in (2.133) is defined by

$$< \overline{w}_{\mathrm{i}}, \phi >= \int_{\gamma(t)} \phi(M) d\vec{M}. \tag{2.134}$$

In other terms, one can write

$$< \overline{w}_{\mathrm{i}}, \phi >= \int_{[0,1]} \phi(\gamma_t(v)){\gamma'}_t(v) dv. \tag{2.135}$$

Let us prove that $\overline{w}_{\mathrm{i}}$ satisfies (2.33). In this part, we denote by a prime the derivative with respect to the variable v and by a point the derivative with respect to the variable t. One has

$$\partial_t < \overline{w}_{\mathrm{i}}, \phi > \;=\; \int_{[0,1]} \partial_t(\phi(\gamma_t(v){\gamma'}_t(v)) dv, \tag{2.136}$$

$$\partial_t < \overline{w}_{\mathrm{i}}, \phi >= \int_{[0,1]} \partial_t(\phi(\gamma_t(v))){\gamma'}_t(v) dv + \int_{[0,1]} \phi(\gamma_t(v))\partial_t({\gamma'}_t(v)) dv. \tag{2.137}$$

We denote by Z_1 the first integral, and by Z_2 the second one. We compute Z_1 the following way (we put j in upper index for the coordinates)

$$Z_1 = \int_{[0,1]} (d\phi)_{\gamma_t(v)} \gamma_t'(v)\dot{\gamma}_t(v) dv.$$

Then one has

$$\begin{aligned}
Z_1 &= \int_{[0,1]} (d\phi)_{\gamma_t(v)} (f(\gamma_t(v)))\gamma'_t(v) dv, \\
&= < \overline{w}_{\mathrm{i}}, d\phi_{(\cdot)}(f) >, \\
&= < \overline{w}_{\mathrm{i}}, \sum_j \frac{\partial \phi}{\partial x_j} f^j >, \\
&= \sum_j < \overline{w}_{\mathrm{i}} f^j, \frac{\partial \phi}{\partial x_j} >, \\
&= -\sum_j < \frac{\partial}{\partial x_j} (\overline{w}_{\mathrm{i}} f^j, \phi >, \\
&= - < \sum_j f^j \frac{\partial}{\partial x_j} \overline{w}_{\mathrm{i}} + \overline{w}_{\mathrm{i}} \sum_j \frac{\partial}{\partial x_j} f^j, \phi >,
\end{aligned}$$

So one finally gets

$$Z_1 = < -(f.\nabla)\overline{w}_{\mathrm{i}} - \overline{w}_{\mathrm{i}} \operatorname{div} f, \phi > . \tag{2.138}$$

We now compute Z_2 (we denote by e_j the j-th vector of the canonical basis of $\mathbb{R}^3$)

$$\begin{aligned}
Z_2 &= \int_{[0,1]} \phi(\gamma_t(v))(df)_{\gamma_t(v)}(\gamma'_t(v)) dv, \\
&= \sum_{j,k} \int_{[0,1]} \phi^k(\gamma_t(v)) \frac{\partial f^k}{\partial x_j} \gamma'^j_t(v) dv, \\
&= \int_{[0,1]} \gamma'_t(v). \sum_{j,k} \phi^k(\gamma_t(v)) \frac{\partial f^k}{\partial x_j} e_j dv, \\
&= < \overline{w}_{\mathrm{i}}, \sum_{j,k} \phi^k \frac{\partial f^k}{\partial x_j} e_j >, \\
&= \sum_j < \overline{w}^j_{\mathrm{i}}, (\sum_k \phi^k \frac{\partial f^k}{\partial x_j}) > \\
&= \sum_{j,k} < \overline{w}^j_{\mathrm{i}} \frac{\partial f^k}{\partial x_j}, \phi^k >, \\
&= < (\overline{w}_{\mathrm{i}}.\nabla) f, \phi > .
\end{aligned} \tag{2.139}$$

We easily deduce from (2.137), (2.138) and (2.139) that

$$\overline{w}_{\mathrm{i}} \text{ satisfies (2.33) in the distribution sense.} \tag{2.140}$$

We come back to the proof of (2.129). For that, we compute, using (2.30),

$$\int_\Omega \zeta^{\mathrm{i}}(1).Q_j dx \;=\; \int_{\Omega\times[0,1]} \partial_t \zeta^{\mathrm{i}}(t).Q_j dxdt.$$

The equations (2.33), (2.34) and (2.38) imply

$$\partial_t \zeta^{\mathrm{i}} + f \wedge \overline{w}_{\mathrm{i}} = \nabla p,$$

from which we deduce

$$\int_\Omega \zeta^{\mathrm{i}}(1).Q_j dx \;=\; -\int_{\Omega\times[0,1]} f \wedge \overline{w}_{\mathrm{i}}.Q_j dxdt.$$

Then, we have

$$\begin{aligned}\int_\Omega \zeta^{\mathrm{i}}(1).Q_j dx \;&=\; -\int_{t\in[0,1]}\int_{\tilde{J}(t)} f \wedge \tau.Q_j d\sigma \\ &=\; -\int_{\tilde{\Sigma}_{\mathrm{i}}} \vec{n}.Q_j d\Sigma,\end{aligned}$$

where n is the unit normal vector on $\tilde{\Sigma}_{\mathrm{i}}$ with an orientation which depends on the sense chosen for ω_0^{i} along J.

Indeed the function $\sigma : [0,1]^2 \to \mathbb{R}^3$, defined by $\sigma(s,t) : \; = \gamma(t)(s)$ describes the surface $\tilde{\Sigma}_{\mathrm{i}}$ with elementary area given by

$$\partial_t J \wedge \partial_s J = \overline{y}^{\mathrm{i}} \wedge \tau. \tag{2.141}$$

As $\tilde{\Sigma}_{\mathrm{i}}$ is equivalent to Σ_{i}, we have finally the following result

$$\int_\Omega \zeta^{\mathrm{i}}(1).Q_j dx \;=\; -\int_{\Sigma_{\mathrm{i}}} \vec{n}.Q_j d\Sigma. \tag{2.142}$$

We deduce from the definition of the functions Q_{i} that

$$\begin{aligned}\int_\Omega Q_{\mathrm{i}}.Q_j dx \;&=\; \int_{\Omega\setminus\Sigma_{\mathrm{i}}} \nabla q_{\mathrm{i}}.\nabla q_j dx, \\ &=\; -\int_{\Sigma_{\mathrm{i}}} (Q_j.n) d\Sigma.\end{aligned}$$

Therefore, we get

$$\int_\Omega \zeta^{\mathrm{i}}(1).Q_j dx \;=\; \int_\Omega Q_{\mathrm{i}}.Q_j dx. \tag{2.143}$$

As by construction of $\overline{y}^{\mathbf{i}}$ and $\overline{w}_{\mathbf{i}}$,

$$\operatorname{curl} \zeta^{\mathbf{i}}(1) = 0,$$

the previous equality implies

$$\zeta^{\mathbf{i}}(1) = \mathcal{Q}_{\mathbf{i}}. \tag{2.144}$$

Now we want to have an equivalent result, but with a more regular vortex repartition. Instead of (2.132), we take here, for $\epsilon < 1$,

$$w^{\mathbf{i}}(\cdot, 0) = \mathcal{M} * \rho_\epsilon \tau, \tag{2.145}$$

where ρ_ϵ is a regularisation kernel with support in the ball with center 0 and radius ϵ in $\mathbb{R}^3$, positive and which moreover satisfies

$$\int_{\mathbb{R}^3} \rho_\epsilon dx = 1. \tag{2.146}$$

Let us take in (2.145) ϵ small enough, in such a way that, if we consider the solution of (2.33), then one has

$$\operatorname{Supp} w^{\mathbf{i}}(\cdot, 1) \subset \tilde{\Omega} \backslash \overline{\Omega}. \tag{2.147}$$

Then by (2.140) we get as a solution of (2.33)

$$w^{\mathbf{i}}(\cdot, t) = \int_{B(0,\epsilon)} \mathcal{M}_{\phi^f(0,t,-x+\gamma)} \tau_{\phi^f(0,t,-x+\gamma)} \rho_\epsilon(x) dx \tag{2.148}$$

In particular, one gets that w is solenoidal. Moreover, using the previous calculus on the linear repartition and (2.146), one deduces that we still get the result that for any $x \in B(0, \epsilon)$

$$\int_{t \in [0,1]} \mathcal{Q}_j . f \wedge \mathcal{M}_{\phi^f(0,t,-x+\gamma)} \tau_{\phi^f(0,t,-x+\gamma)} = - \int_\Omega \mathcal{Q}_i . \mathcal{Q}_j,$$

so after integration in variable x, with (2.146), one gets the same way as previously that

$$\int_{t \in [0,1]} f \wedge w^i = - \int_\Omega \mathcal{Q}_i . \mathcal{Q}_j,$$

which leads all the same to

$$\zeta^{\mathbf{i}}(1) = \mathcal{Q}_{\mathbf{i}},$$

which completes the proof.

2.8 Proof of Lemma 2.6

We argue by contradiction. If this proposition was not true, then for some $\overline{x} \in \partial\Omega$, there would exist a vector $V \in T_{\overline{x}}(\partial\Omega)\backslash\{0\}$, such that for any ϕ in $C^{\infty}(\overline{\Omega};\mathbb{R})$, with

$$\Delta\phi = 0 \text{ in } \overline{\Omega},$$
$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \Gamma_0,$$

stands

$$V.\nabla\phi(\overline{x}) = 0.$$

In particular, for $\Omega^{\#}$ as in figure 2.1, we will consider functions $\phi^{\overline{a},a}$ defined for $a, \overline{a} \in \Omega^{\#}$ by

$$\begin{cases} \Delta\phi^{\overline{a},a} = 4\pi(\delta_a - \delta_{\overline{a}}) \text{ in } \Omega \cup \Omega^{\#}, \\ \partial_n\phi^{\overline{a},a} = 0 \text{ in } \partial(\overline{\Omega \cup \Omega^{\#}}), \\ \int_{\Omega} \phi^{\overline{a},a} = 0. \end{cases} \tag{2.149}$$

Because of the analyticity in $a \in \overline{\Omega \cup \Omega^{\#}}\backslash\partial(\Omega \cup \Omega^{\#})$ of $V.\nabla\phi^{\overline{a},a}(\overline{x})$, the relation

$$V.\nabla\phi^{\overline{a},a}(\overline{x}) = 0, \tag{2.150}$$

which holds for $a \in \Omega^{\#}$, remains true when $a \in \Omega$.

Let $\overline{a}$ be constant in $\Omega^{\#}$, so we will note simply $\phi^a : = \phi^{a,\overline{a}}$. We want to prove that (2.150) is false, by making a approach $\overline{x}$.

Let us denote by $\mathfrak{S}$ the orthogonal symmetry with respect to $T_{\overline{x}}(\partial\Omega)$.

We now consider the functions $\hat{\phi}^a$ defined by

$$\hat{\phi}^a(x) = \frac{1}{|x-a|} + \frac{1}{|x-\mathfrak{S}(a)|} - \frac{1}{|x-\overline{a}|} \text{ in } \mathbb{R}^3. \tag{2.151}$$

This corresponds to the 3D-potential created by 3 particles placed in a, $\mathfrak{S}(a)$ and $\overline{a}$, with respective charge 1, 1 and -1. We also have

$$\Delta\hat{\phi}^a = 4\pi(\delta_a + \delta_{\mathfrak{S}(a)} - \delta_{\overline{a}}) \text{ in } \mathbb{R}^3. \tag{2.152}$$

Let $N(a) : = |a - \overline{x}|^2$. We normalize our functions ϕ^a and $\hat{\phi}^a$ by setting

$$\begin{cases} \psi^a : = N(a)\phi^a/2, \\ \hat{\psi}^a : = N(a)\hat{\phi}^a/2. \end{cases} \tag{2.153}$$

We now want to prove that for a "close" to $\overline{x}$, $\nabla\psi^a(\overline{x})$ is "close"' to $\nabla\hat{\psi}^a(\overline{x})$, and that, if wanted, $\nabla\hat{\psi}^a(\overline{x}).V$ is close to $\|V\|$.

This last point is easy to see. Indeed,

$$\nabla\hat{\psi}^a(x) = N(a)\left(\frac{a-x}{|x-a|^3} + \frac{\mathfrak{S}(a)-x}{|x-\mathfrak{S}(a)|^3} - \frac{\overline{a}-x}{|x-\overline{a}|^3}\right) \text{ in } \mathbb{R}^3, \tag{2.154}$$

so at the point $\overline{x}$ we get, as $a \to \overline{x}$,

$$\nabla\hat{\psi}^a(\overline{x}) = \frac{\overline{x}-P(a)}{|\overline{x}-a|} + o(1), \tag{2.155}$$

where P is the orthogonal projector on $T_{\overline{x}}(\partial\Omega)$. So we can approach by $\nabla\hat{\psi}^a(\overline{x})$ any unit vector of the tangent plane, in particular $\frac{V}{\|V\|}$.

★

In the rest of the proof, a will converge to $\overline{x}$ along straight lines passing through $\overline{x}$ and non tangent to $\partial\Omega$ at point $\overline{x}$, in such a way that

$$d(a, \partial\Omega) \geq c d(a, \overline{x}), \tag{2.156}$$

when $a \to \overline{x}$, for a certain $c > 0$, which depends on the chosen direction.

Now to prove Lemma 2.6, we have left to prove that $\nabla\psi^a(\overline{x})$ is "close"' to $\nabla\hat{\psi}^a(\overline{x})$.

This property relies on the fact that $\partial_n\hat{\psi}^a$ is small for a close to $\overline{x}$ in an Hölder norm $C^{0,\alpha}$ with α small enough (for example $\alpha < \frac{1}{6}$).

Indeed, we clearly have

$$\Delta(\phi^a - \hat{\phi}^a) = 0 \text{ in } \tilde{\Omega}.$$

Hence we can bound $(\phi^a - \hat{\phi}^a)$ the following way, using an elliptic regularity property with $C^{0,\alpha}$ Neumann boundary condition (for which we refer to [20]):

$$\|\nabla\psi^a - \nabla\hat{\psi}^a\|_{C^{0,\alpha}(\Omega)} \leq C\|\partial_n\hat{\psi}^a\|_{C^{0,\alpha}(\partial\Omega)},$$

and consequently

$$\|\nabla\psi^a - \nabla\hat{\psi}^a\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\partial_n\hat{\psi}^a\|_{C^{0,\alpha}(\partial\Omega)}.$$

Let us now prove that

$$\partial_n\hat{\psi}^a \overset{C^{0,\alpha}(\partial\Omega)}{\longrightarrow} 0, \tag{2.157}$$

as $a \to \overline{x}$.

We denote by $u^a(x)$ the unit vector $(x-a)/|x-a|$ and by $\rho^a(x)$ the scalar function $(1/|x-a|^2)$. Hence, we have

$$\partial_n\hat{\phi}^a(x) = (\rho^a(x)u^a(x) + \rho^{\mathfrak{S}(a)}(x)u^{\mathfrak{S}(a)}(x)).n(x) \text{ on } \partial\Omega.$$

From the previous equation, we deduce:

$$\begin{aligned}
\|\partial_n \hat{\phi}^a\|_{0,\alpha,X} &\leq \|\rho^a\|_{0,\alpha,X} \cdot \|(u^a.n)\|_{0,X} \\
&+ \|\rho^a\|_{0,X} \cdot \|(u^a.n)\|_{0,\alpha,X} \\
&+ \|\rho^{\mathfrak{S}(a)}\|_{0,\alpha,X} \cdot \|(u^{\mathfrak{S}(a)}.n)\|_{0,X} \\
&+ \|\rho^{\mathfrak{S}(a)}\|_{0,X} \cdot \|(u^{\mathfrak{S}(a)}.n)\|_{0,\alpha,X}),
\end{aligned} \tag{2.158}$$

for any open part X of the boundary. Equation (2.158) will be useful for points far from $\overline{x}$.

We consequently estimate $\|\rho^a\|_{0,X}$ and $\|\rho^a\|_{0,\alpha,X}$.

It is quite clear that

$$\|\rho^a\|_{0,X} \leq C \frac{1}{d^2(a,X)}.$$

We now estimate $\|\rho^a\|_{0,\alpha,X}$ and $\|\rho^{\mathfrak{S}(a)}\|_{0,\alpha,X}$. It is clear that it is sufficient to estimate the first norm, because the same estimate will hold also for the second one.

We introduce the functions, defined for any $x \in X$ (in fact for any x such that $d(x,a) \geq d(X,a)$) as follows

$$\mathfrak{U}_y^x = \frac{\rho^a(x) - \rho^a(y)}{|x-y|^\alpha}, \quad \forall y \neq x. \tag{2.159}$$

Let $d := d(X,a)$. We now want to estimate $\mathfrak{U}_y^x$. For x with $d(x,a) \geq d(X,a)$, the value

$$\max_{\{y \ / \ |x-y|=h, \ d(y,a) \ \geq d/2\}} \left(\frac{1}{|x-a|^2} - \frac{1}{|y-a|^2}\right)$$

is less or equal than the maximum of the value obtained for

$$y = x - d/2(x-a)/|x-a|$$

(for $h > d/2$), and of the one obtained for

$$y = x - h(x-a)/|x-a|$$

(for $h \leq d/2$). Consequently, we now have left to estimate

$$m_1 = \max_{0<h<d/2} h^{-\alpha}\left(\frac{1}{|x-a|^2} - \frac{1}{|x-h((x-a)/|x-a|)-a|^2}\right),$$

and

$$m_2 = \max_{h>d/2} h^{-\alpha}\left(\frac{1}{|x-a|^2} - \frac{1}{|x-d/2((x-a)/|x-a|)-a|^2}\right).$$

A simple study of these numerical functions shows that these maximum are obtained in the first case for h satisfying the equation

$$-\alpha|1-h|^3+\alpha|1-h|-h=0,$$

which is a value independent from the choice of x in the suitable set. In the second case, one obtains that the maximum is obtained for $h=d/2$.

As a consequence, we can deduce an inequality (with C a constant independent of x)

$$\max_y\left(\frac{1}{|x-a|^2}-\frac{1}{|y-a|^2}\right)/(|x-y|^\alpha)\leq C(d(X,a))^{-2-\alpha}. \tag{2.160}$$

Finally, we can deduce from the previous inequalities that

$$\|\rho^a\|_{0,\alpha,\partial\Omega}\leq C[d(a,\partial\Omega)]^{-2-\alpha}. \tag{2.161}$$

⋆

We consider next $\|u^a\|_{0,X}$ and $\|u^a\|_{0,\alpha,X}$. Of course $\|u^a\|_{0,X}\leq 1$.

We now consider $\|u^a\|_{0,\alpha,X}$. As $u^a(x)$ is constant along the half-lines with origin a, it is obvious that the greatest values of the quotient

$$\frac{|u^a(x)-u^a(y)|}{|x-y|^\alpha},$$

for $x\neq y$ and $\min(d(a,x),d(a,y))\leq d$, are obtained when $d(a,x)=d(a,y)=d$.

By using polar coordinates with center a, we can see also that for some $C>0$ independent from a,

$$\|u^a\|_{0,\alpha,X}\leq C\frac{1}{d(X,a)^\alpha}, \tag{2.162}$$

which is in fact still valid when $\alpha=1$.

⋆

We now deal with $\|\nabla\psi^a.n\|_{0,X}$ and $\|\nabla\psi^a.n\|_{0,\alpha,X}$ directly. The estimate we get here will be used for points near $\overline{x}$. Of course, $\|(\nabla\psi^a.n)\|_{0,X}\leq 1$, and the norm $\|(\nabla\psi^a.n)\|_{0,\alpha,X}$ will get our whole attention.

Here, for $d>0$, we introduce $X^d:=\partial\Omega\cap B(\overline{x},d)$. Let us show the next formula, for a certain $C>0$ independent from a and d:

$$\|\nabla\psi^a.n\|_{0,\alpha,X^d}\leq C\left(d^{1-\alpha}+\frac{d^{3-\alpha}}{|\overline{x}-a|^2}\right). \tag{2.163}$$

For $x,y\in X^d$ with $x\neq y$, we set:

$$I_{x,y}=(\nabla\psi^a(y).n(y)-\nabla\psi^a(x).n(x))|x-y|^{-\alpha},$$

and we have

$$
\begin{aligned}
I_{x,y} &= \nabla\psi^a(x).(n(y)-n(x))|x-y|^{-\alpha} \\
&+ (n(y)-n(\overline{x}))(\nabla\psi^a(y)-\nabla\psi^a(x))|x-y|^{-\alpha} \\
&+ n(\overline{x}).(\nabla\psi^a(y)-\nabla\psi^a(x))|x-y|^{-\alpha}. \qquad (2.164)
\end{aligned}
$$

We evaluate the first term. Clearly,

$$
\begin{aligned}
\left|\nabla\psi^a(x).(n(y)-n(x))|x-y|^{-\alpha}\right| &\leq |\nabla\psi^a(x)|\,\|n\|_{1,\partial\Omega}|x-y|^{1-\alpha} \\
&\leq Cd^{1-\alpha}. \qquad (2.165)
\end{aligned}
$$

We then evaluate the second one. We compute

$$
\begin{aligned}
&\left|(n(y)-n(\overline{x}))(\nabla\psi^a(y)-\nabla\psi^a(x))|x-y|^{-\alpha}\right| \\
&\qquad\qquad \leq C(\Omega)|\overline{x}-y|\|\nabla\psi^a\|_{1,X^d}|x-y|^{1-\alpha}.
\end{aligned}
$$

Let us interest ourselves to the term "$\|\nabla\psi^a\|_{1,X^d}$". From (2.151) and (2.153), one deduces that for some $C>0$, one has

$$\|\nabla\psi^a\|_{1,X^d} \leq \frac{C}{|a-\overline{x}|}.$$

Hence, taking (2.156) in account, we get

$$\left|(n(y)-n(\overline{x}))(\nabla\psi^a(y)-\nabla\psi^a(x))|x-y|^{-\alpha}\right| \leq C(\Omega,c)\frac{d^{2-\alpha}}{|a-\overline{x}|}. \qquad (2.166)$$

We finally evaluate the third term in (2.164). One has

$$\left|n(\overline{x}).\left(\nabla\psi^a(x)-\nabla\psi^a(y)\right)|x-y|^{-\alpha}\right| \leq \|\nabla\psi^a(\cdot).n(\overline{x})\|_{C^1(X^d)}|x-y|^{1-\alpha}$$

We remark that $\nabla\psi^a(x).n(\overline{x})=0$ for any x in $T_{\overline{x}}\partial\Omega$. Then

$$
\begin{aligned}
\|\nabla\psi^a(\cdot).n(\overline{x})\|_{C^1(X^d)} &\leq \|\nabla\psi^a.n(\overline{x})\|_{2,X^d}\ d(X^d,T_{\overline{x}}\partial\Omega) \\
&\leq \|\nabla\psi^a.n(\overline{x})\|_{2,X^d}\ d^2 \\
&\leq C\frac{d^2}{|a-\overline{x}|^2}.
\end{aligned}
$$

So one has

$$\left|n(\overline{x}).\left(\nabla\psi^a(x)-\nabla\psi^a(y)\right)|x-y|^{-\alpha}\right| \leq C\frac{d^{3-\alpha}}{|a-\overline{x}|^2}. \qquad (2.167)$$

Putting together estimates (2.165), (2.166) and (2.167), we get (2.163).

⋆

Let us now prove (2.157). We consider for $\epsilon > 0$

$$\beta : = \frac{2+\epsilon}{3-\alpha}. \tag{2.168}$$

We choose ϵ small enough (for example $\epsilon < \frac{1}{6}$) such that one has

$$2 - \beta(2+\alpha) > 0. \tag{2.169}$$

We distinguish the points of $\partial\Omega$, located inside and outside the open ball $B(a, |a-\overline{x}|^{\beta})$ (which obviously meets $\partial\Omega$ for $N(a) < 1$).

If $x \notin B(a, N(a)^{\frac{\beta}{2}})$, then we have $|x-a| \geq |a-\overline{x}|^{\beta}$ and consequently, with (2.154), (2.158), (2.161) and (2.162), we get

$$\|\partial_n \hat{\psi}^a\|_{0,\alpha,[\partial\Omega\setminus B(a,N(a)^{\frac{\beta}{2}})]} \leq C|\overline{x}-a|^{2-\beta(2+\alpha)}. \tag{2.170}$$

We have to treat the case when $x \in B(a, N(a)^{\frac{\beta}{2}})$. We omit the term concerning $\overline{a}$ in (2.154), which obviously has no importance to estimate $\|\partial_n \hat{\psi}^a\|$. We use (2.163) and get, with (2.168),

$$\|\partial_n \hat{\psi}^a\|_{0,\alpha,[\partial\Omega\cap B(a,N(a)^{\frac{\beta}{2}})]} \leq C(|a-\overline{x}|^{\beta(1-\alpha)} + |a-\overline{x}|^{\epsilon}).$$

Finally, we conclude that $\|\partial_n \hat{\psi}^a\|_{C^{0,\alpha}}$ tends to 0 and that ends the proof.

2.9 Proof of Lemma 2.7

2.9.1 Preliminary lemmas

The main purpose of section 2.9 is to prove the first statement of Lemma 2.7, that is the one relative to $\tilde{J}$. At the end of the section, we will come back to the case when we wish to have a control on the whole curve J_0.

We first establish a lemma which corresponds to the control of paths inside Ω. Precisely, we prove

Lemma 2.8 *Let $\gamma : [0, L] \to \overline{\Omega}$ be a smooth path, injective, with $\gamma((0, L)) \subset \Omega$, and $\gamma(0), \gamma(L) \in \partial\Omega$, and such that for any s in $[0, L]$, $|\dot{\gamma}(s)| = 1$. Suppose also that $\dot{\gamma}(s) \notin T_{\gamma(s)}\partial\Omega$ for $s = 0, L$.*

Let $y \in C^{\infty}(\gamma, \mathbb{R}^3)$ be a vector field defined on γ such that

$$y(\gamma(0)).n(\gamma(0)) = y(\gamma(L)).n(\gamma(L)) = 0. \tag{2.171}$$

Then for any $\epsilon > 0$, there exists $\theta \in C_0^{\infty}(\mathbb{R}^3, \mathbb{R})$ such that the following properties hold:

$$\Delta\theta = 0 \ in \ \Omega, \tag{2.172}$$

$$\partial_n \theta = 0 \ on \ \partial\Omega \backslash \Gamma_0, \tag{2.173}$$

$$\|\nabla\theta - y\|_{C^0(\gamma)} < \epsilon. \tag{2.174}$$

Actually, we need

Corollary 2.1 *Given $\gamma : [0, L] \to \overline{\Omega}$ a smooth path, injective, satisfying $\gamma((0, L)) \subset \Omega$, and $\gamma(0), \gamma(L) \in \partial\Omega$, and given $\epsilon > 0$, there exists $\delta(\gamma, \epsilon) > 0$, and $\mathcal{W}(\gamma, \epsilon) > 0$, such that for any $\tilde{\gamma}$ such that $\|\gamma - \tilde{\gamma}\|_{C^0} < \delta$, for any $\tilde{y}$ in $C^1([0, L])$ satisfying (2.171), one can find θ such that (2.172) and (2.173) hold, and such that*

$$\|\tilde{y} \circ \tilde{\gamma} - \nabla\theta \circ \tilde{\gamma}\|_{C^0([0,L])} < \epsilon \|\tilde{y}\|_{C^0([0,L])}, \tag{2.175}$$

$$\|\theta\|_{C^2(\Omega)} \leq \mathcal{W} \|\tilde{y}\|_{C^1([0,L])}. \tag{2.176}$$

2.9.2 Proof of Lemma 2.8

Let us denote $\Gamma : = \gamma([0, L])$. We first remark that the map which associates to θ satisfying (2.172) and (2.173), the function $(\nabla\theta_{|\Gamma}) \in C^0(\Gamma)$ is linear.

We argue by contradiction and suppose that there exists a non-zero measure $M \in \mathcal{M}(\Gamma, \mathbb{R}^3)$ (viz. the Radon measures defined on Γ), precisely belonging to the set

$$\begin{aligned} \{y \in C^0(\Gamma, \mathbb{R}^3) \ / \ y(\gamma(0)).n(\gamma(0)) = y(\gamma(L)).n(\gamma(L)) = 0\}' \\ = \frac{\mathcal{M}(\Gamma, \mathbb{R}^3)}{\text{Vect}\{\delta_{\gamma(0)}.n(\gamma(0)), \ \delta_{\gamma(1)}.n(\gamma(L))\}}, \end{aligned} \tag{2.177}$$

such that:

$$< M, \nabla\theta >_{\mathcal{M}(\Gamma) \times C^0(\Gamma)} = 0, \tag{2.178}$$

for all $\theta \in C^\infty(\Omega, \mathbb{R})$ satisfying (2.172) and (2.173).

To find a contradiction, we introduce, as for Lemma 2.1, an "over-domain" $\Omega^\#$ described by figure 2.1, and the functions $\phi^{\overline{a},a}$ defined on $\Omega^\#$ by

$$\begin{cases} \Delta\phi^{a,\overline{a}} = 4\pi(\delta_a - \delta_{\overline{a}}) \text{ in } \Omega^\#, \\ \partial_n \phi^{a,\overline{a}} = 0 \text{ on } \partial\Omega^\#, \\ \int_{\Omega^\#} \phi^{a,\overline{a}} = 0. \end{cases}$$

As in Lemma 2.6, relation (2.178), which is true for $\theta = \phi^{a,\overline{a}}$ when a is in $\Omega^\# \backslash \overline{\Omega}$, is still valid when a is in Ω, as a consequence of the analyticity of $< M, \nabla\phi^{\overline{a},a} >_{\mathcal{M}(\Gamma) \times C^0(\Gamma)}$ in variable a.

In the rest of the proof, we will omit $\overline{a}$ whose corresponding term has not importance, and generally, instead of dealing with the function $\phi^{a,\overline{a}}$, we will work on the function

$$x \mapsto \frac{1}{|x-a|},$$

which is close to $\phi^{a,\overline{a}}$, for x close to a in Ω. This can be done because effects of $\overline{a}$ and of the Neumann condition on the boundary are negligeable.

In a first step, we prove that the measure M is necessarily tangent on Γ. Let τ be defined on Γ by $\tau(\gamma(s)) := \frac{d}{ds}\gamma(s)$. We also introduce a C^∞ function $\nu(x)$ of unit normal vectors defined on Γ. We consider a fixed point $x_0 := \gamma(s_0)$ of $\Gamma\backslash\{\gamma(0),\gamma(1)\}$ and for any $\lambda \in \mathbb{R}$ the point $a(x_0,\lambda) := x_0 - \lambda\nu(x_0)$ (placed as described in figure 2.9), which we will denote by $a(\lambda)$ or even a when there is no possible ambiguity. We use as a parameterization of Γ the arc length from x_0, which we denote again by s.

First, let us prove that, for all vectorial continuous function f defined on Γ, one has

$$\lambda \int_\Gamma f(x).\nabla\phi^{a(x_0,\lambda)} d\tau \longrightarrow 2f(x_0).\nu(x_0), \text{ as } \lambda \to 0. \tag{2.179}$$

In that purpose, we introduce a new function η, defined once x_0 is fixed,

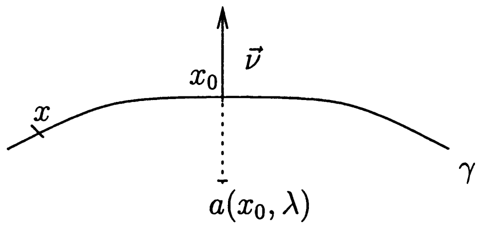

FIG. 2.9: *The position of the particle approaching x_0.*

and depending on the variables s and λ. It is indeed quite easy to see that :

$$|a(\lambda)-\gamma(s)|^2 = |a(\lambda)-x_0|^2 + s^2\eta(s,\lambda), \tag{2.180}$$

where η is a continuous uniformly bounded function defined for s in a certain interval $[-\alpha_0, \alpha_0]$ and for λ in $[0,\hat{\lambda}]$. Moreover, there exists some $m > 0$ such that, for α small enough, one has

$$|\eta(s,a)| \geq m \text{ on } \gamma_\alpha, \text{ for all } \lambda \in [0,\hat{\lambda}]. \tag{2.181}$$

We remark that $\eta(0,\lambda) = 1$ for all $\lambda \in [0,\hat{\lambda}]$.

For $\alpha > 0$, let us denote by Γ_α the arc $\gamma([s_0-\alpha, s_0+\alpha])$.

We fix $\epsilon > 0$. We deduce from the continuity of the function $(s, \lambda) \mapsto f(s).(\gamma(s) - a(x_0, \lambda)$, that, for a certain $\lambda_0 > 0$, and a certain $\alpha > 0$ one has

$$|[(\gamma(s) - a(\lambda)).f(\gamma(s))] - [(x_0 - a(\lambda)).f(x_0)]| < \epsilon, \qquad \forall s \in [-\alpha, \alpha],\ \forall \lambda < \lambda_0. \tag{2.182}$$

In this expression, we remark that $(x_0 - a).f(x_0) = \lambda(a)(f.\nu)(x_0)$.

We introduce the function $\Theta_\lambda : = \lambda\phi^{a(x_0,\lambda)}$, so that we are interested in the limit of

$$\int_\Gamma f(x).\nabla\Theta_\lambda d\tau, \tag{2.183}$$

when $\lambda \to 0$. We cut this integral the following way:

$$\int_\Gamma f.\nabla\Theta_\lambda d\sigma = \int_{\Gamma_\alpha} f.\nabla\Theta_\lambda d\sigma + \int_{\Gamma\backslash\Gamma_\alpha} f.\nabla\Theta_\lambda d\sigma. \tag{2.184}$$

Let us prove the second of these two integrals tends to zero as $\lambda \to 0$. Indeed, we know that on $\Gamma\backslash\Gamma_\alpha$

$$\nabla\phi^{a(\lambda)}(x) \longrightarrow \nabla\phi^{x_0}(x)$$

as $\lambda \to 0$, uniformly in x. So, considering integrals, we get that

$$\int_{\Gamma\backslash\Gamma_\alpha} \nabla\phi^{a(\lambda)} d\sigma \text{ is bounded,}$$

and hence multiplying it by λ, we obtain an integral which converges to 0 as $\lambda \to 0$.

We are now interested in the first integral in the right hand side of (2.184), which we call I_1. First, as $x_0 \notin \partial\Omega$, the difference between the two functions

$$\nabla\phi^{a(\lambda)} \text{ and } x \mapsto \frac{x - a}{|x - a|^3},$$

is bounded as λ tends to 0. So considering integrals and multiplying by λ, we get the existence of $\lambda_1 < \lambda_0$ such that, if we set

$$I_2 : = \int_{\Gamma_\alpha} \frac{\lambda(x - a(\lambda)).f}{|x - a(\lambda)|^3} d\sigma,$$

then one has, for $\lambda < \lambda_1$,

$$|I_1 - I_2| < \epsilon. \tag{2.185}$$

Using (2.180) and (2.182), we get

$$|I_2 - \int_{-\alpha}^{\alpha} \frac{(f.\nu)(x_0)\lambda^2}{(\lambda^2 + s^2\eta(a,s))^{\frac{3}{2}}} ds| \leq \epsilon \int_{-\alpha}^{\alpha} \frac{\lambda^2}{(\lambda^2 + s^2\eta(a,s))^{\frac{3}{2}}} ds, \tag{2.186}$$

and then,

$$|I_2 - \int_{-\alpha}^{\alpha} \frac{(f.\nu)(x_0)}{(1+(\frac{s}{\lambda})^2\eta(a,s))^{\frac{3}{2}}} d(\frac{s}{\lambda})| \leq \epsilon \int_{-\alpha}^{\alpha} \frac{1}{(1+(\frac{s}{\lambda})^2\eta(a,s))^{\frac{3}{2}}} d(\frac{s}{\lambda}). \quad (2.187)$$

Now we use (2.181) and consequently get

$$|I_2 - \int_{-\alpha}^{\alpha} \frac{(f.\nu)(x_0)}{(1+(\frac{s}{\lambda})^2\eta(a,s))^{\frac{3}{2}}} d(\frac{s}{\lambda})| \leq \epsilon \int_{-\infty}^{\infty} \frac{1}{(1+mt^2)^{\frac{3}{2}}} dt. \quad (2.188)$$

We introduce the new notation

$$I_3 : = \int_{-\alpha}^{\alpha} \frac{(f.\nu)(x_0)}{(1+(\frac{s}{\lambda})^2\eta(a,s))^{\frac{3}{2}}} d(\frac{s}{\lambda}).$$

We now want to know the limit of I_3. We compute

$$I_3 = \int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{(f.\nu)(x_0)}{(1+t^2\eta(a,\lambda t))^{\frac{3}{2}}} dt. \quad (2.189)$$

We extend η to $\mathbb{R}$ by making it equal to $\eta(-\alpha_0)$ on $(-\infty, -\alpha_0]$ and to $\eta(\alpha_0)$ on $[\alpha_0, +\infty)$, for any λ. Then, we remark that

$$I_4 : = \int_{-\infty}^{\infty} \frac{(f.\nu)(x_0)dt}{(1+t^2\eta(a,\lambda t))^{\frac{3}{2}}} \leq \int_{-\infty}^{\infty} \frac{|(f.\nu)(x_0)|dt}{(1+mt^2)^{\frac{3}{2}}} < +\infty.$$

As the following inequality stands

$$|I_4 - I_3| \leq \cdot \int_{\mathbb{R}\setminus(-\frac{\alpha}{\lambda},\frac{\alpha}{\lambda})} \frac{(f.\nu)(x_0)dt}{(1+t^2m)^{\frac{3}{2}}}, \quad (2.190)$$

the difference $|I_4 - I_3|$ goes to 0 when $\lambda \to 0$.

We have now to search for the limit of the integral I_4 as $\lambda \to 0$. But given $\epsilon > 0$, using m, one can find a positive number A depending only on η, (in particular, independent from d and from the former choice of α) such that

$$|\int_{-\infty}^{\infty} \frac{(f.\nu)(x_0)}{(1+t^2\eta(a,\lambda t))^{\frac{3}{2}}} dt - \int_{-A}^{A} \frac{(f.\nu)(x_0)}{(1+t^2\eta(a,\lambda t))^{\frac{3}{2}}} dt| < \epsilon, \quad (2.191)$$

$$|\int_{-\infty}^{\infty} \frac{(f.\nu)(x_0)}{(1+t^2)^{\frac{3}{2}}} dt - \int_{-A}^{A} \frac{(f.\nu)(x_0)}{(1+t^2)^{\frac{3}{2}}} dt| < \epsilon, \quad (2.192)$$

for all λ. Then given A, one can find $\lambda_2 \in (0, \lambda_1)$ and $\tilde{\alpha}$ small enough such that

$$|\eta(a,\lambda t) - 1| < \frac{\epsilon}{2A} \text{ for } t \in (-\frac{\tilde{\alpha}}{\lambda}, \frac{\tilde{\alpha}}{\lambda}) \text{ and } \lambda < \lambda_2. \quad (2.193)$$

Then, using (2.188), (2.189), (2.190) and (2.191), we get

$$|\int_{-\infty}^{\infty} \frac{(f.\nu)(x_0)}{(1+t^2\eta(a,\lambda t))^{\frac{3}{2}}}dt - \int_{-\infty}^{\infty} \frac{(f.\nu)(x_0)}{(1+t^2)^{\frac{3}{2}}}dt| < 3\epsilon.$$

It is easy to compute that

$$\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^{\frac{3}{2}}}dt = 2.$$

As the choice of A depends only on η, so does the one of $\tilde{\alpha}$, and we can have chosen in (2.182) the constant α so that $\alpha < \tilde{\alpha}$.

So finally, we have proved that

$$\int_\Gamma \nabla\Theta_\lambda.f d\sigma \longrightarrow 2f(x_0).\nu(x_0), \text{ as } \lambda \to 0,$$

that is, exactly (2.179).

We go back to the proof that $M_{|\Gamma((0,L))}$ is a tangent measure on Γ.

We argue by contradiction and suppose that $M.\nu(x) \neq 0$ on the arc $\gamma([x_0-\alpha, x_0+\alpha])$, for a certain α small to be determined. We still denote by Γ_α the arc $\gamma([x_0-\alpha, x_0+\alpha])$.

We can suppose without loss of generality that $\|M.\nu\|_{\mathcal{M}(\Gamma_\alpha)} = 1$.

Then given $\epsilon > 0$, there exists $f \in C^0(\Gamma_\alpha)$ such that

$$\|f - M.\nu\|_{\mathcal{M}(\Gamma_\alpha)} < \epsilon, \tag{2.194}$$

$$\|f\|_{\mathcal{M}(\Gamma_\alpha)} = 1. \tag{2.195}$$

From (2.178) and (2.194), we get that

$$|\int_\Gamma f(x).\nabla\psi d\tau| < \epsilon\|\nabla\psi\|_{C^0(\Gamma)}, \tag{2.196}$$

for any ψ satisfying (2.172) and (2.173).

Let us describe the form of functions ψ which will allow us to conclude. Given $\nu(x)$, we consider the curve $\tilde{\gamma}$ obtained by associating to x the point situated at a distance λ in the direction $-\nu(x)$ (a kind of "wave front") such as described in figure 2.10. We consider also as a parametrisation of $\tilde{\gamma}$ the arc length $\int$ with origin at a certain $\tilde{x}_0 : = x_0 - \lambda\nu(x_0)$. As previously, the image of $\tilde{\gamma}$ is denoted by $\tilde{\Gamma}$.

We then consider any function $\mathcal{R}$ of $\mathcal{D}([-\alpha,\alpha])$, with α small to be determined, and define ψ as the integral

$$\psi(x) : = \int_{s\in[-\alpha,\alpha]} \mathcal{R}(s)\lambda\phi^{\tilde{\gamma}(s)}(x)ds. \tag{2.197}$$

Note that ψ satisfies (2.178). Let us prove that ψ is bounded on Γ as $\lambda \to 0$.

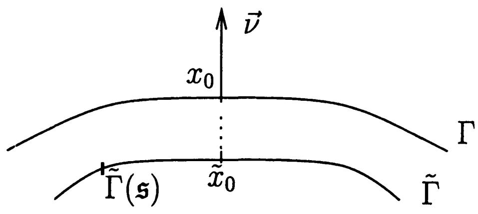

FIG. 2.10: *The curve $\tilde{\Gamma}$ which supports $\Delta\psi$.*

For x_0 in Γ, we have for a certain m which can be found independent (locally at least) from x_0, for all $x \in \Gamma$

$$\begin{aligned}\int_{\tilde{\Gamma}} \frac{\lambda dy}{|y-x|^2} &\leq \int_{-s_0}^{L-s_0} \lambda \frac{ds}{\lambda^2 + ms^2}, \\ &\leq \int_{-\infty}^{+\infty} \frac{d\mathfrak{x}}{1+m\mathfrak{x}^2} \\ &\leq C(\gamma). \end{aligned} \tag{2.198}$$

But as we noticed for (2.185), as $x_0 \notin \partial\Omega$, if we take λ small enough (say, inferior to a certain λ_3), then one has

$$|\int_{\tilde{\Gamma}} |\lambda \nabla \phi^y(x)| dy - \int_{\tilde{\Gamma}} \frac{\lambda dy}{|y-x|^2}| < \epsilon.$$

So actually we get that

$$|\nabla\psi(x)| \leq C(\gamma) \|\mathcal{R}\|_{C_0(\Gamma)}, \tag{2.199}$$

for x near x_0, as $\lambda \to 0$.

Considering (2.196) and (2.199), we get that, at least if $\mathcal{R}$ has a small enough support,

$$| < f, \nabla\psi >_{\mathcal{M}(\Gamma)\times C^0(\Gamma)} | < C(\gamma) \|\mathcal{R}\|_{C_0(\Gamma)} \epsilon. \tag{2.200}$$

With (2.179), (2.199), (2.200) and the Lebesgue convergence theorem, we get that

$$< f.\nu, \mathcal{R} >_{\mathcal{M}(\Gamma)\times C^0(\Gamma)} \leq C(\gamma) \|\mathcal{R}\|_{C_0(\Gamma)} \epsilon. \tag{2.201}$$

Consequently, with (2.194) we get that

$$| < (M.\nu), \mathcal{R} >_{\mathcal{M}(\Gamma)\times C^0(\Gamma)} | < (1 + C(\gamma)) \|\mathcal{R}\|_{C_0(\Gamma)} \epsilon. \tag{2.202}$$

This implies $\|M_{|\Gamma((0,L))}.\nu\|_{\mathcal{M}(\Gamma)} < C'(\gamma)\epsilon$ for every $\epsilon > 0$. So we must have $M_{|\Gamma((0,L))}.\nu = 0$.

Possibly, one could have $M.\nu = \rho_0 \delta_{\gamma(0)}.\nu + \rho_1 \delta_{\gamma(L)}.\nu$. Let us explain why necessarily, $\rho_0 = \rho_1 = 0$. This is a simple consequence of the proof of Lemma 2.6. Indeed, considering as a "θ" in (2.178) the function $|a - \gamma(0)|^2 \phi^{a,\overline{a}}$ and $a \to \gamma(0)$, one could get as a limit value at the point $\gamma(0)$ any tangent vector.

As Γ is transverse to $\partial\Omega$ at the point $\gamma(0)$, this implies that $\rho_0 = 0$. The same can be done for $\gamma(L)$.

Since the previous proof is valid whatever the choice of ν, we deduce that M is tangent everywhere on Γ.

As the second step of the proof of Lemma 2.8, we now check that actually $M = 0$. First, we prove $M_{|\gamma((0,L))} = 0$. As in the previous step, we suppose $M_{|\gamma([x_0-\alpha, x_0+\alpha])} \neq 0$.

We introduce a function $f \in C^1(\Gamma)$ with compact support in Γ_α so that

$$\|f - M\|_{\mathcal{M}(\Gamma_\alpha)} < \tilde{\epsilon}, \tag{2.203}$$

and

$$\|f\|_{\mathcal{M}(\Gamma_\alpha)} \geq \frac{1}{2}, \tag{2.204}$$

and moreover that

$$f \text{ is tangent everywhere on } \Gamma. \tag{2.205}$$

In the rest of the proof, we will make no difference between the vectorial function f and the scalar function, which multiplication by the unit tangent vector along Γ is f.

We now want to prove that $\|f\|_{\mathcal{M}(\Gamma)} < C'(\gamma)\tilde{\epsilon}$, for some constant $C'(\gamma)$. We proceed as previously, and prove that

$$\frac{-1}{\log(\lambda)} \int_\Gamma (\partial_\tau f(x)) \phi^{a(\lambda)} d\tau \longrightarrow \partial_\tau f(x_0), \text{ as } \lambda \to 0, \tag{2.206}$$

where $a(\lambda)$ is defined (as in the first step) in accordance with figure 2.9. As previously, we will use the same arc length s as a parameter on Γ. Let ϵ be a positive number. One can find $\alpha > 0$, such that

$$|\partial_\tau f(x) - \partial_\tau f(x_0)| < \epsilon \text{ on } \Gamma_\alpha. \tag{2.207}$$

Reducing α if necessary, we demand that for all $s \in (-\alpha, \alpha)$ and for all λ less than some λ_0, one has

$$|\eta(s,\lambda) - \eta(0,\lambda)| < \epsilon. \tag{2.208}$$

Let us denote by $\tilde{\Theta}^\lambda$ the function $-\frac{\phi^{a(\lambda)}}{\log \lambda}$. As for (2.184), one can write

$$\int_\Gamma (\partial_\tau f(x)) \tilde{\Theta}^\lambda = \int_{\Gamma_\alpha} (\partial_\tau f(x)) \tilde{\Theta}^\lambda + \int_{\Gamma \backslash \Gamma_\alpha} (\partial_\tau f(x)) \tilde{\Theta}^\lambda. \tag{2.209}$$

For that α, one can find $\lambda_1 \in (0, \lambda_0]$ such that

$$|\int_{\Gamma\backslash\Gamma_\alpha} (\partial_\tau f(x))\tilde{\Theta}^\lambda d\tau| < \epsilon, \tag{2.210}$$

for all $\lambda < \lambda_1$. The existence of λ_1 is a simple consequence of the uniform convergence of the function ϕ^a to 0 on $\Gamma\backslash\Gamma_\alpha$ as $a \to x_0$.

From now, we will impose λ to be strictly inferior to λ_0, λ_1 and 1. Let us then consider the integral of $\tilde{\Theta}^\lambda \partial_\tau f$ along Γ_α, which we will denote by $\tilde{I}_1$. This integral can be estimated the following way:

$$\begin{aligned}
\tilde{I}_1 &= \frac{-1}{\log|\lambda|}\int_{-\alpha}^{\alpha} \frac{\partial_\tau f(x(s))}{\sqrt{\lambda^2 + s^2\eta(s,\lambda)}}ds && (2.211)\\
&= \frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{(\partial_\tau f)(x((t/\lambda))}{\sqrt{1 + t^2\eta(\lambda t,\lambda)}}dt. && (2.212)
\end{aligned}$$

We now want to find the limit of this integral. Note that

$$\begin{aligned}
&\left|\frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x((t/\lambda))}{\sqrt{1 + t^2\eta(\lambda t,\lambda)}}dt - \frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x_0)}{\sqrt{1 + t^2\eta(\lambda t,\lambda)}}dt\right|\\
&\qquad\qquad \le \epsilon\left(2\,lim_{\lambda\to 0}\frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{1}{\sqrt{1+t^2m}}dt\right),
\end{aligned}$$

if λ_1 is small enough. As

$$\lim_{\lambda\to 0}\frac{-1}{\log|\lambda|}\int_{-\frac{K}{\lambda}}^{\frac{K}{\lambda}} \frac{1}{\sqrt{1+t^2m}}dt = \frac{1}{\sqrt{m}},$$

we can deduce the estimate

$$\begin{aligned}
&\left|\frac{-1}{\log|\lambda|}(\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x((t/\lambda))}{\sqrt{1 + t^2\eta(\lambda t,\lambda)}}dt - \int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x_0)}{\sqrt{1 + t^2\eta(\lambda t,\lambda)}}dt)\right|\\
&\qquad\qquad \le 2\epsilon\sqrt{m}^{-1}. && (2.213)
\end{aligned}$$

Let us define

$$\tilde{I}_2 : = -\frac{1}{|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x_0)}{\sqrt{1 + t^2\eta(\lambda t,\lambda)}}dt.$$

Using (2.208), we can deduce:

$$\frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x_0)}{\sqrt{1+(1+\epsilon)t^2}}dt \le I_2 \le \frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}} \frac{\partial_\tau f(x_0)}{\sqrt{1+(1-\epsilon)t^2}}dt,$$

if $\partial_\tau f(x_0) > 0$ and in the other way around otherwise. Consequently,

$$(1-\epsilon)\lim_{\lambda\to 0}\frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}}\frac{\partial_\tau f(x_0)dt}{\sqrt{1+(1+\epsilon)t^2}} \le I_2,$$

and

$$I_2 \le (1+\epsilon)\lim_{\lambda\to 0}\frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}}\frac{\partial_\tau f(x_0)dt}{\sqrt{1+(1-\epsilon)t^2}}, \tag{2.214}$$

which implies

$$(1-3\epsilon)\partial_\tau f(x_0) \le \frac{-1}{\log|\lambda|}\int_{-\frac{\alpha}{\lambda}}^{\frac{\alpha}{\lambda}}\frac{\partial_\tau f(x_0)}{\sqrt{1+t^2\eta(\lambda t,\lambda)}}dt \le (1+3\epsilon)\partial_\tau f(x_0) \tag{2.215}$$

From (2.210), (2.213) and (2.215) we get that for $\lambda < \lambda_1$,

$$|\int_\Gamma \tilde{\Theta}(x)\partial_\tau f(x)dx - \partial_\tau f(x_0)| < (1+(3+2\sqrt{m}^{-1})\|\partial_\tau f\|_{C^0})\epsilon. \tag{2.216}$$

So we have obtained (2.206).

We go back to the proof that $M = 0$. We integrate by parts to obtain the general equality

$$\int_\Gamma f.\nabla\psi = f(\gamma(0))\psi(\gamma(0)) - f(\gamma(L))\psi(\gamma(L)) + \int_\Gamma \psi\partial_\tau f d\tau. \tag{2.217}$$

Considering $|a-\gamma(0)|\phi^a$ as a function ψ, and making a converge to $\gamma(0)$, we obtain easily that

$$|f(\gamma(0))| < \epsilon, \tag{2.218}$$

and the same for $\gamma(L)$ (indeed this family of functions is uniformly bounded by 1 and converges to the characteristic function of $\gamma(0)$ when $a \to \gamma(0)$).

After that we consider a function ψ defined exactly as in the previous step, with a support described by figure 2.10. Again we introduce a smooth function $\mathcal{R}$ with support $[-\alpha,\alpha]$ (α small). Then we define the function ψ by

$$\psi(x) : = \int_{\mathfrak{s}\in[-\alpha,\alpha]} \mathcal{R}(\mathfrak{s})\frac{-1}{\log\lambda}\phi^{\tilde{\gamma}(\mathfrak{s})}(x)d\mathfrak{s}. \tag{2.219}$$

We can prove that ψ is bounded when $d \to 0$, in the same way as is the first step.

We get then by (2.196), (2.206), (2.218) and (2.219), using the same deduction as in the previous step, that

$$\|f\|_{\mathcal{M}(\Gamma)} < C'(\gamma)\epsilon,$$

which is a contradiction with (2.204) for ϵ small enough.

2.9.3 Proof of Corollary 2.1

Before precisely proving corollary 2.1, we state the following intermediate result:

Corollary 2.2 *Given γ as in Lemma 2.8, for any $\epsilon > 0$, there exists some positive constant $W(\gamma, \epsilon)$ such that for any y satisfying the assumptions of Lemma 2.8, which is moreover in $C^1(\Gamma)$, there exists a function θ satisfying (2.172)-(2.173) such that*

$$\|y - \nabla\theta\|_{C^0(\Gamma)} < \epsilon\|y\|_{C^0(\Gamma)}, \tag{2.220}$$

and furthermore the inequality

$$\|\theta\|_{C^2(\overline{\Omega})} \leq W(\gamma, \epsilon)\|y\|_{C^1(\Gamma)}. \tag{2.221}$$

Let us first remark that, if we do not take into account (2.221), there is no difference between corollary 2.2 and Lemma 2.8.

To prove this corollary, we argue by contradiction, and suppose that there exists a certain $\epsilon_0 > 0$, such that for any $n \in \mathbb{N}$, one can find $y_n \in C^1(\Gamma)$, such that for any θ satisfying (2.172)-(2.173) and (2.220) with $y = y^n$ and $\epsilon = \epsilon_0$, then one has

$$\|\theta\|_{C^2(\overline{\Omega})} \geq n\|y^n\|_{C^1(\Gamma)}. \tag{2.222}$$

We can obviously suppose that $\|y^n\|_{C^1(\Gamma)} = 1$, by "homogeneity" of formula (2.220).

By Ascoli's theorem, we can suppose that

$$y^n \overset{C^0(\Gamma)}{\longrightarrow} Y, \text{ as } n \to \infty. \tag{2.223}$$

We use Lemma 2.8 with $y = Y$ and $\epsilon = \frac{1}{2}\epsilon_0$, and hence we get a certain $\overline{\theta}$. Then, by (2.223), for n large enough, one has $\|\nabla\overline{\theta} - y^n\|_{C^0(\Gamma)} < \epsilon_0\|y^n\|_{C^0(\Gamma)}$.

By (2.222), we deduce

$$\|\overline{\theta}\|_{C^2(\overline{\Omega})} \geq n,$$

for all $n \in \mathbb{N}$, which is obviously absurd.

Our goal is now, in order to prove corollary 2.1, to check that one can find in the constant $W(\gamma, \epsilon)$ independently from γ (at least for curves "close" to γ).

Let us indeed consider near γ a different arc $\tilde{\gamma}$ such that $\|\gamma - \tilde{\gamma}\|_{C^0} < \beta$ for a certain $\beta > 0$. We consider a certain $\tilde{y}$ in $C^1(\tilde{\Gamma})$ which satisfies the same assumption as y in Lemma 2.8. Then one can extend $\tilde{y}$ to a C^1 vector field in $\overline{\Omega}$, in such a way that $\|y\|_{C^1(\overline{\Omega})} = \|\tilde{y}\|_{C^1(\tilde{\Gamma})}$.

Now we consider the θ given by corollary 2.2 on γ with a given ϵ, and the function $y_{|\gamma}$.

We want to show this θ solves a similar problem on $\tilde{\gamma}$. For any $s \in [0, L]$, one can compute

$$\begin{aligned}|\nabla\theta(\tilde{\gamma}(s)) - \tilde{y}(\tilde{\gamma}(s))| &\leq |\nabla\theta(\tilde{\gamma}(s)) - \nabla\theta(\gamma(s))| \\ &+ |\nabla\theta(\gamma(s)) - y(\gamma(s))| \\ &+ |y(\gamma(s)) - \tilde{y}(\tilde{\gamma}(s))|. \end{aligned} \tag{2.224}$$

Using the second derivatives of θ, one can deduce

$$|\nabla\theta(\gamma(s)) - \nabla\theta(\tilde{\gamma}(s))| < W(\epsilon)\beta\|y\|_{C^1(\tilde{\Gamma})}. \tag{2.225}$$

By the choice of y, we have

$$|y(\gamma(s)) - \tilde{y}(\tilde{\gamma}(s))| < \|\tilde{y}\|_{C^1(\tilde{\Gamma})}\beta. \tag{2.226}$$

And θ was chosen such that

$$|\nabla\theta(\gamma(s)) - y(\gamma(s))| < \epsilon\|y\|_{C^0(\Gamma)}. \tag{2.227}$$

Consequently, we have finally

$$|\nabla\theta(\tilde{\gamma}(s)) - \tilde{y}(\tilde{\gamma}(s))| < (W(\epsilon)\beta + 2\beta)\|\tilde{y}\|_{C^1(\tilde{\Gamma})} + \epsilon\|y\|_{C^0(\Gamma)}. \tag{2.228}$$

Taking $\beta : = \frac{\epsilon}{W(\epsilon)+2}$, we obtain corollary 2.1.

2.9.4 Back to the proof of Lemma 2.7

Before defining $\overline{\mathbf{v}} : = \nabla\theta$, we would like to make a small transformation on $\mathbf{v}$, which one can perform for any $q \in \mathbb{N}^*$ and any $\tau < \frac{1}{3q}$.

We introduce a function $p : \mathbb{R} \to \mathbb{R}$, such that

$$p = 1 \text{ in } (-\infty, -\frac{3}{2}] \cup [0, +\infty), \tag{2.229}$$

$$p = 0 \text{ in } [-1, -\frac{1}{2}], \tag{2.230}$$

$$\|p\|_{C^0} \leq 1, \quad \|p\|_{C^1} \leq 10. \tag{2.231}$$

We also introduce a function j such that

$$j = 0 \text{ on } (-\infty, 0] \cup [1/2, +\infty), \tag{2.232}$$

$$0 \leq j \leq 10 \text{ in } \mathbb{R}, \tag{2.233}$$

$$\int_{\mathbb{R}} j = 1. \tag{2.234}$$

We consider times $t_i := \frac{i}{q}$, for $i \in \{0, \dots, q\}$. Then we define $\hat{\mathbf{v}}_{\tau,q}$ on the time interval $[t_i, t_{i+1}]$, $i \in \{0, \dots, q-1\}$ by

$$\hat{\mathbf{v}}_{\tau,q}(t,x) = p(\frac{t - t_{i+1}}{\tau})\mathbf{v}(t,x) \text{ in } [t_i, t_{i+1}] \times \overline{\Omega}, \tag{2.235}$$

with τ to be chosen small enough. Note in particular that

$$\|\hat{v}\|_{C^0([0,1],C^1(\overline{\Omega}))} \leq \|v\|_{C^0([0,1],C^1(\overline{\Omega}))}. \tag{2.236}$$

Let us prove that, given q, one has $|\phi^{\mathbf{v}}(0,t,x) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(0,t,x)|$ uniformly small if τ is small. This is done by an induction method.

We consider the evolution of both flows of $\mathbf{v}$ and $\hat{\mathbf{v}}_{\tau,q}$ between times t_i and $t_{i+1} - \frac{3\tau}{2}$. For these times, one has $\hat{\mathbf{v}}_{\tau,q} = \mathbf{v}$, so by Gronwall's lemma we obtain the inequality for all $t \in [t_i, t_{i+1} - \frac{3\tau}{2}]$

$$|\phi^{\mathbf{v}}(x,0,t) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(x,0,t)| \leq |\phi^{\mathbf{v}}(0,t_i,x) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(0,t_i,x)|te^{t\|\mathbf{v}\|_{C^0}}. \tag{2.237}$$

But is is quite clear that for all t in $[t_{i+1} - \frac{3\tau}{2}, t_{i+1}]$, one has

$$\begin{aligned} |\phi^{\mathbf{v}}(x,0,t) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(x,0,t)| \ &\leq \ |\phi^{\mathbf{v}}(0,t_{i+1} - \frac{3\tau}{2},x) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(0,t_{i+1} - \frac{3\tau}{2},x)| \\ &+ \ 2\tau\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}. \end{aligned} \tag{2.238}$$

If we consider the sequence (u_n) defined by

$$\left\{ \begin{array}{l} u_0 = 0, \\ u_{n+1} = u_n e^{\frac{\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}}{q}} + 2\tau\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}, \end{array} \right. \tag{2.239}$$

then we get

$$|\phi^{\mathbf{v}}(x,0,t) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(x,0,t)| \leq u_q, \tag{2.240}$$

for all $t \in [0,1]$.

Let

$$b := e^{\frac{\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}}{q}} \text{ and } c := 2\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}\tau.$$

We have

$$u_n = \frac{c}{1-b} + (\frac{c}{b-1})b^n. \tag{2.241}$$

As $b^q = e^{\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}}$, we obtain that u_q is small when τ is small, that is what we intended to prove. Precisely, if one expects

$$|\phi^{\mathbf{v}}(x,0,t) - \phi^{\hat{\mathbf{v}}_{\tau,q}}(x,0,t)| < \frac{\epsilon}{2}, \tag{2.242}$$

then it is sufficient to take as τ the real number (if k is large)

$$\tau(\epsilon,k) : = \frac{\epsilon}{4k}. \tag{2.243}$$

We go back to the problem of approximation of $\mathbf{v}$ by $\overline{\mathbf{v}}$. Let us denote by $J(t)$ the arc $\phi^{\mathbf{v}}(J,0,t)$. It is parametrized by $s \in [0,1]$. We consider $\epsilon \in (0,1/400]$. For every $t \in [0,1]$, one can deduce by corollary 2.1 a certain $\delta(J(t),\epsilon/2)$ and a certain $\mathcal{W}(J(t),\epsilon/2)$.

Let us remark that, if $\tilde{t}$ is close enough to t, then one can expect the relation $\|J(t) - J(\tilde{t})\|_{C^0([0,1],C^1(\overline{\Omega}))} < \delta(\epsilon/2, J(t))/2$.

So for fixed t, if we increase $\mathcal{W}$ and decrease δ, we can consider them valid for the arcs $\gamma(\tilde{t})$, with $\tilde{t}$ in a small open set around t. Then by compacity of the interval $[0,1]$, one can find $\delta(\epsilon)$ and $\mathcal{W}(\epsilon)$ valid for all γ in the set $\{J(t),\ t \in [0,1]\}$. We furthermore impose

$$\delta < \epsilon/20 \text{ and } \delta\mathcal{W} < \epsilon/2, \tag{2.244}$$

reducing δ if needed.

Then, we fix k in $\mathbb{N}$ such that

$$(20\epsilon\delta + \frac{2\epsilon\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}}{k})e^{\frac{\mathcal{W}\|\mathbf{v}\|_{C^0([0,1],C^1(\overline{\Omega}))}}{k}} < \frac{\delta}{20}. \tag{2.245}$$

We also require from k that for any $t,t' \in [0,1]$ such that $|t-t'| < \frac{1}{k}$, one has

$$|\mathbf{v}(t,\cdot) - \mathbf{v}(t',\cdot)|_{C^0(\overline{\Omega})} < \epsilon\|\mathbf{v}\|_{C^0}, \tag{2.246}$$

which can be obtained by an argument of uniform continuity.

We cut the time interval $[0,1]$ in k parts $[\frac{l}{k}, \frac{l+1}{k}]$, with $l = 0,\ldots,k-1$. From now we denote by t_l the time l/k. For that k, we introduce a τ such that $|(\phi^{\hat{\mathbf{v}}_{k,\tau}} - \phi^{\mathbf{v}})(x,0,t)| < \epsilon$ for any $(t,x) \in [0,1]\times\overline{\Omega}$ (expressed by (2.243)). From now, we will consider $\hat{\mathbf{v}}_{k,\tau}$ instead of $\mathbf{v}$ and particularly we will set $\hat{J}(t) : = \phi^{\hat{\mathbf{v}}_{k,\tau}}(0,t,J_0)$.

We will define $\overline{\mathbf{v}}$ inductively on these time intervals, so that it will be close to $\hat{\mathbf{v}}_{k,\tau}$.

We will denote by $\mathcal{T}(\gamma,y,\epsilon)$ a function θ given by corollary 2.1 with γ, y and ϵ as variables. At each time, we introduce the arc

$$\mathcal{J}(t) : = \phi^{\overline{\mathbf{v}}}(J_0,0,t), \tag{2.247}$$

and we will denote by $G(s,t)$ the function defined on $[0,1]\times[0,1]$ by

$$G(\cdot,t)=\frac{\hat{J}(t)-\mathcal{J}(t)}{\tau}. \tag{2.248}$$

Both functions $\mathcal{J}$ and G are well defined along the construction of $\overline{\mathbf{v}}$.

During the time interval $[0,\frac{1}{k}-\tau]$, we set

$$\overline{\mathbf{v}}(t,x)=p(\frac{t-t_1}{\tau})\mathcal{T}(J_0,\mathbf{v}(0,\cdot),\epsilon/2). \tag{2.249}$$

Then we define $\overline{\mathbf{v}}$ during the second time interval $[\frac{1}{k}-\tau,\frac{1}{k}-\frac{\tau}{2}]$ by

$$\overline{\mathbf{v}}(t,x)=j(\frac{t-t_1+\tau}{\tau})\mathcal{T}(\mathcal{J}(t_1-\tau),G(\cdot,t_1-\tau),\epsilon/2). \tag{2.250}$$

We then define $\overline{\mathbf{v}}$ by induction on the intervals $[\frac{l}{k}-\frac{\tau}{2},\frac{l+1}{k}-\tau]$ by

$$\overline{\mathbf{v}}(t,x) \;=\; p(\frac{t-t_{l+1}}{\tau})\mathcal{T}(\mathcal{J}(t_l),\mathbf{v}(t_l,\cdot),\epsilon/2), \tag{2.251}$$

and during the intervals: $[\frac{l+1}{k}-\tau,\frac{l+1}{k}-\frac{\tau}{2}]$, we set

$$\overline{\mathbf{v}}(t,x) \;=\; j(\frac{t-t_l+\tau}{\tau})\mathcal{T}(\mathcal{J}(t_l-\tau),G(\cdot,t_l-\tau),\epsilon/2). \tag{2.252}$$

When $k=l-1$, we extend this formula till $t=1$.

Note that this can be done, because the assumptions on γ (injectivity in particular) in corollary 2.1 are preserved by the flow of $\mathbf{v}$.

Now we have to prove that the $\overline{\mathbf{v}}$ constructed above has the required properties. We want to estimate the norm $\|\phi^{\overline{\mathbf{v}}}(0,t,J_0)-\phi^{v}(0,t,J_0)\|_{C^0[0,1]}$ (as the constants k and τ are now fixed, we will no longer precise them for $\hat{\mathbf{v}}$).

We compute

$$\frac{d}{dt^+}|\mathcal{J}(t)(s)-\hat{J}(t)(s)|\le|\overline{\mathbf{v}}(\mathcal{J}(t)(s))-\hat{\mathbf{v}}(\hat{J}(t)(s))|. \tag{2.253}$$

We want to prove that for all $t\in[0,1]$, one has

$$|G(\cdot,t)|<\frac{\delta}{\tau}, \tag{2.254}$$

and

$$|G(\cdot,t_l-\frac{\tau}{2})|<\frac{20\epsilon\delta}{\tau}. \tag{2.255}$$

We prove jointly that (2.254) is true on the intervals $[t_l-\frac{\tau}{2},t_{l+1}-\frac{\tau}{2}]$ and (2.255) by induction on l.

For $l = 0$, as we have chosen the time support of $\mathbf{v}$ placed after t_1, we have nothing to prove.

Let us suppose (2.254) true on the interval $[t_{l-1} - \frac{\tau}{2}, t_l - \frac{\tau}{2}]$ and also (2.255) at rank l. Then, we first consider what happens during the time interval $[t_l - \frac{\tau}{2}, t_{l+1} - \tau]$. By (2.253), one has

$$\begin{aligned}\frac{d}{dt^+}|\mathcal{J}(t)(s) - \hat{J}(t)(s)| &\leq |\overline{\mathbf{v}}(\mathcal{J}(t)(s)) - \overline{\mathbf{v}}(\hat{J}(t_l - \frac{\tau}{2})(s))| \\ &+ |\overline{\mathbf{v}}(\hat{J}(t_l - \frac{\tau}{2})(s)) - \hat{\mathbf{v}}(\hat{J}(t_l - \frac{\tau}{2})(s))| \\ &+ |\hat{\mathbf{v}}(\hat{J}(t_l - \frac{\tau}{2})(s)) - \hat{\mathbf{v}}(\hat{J}(t)(s))|. \qquad (2.256)\end{aligned}$$

Consequently, one has

$$\frac{d}{dt^+}|\mathcal{J}(t)(s) - \hat{J}(t)(s)| \leq \mathcal{W}\|\mathbf{v}\||\mathcal{J}(t)(s) - \hat{J}(t)(s)| + \epsilon\|\mathbf{v}\| + \epsilon\|\mathbf{v}\|,$$

where the norm considered is the norm in $C^0([0,1], C^1(\overline{\Omega}))$. This gives, when integrated, for all $t \in [t_l - \frac{\tau}{2}, t_{l+1} - \tau]$,

$$\begin{aligned}|\mathcal{J}(t)(s) - \hat{J}(t)(s)| &\leq |\mathcal{J}(t_l - \frac{\tau}{2})(s) - \hat{J}(t_l - \frac{\tau}{2})(s)| + \frac{2\epsilon\|\mathbf{v}\|}{k} \\ &+ \int_{t_l - \frac{\tau}{2}}^{t} \mathcal{W}\|\mathbf{v}\||\mathcal{J}(t)(s) - \hat{J}(t)(s)|,\end{aligned}$$

which implies by the induction hypothesis and Gronwall's lemma

$$|\mathcal{J}(t)(s) - \hat{J}(t)(s)| \leq (20\epsilon\delta + \frac{2\epsilon\|\mathbf{v}\|}{k})e^{\mathcal{W}\|\mathbf{v}\|t - t_l + \frac{\tau}{2}}.$$

Together with (2.245), this gives us for $t \in [t_l - \frac{\tau}{2}, t_{l+1} - \tau]$

$$|\mathcal{J}(t)(s) - \hat{J}(t)(s)| < \frac{\delta}{20}, \qquad (2.257)$$

which implies a fortiori relation (2.254) on the interval $[t_l - \frac{\tau}{2}, t_{l+1} - \tau]$. Let us prove it on $[t_{l+1} - \frac{3\tau}{2}, t_{l+1} - \frac{\tau}{2}]$.

We make the same computation as in (2.256) and get for t in $[t_{l+1} - \tau, t_{l+1} - \frac{\tau}{2}]$

$$\frac{d}{dt^+}|\mathcal{J}(t)(s) - \hat{J}(t)(s)| \leq \mathcal{W}\|j(t)G(t_l - \frac{\tau}{2})\||\mathcal{J}(t)(s) - \hat{J}(t)(s)| + \epsilon\|G\|.$$

Therefore

$$\begin{aligned}|\mathcal{J}(t)(s) - \hat{J}(t)(s)| &\leq \frac{\delta}{20} + \tau\epsilon\|j(t_l - \frac{\tau}{2})G(t_l - \frac{\tau}{2})\| \\ &+ \int_{t_l - \frac{\tau}{2}}^{t} 10\mathcal{W}\|G\||\mathcal{J}(t)(s) - \hat{J}(t)(s)|,\end{aligned}$$

which implies, by Gronwall's lemma,

$$\begin{aligned} |\mathcal{J}(t)(s) - \hat{J}(t)(s)| &\leq (\frac{\delta}{10})e^{10\mathcal{W}\|G\|(t-t_l+\frac{\tau}{2})}, \\ &\leq \frac{\delta}{10}e^{10W\delta}, \\ &\leq \frac{\delta}{2}, \end{aligned}$$

and hence, (2.254) holds on the interval $[t_{l+1}-\tau, t_{l+1}-\frac{\tau}{2}]$.

Then, in order to prove (2.255) at rank $l+1$, we consider the difference for $t \in [t_{l+1}-\tau, t_{l+1}-\frac{\tau}{2}]$:

$$|\phi^{\overline{\mathbf{v}}}(t_l-\tau, t, \mathcal{J}(t_l-\tau)) - \phi^{G(s,t)}(t_l-\tau, t, \mathcal{J}(t_l-\tau))|$$

But one has

$$\begin{aligned} \|\overline{\mathbf{v}}(\mathcal{J}(t)(s), t) - j(t)G(s,t)\| &\leq 10\epsilon\|G\| + 10\mathcal{W}\|G\||\mathcal{J}(t)(s) - \hat{J}(t)(s)|, \\ &\leq 10(\epsilon + \mathcal{W}\delta)\|G\| \end{aligned}$$

from what we deduce

$$\begin{aligned} &|\phi^{\overline{\mathbf{v}}}(t_l-\tau, t, \mathcal{J}(t_l-\tau)) - \phi^{j(t)G(s,t)}(t_l-\tau, t, \mathcal{J}(t_l-\tau))| \\ &\qquad \leq (\epsilon + \mathcal{W}\delta)10\|G\|\tau, \\ &\qquad \leq 10\delta(\epsilon + \mathcal{W}\delta), \end{aligned} \tag{2.258}$$

which gives us (cf (2.244)), if we take $t = t_{l+1} - \frac{\tau}{2}$, relation (2.255) at rank l.

Note that we could use the estimate of corollary 2.1 because $\mathcal{J}(t)$ was not too far from $\hat{J}(t)$.

Then (2.254) implies

$$|\hat{J}(t) - \mathcal{J}(t)| < \epsilon,$$

for all t in $[0, 1]$.

This concludes the proof of the first statement of Lemma 2.7.

2.9.5 The second statement of Lemma 2.7

In this case the demonstration of an equivalent Lemma 2.8 is the same. First, one shows that the measure M is necessarily tangent on J_0, then that it must be a linear vortex on J_0 exactly in the same way. But this time the conclusion that M is zero does not come from a particular study of points $\gamma(0)$ and $\gamma(1)$, but from the additional assumption (2.131).

Then, the conclusions given by corollary 2.1 and subsection 9.4 are still valid, which completes the proof of Lemma 2.7.

Chapitre 3

Contrôle des fluides bidimensionnels

Résumé

Dans [6], J.-M. Coron a établi un résultat de contrôlabilité approchée du système d'Euler pour les fluides parfaits incompressibles, dans les espaces L^p pour $p < \infty$. Lorsque la partie du bord sur laquelle s'applique le contrôle ne rencontre pas toutes les composantes connexes du bord du domaine, on ne peut pas en général obtenir la contrôlabilité L^∞, car la loi de Kelvin impose un certain nombre d'invariants durant le processus. Dans ce papier, nous prouvons que ces invariants sont les seules objections à la contrôlabilité $W^{1,p}$ pour $p < \infty$. Sous une hypothèse naturelle supplémentaire sur les profils de vitesse à connecter, on peut assurer un résultat de contrôlabilité approchée $W^{2,p}$.

Abstract

In [6], J.-M. Coron established a result of approximate controllability of the 2D Euler system for incompressible inviscid fluids in the L^p spaces for $p < +\infty$. When the controlled part of the boundary does not meet every connected component of the boundary of the domain, one cannot in general extend the result to the L^∞ controllability, because the Kelvin law guarantees some invariants during the process. Here we prove that these invariants are the only objection for the $W^{1,p}$ controllability. Under supplementary natural assumption on the flows we want to connect, we can improve the result to the $W^{2,p}$ approximate controllability.

3.1 Introduction

3.1.1 Statement of the results

Consider Ω an open set in $\mathbb{R}^2$, nonempty, bounded, regular (precisely which boundary is composed of a finite number of C^∞ Jordan curves), and not simply connected. Let Σ be a nonempty open part of its boundary $\partial\Omega$, which does not intersect every connected component of $\partial\Omega$.

The general controllability theorem concerning the 2D Euler system for incompressible inviscid fluids – answering to a problem raised by J.-L. Lions in [25] – was established by par J.-M. Coron in [6]. This result proves that this system is "approximately controllable" on Ω, with respect to the $L^p(\Omega)$ topology, for all p in $[1,+\infty)$.

Precisely, for every $T > 0$ and for all y_0 and y_1 in $C^\infty(\overline{\Omega},\mathbb{R}^2)$ such that

$$\operatorname{div} y_0 = \operatorname{div} y_1 = 0 \text{ in } \Omega, \tag{3.1}$$

$$y_0.\nu = y_1.\nu = 0 \text{ on } \partial\Omega\backslash\Sigma, \tag{3.2}$$

(where we denote by ν the unit exterior normal vector on the boundary), there exists a sequence $(y^n)_{n\in\mathbb{N}}$ of functions in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$, which is composed of solutions of the Euler system for 2D incompressible inviscid fluids, that is:

$$\operatorname{div} y^n(t,x) = 0,\ \forall (t,x)\in[0,T]\times\Omega, \tag{3.3}$$

$$\partial_t y^n(t,x) + (y^n(t,x).\nabla)y^n(t,x) = \nabla P(t,x),\ \forall (t,x)\in[0,T]\times\Omega, \tag{3.4}$$

(for some function P in $C^\infty(\overline{\Omega}\times[0,T])$), which satisfies the condition on the boundary

$$y^n(t,x).\nu(x) = 0,\quad \forall t\in[0,T],\ \forall x\in\partial\Omega\backslash\Sigma, \tag{3.5}$$

and moreover

$$y^n_{|t=0} = y_0 \text{ on } \Omega, \tag{3.6}$$

$$y^n_{|t=T} \longrightarrow y_1 \text{ with respect to the } L^p(\Omega) \text{ norm when } n\to+\infty, \tag{3.7}$$

for all $1\leq p<+\infty$. One may furthermore require that y^n should coincide with y_1 at points situated at a distance superior to $1/n$ from the components of $\partial\Omega$ which do not intersect Σ.

Remark 3.1 *If we were not in the case where Σ does not meet every connected component of the boundary (in particular, this remark is valid when $\partial\Omega$*

is simply connected), but on the contrary in the case where it meets every one of them, the system would be exactly controllable (as shown in [6]). That is, we could substitute to the result (3.7) the following one:

$$y_{|t=T} = y_1 \text{ in } \Omega. \tag{3.8}$$

But in our precise situation, we cannot even obtain a better convergence result than the L^p convergence for all $p < +\infty$. For example, we have a negative result for the L^∞ approximate controllability problem.

Indeed, let us denote by $\Gamma_1, \ldots, \Gamma_k$ the connected components of $\partial\Omega$ which meet Σ, and $\Gamma_{k+1}, \ldots, \Gamma_g$ the ones which do not meet Σ. Let also Γ^b be the union of all connected components of $\partial\Omega$ which do not intersect Σ, i.e. $\cup_{i=k+1}^{g}\Gamma_i$.

Now consider $i \in \{k+1, \ldots, g\}$. Then if one choses y_0 and y_1 such that

$$\int_{\Gamma_i} y_0.d\tau \neq \int_{\Gamma_i} y_1.d\tau,$$

the Kelvin law, which ensures

$$\int_{\Gamma_i} y_0.d\tau = \int_{\Gamma_i} y^n_{|t=T}.d\tau,$$

for a solution of the Euler system satisfying (3.5) (because the loop Γ_i does not change when following the flow of the velocity), makes the L^∞ convergence impossible.

But if we restrict the problem to y_0 and y_1 satisfying

$$\int_{\Gamma_i} y_0.d\tau = \int_{\Gamma_i} y_1.d\tau, \quad \forall i \in \{k+1, \ldots, g\}, \tag{3.9}$$

one can wonder if we can expect a better convergence of the sequence (y^n).

The purpose of this paper is to show that, indeed, if we are in the case described by (3.9), one can find a sequence (y^n), satisfying (3.3), (3.4), (3.5) and (3.6), and whose final value $y^n_{|t=T}$ converges to y_1 in the $W^{1,p}(\Omega)$ sense, for all p in $[1, +\infty)$. Moreover, one can require in addition the above coincidence property.

Let us remark that one cannot expect a really better convergence than this one, because the vorticity of y_0, viz. $\operatorname{curl} y_0$, in the process (3.3)-(3.4), is transported by the flow of y^n. In particular, $\operatorname{curl} y_{0|\Gamma^b}$ is transported inside each connected component of Γ^b. But $\operatorname{curl} y_{1|\Gamma^b}$ may be very different from any function obtained this way.

To have a precise counter-example, one may choose as a domain the annulus $B(0,2)\backslash\overline{B}(0,1)$, and take $\Sigma := \partial B(0,2)$ as a control zone. We choose $y_0 = 0$ and define y_1 the following way: let ψ_1 be defined by

$$\begin{cases} \Delta\psi_1 = 1 \text{ in } \Omega, \\ \psi_1 = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.10}$$

Now consider the function τ_1 defined by

$$\begin{cases} \Delta\tau_1 = 0 \text{ in } B(0,2)\backslash\overline{B}(0,1), \\ \tau_1 = 1 \text{ on } \partial B(0,1), \\ \tau_1 = 0 \text{ on } \partial B(0,2). \end{cases} \tag{3.11}$$

We then set λ such that

$$\lambda \int_\Omega |\nabla\tau_1|^2 = -\int_\Omega \tau_1. \tag{3.12}$$

Then we choose

$$y_1 : = \nabla^\perp\psi_1 + \lambda\nabla^\perp\tau_1. \tag{3.13}$$

One easily checks that y_0 and y_1 satisfy (3.9). (For that, remark that

$$\int_{\Gamma_i} \nabla^\perp\psi.d\vec{\tau} = \int_{\partial\Omega} \tau_i\partial_n\psi dx,$$

and then integrate by parts.) But the $W^{1,\infty}$ controllability does not occur, because $\operatorname{curl} y_0 = 0$ on $\partial B(0,1)$, whereas $\operatorname{curl} y_1 = 1$ on $\partial B(0,1)$.

In consequence we set up the

Theorem 3.1 *Let $T > 0$, and let y_0 and y_1 be two functions in $C^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying (3.1), (3.2) and (3.9). Then there exists a sequence (y^n) of functions in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$ which satisfy (3.3), (3.4), (3.5) and (3.6), and moreover*

$$y^n_{|t=T} \longrightarrow y_1 \text{ in norm } W^{1,p}(\Omega), \tag{3.14}$$

for all p such that $1 \leq p < +\infty$. In addition to that, one can choose y^n in order that it satisfies

$$y^n(T,x) = y_1(x) \text{ for all } x \text{ in } \Omega \text{ such that } \operatorname{dist}(x,\Gamma^b) \geq \frac{1}{n}. \tag{3.15}$$

Now one can wonder if the fact that during the process $\operatorname{curl} y_0$ is transported by the flow of the velocity of the fluid along any component of Γ^b is the only objection – in addition to (3.9) – to the $W^{2,p}$ approximate controllability. This is the aim of our second result. Precisely, we show the following theorem:

Theorem 3.2 *Let $T > 0$, and y_0 and y_1 two functions in $C^\infty(\overline{\Omega})$ satisfying (3.1), (3.2), (3.9), and moreover the condition*

"there exists $g-k$ diffeomorphisms $\mathcal{A}_{k+1},\ldots,\mathcal{A}_g$, $\mathcal{A}_i : \Gamma_i \longrightarrow \Gamma_i$, preserving orientation, such that:

$$\forall i \in \{k+1,\ldots,g\}, \operatorname{curl} y_1 = \operatorname{curl} y_0 \circ \mathcal{A}_i \text{ on } \Gamma_i\text{''}. \tag{3.16}$$

Then there exists a sequence (y^n) *of functions in* $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$ *which satisfy (3.3), (3.4), (3.5) and (3.6), and moreover*

$$y^n_{|t=T} \longrightarrow y_1 \text{ in norm } W^{2,p}(\Omega), \tag{3.17}$$

for all p *such that* $1 \leq p < +\infty$. *In addition to that, one can choose* y^n *in order that it satisfies (3.15).*

Again, one cannot expect any better convergence, particularily the $W^{2,\infty}$ one. To have a counter-example, one can for example consider in the same way the annulus $B(0,2)\backslash\overline{B}(0,1)$ as a domain Ω, and all the same consider a control distributed on $\Sigma: =\partial B(0,2)$. Then take $y_1: =\nabla^\perp\psi_1$ where ψ_1 is defined by (3.10), and $y_0: =\nabla^\perp\psi_0$, where ψ_0 is chosen in order that

$$\left\{\begin{array}{l} \Delta\psi_0 = 2-r, \text{ in } B(0,3/2)\backslash\overline{B}(0,1), \\ \displaystyle\int_\Omega \tau_1\Delta\psi_0 dx = -\int_\Omega \tau_1 dx, \\ \psi_0 = 0 \text{ on } \partial\Omega, \end{array}\right. \tag{3.18}$$

where $r=|x|$ and τ_1 is defined by (3.11).

From the construction, (3.9) and (3.16) are satisfied. But one cannot expect the $W^{2,\infty}$ (and hence the C^2) approximate controllability, because this would imply for any $\epsilon>0$ the existence of an orientation and area-preserving diffeomorphism $\mathcal{A}_\epsilon$ from a neighbourhood of $\partial B(0,1)$ of Ω into another one, say $B(0,1+\tilde{\epsilon})\backslash\overline{B}(0,1)$ such that

$$\|\operatorname{curl} y_0 - \operatorname{curl} y_1\circ\mathcal{A}_\epsilon\|_{C^1(B(0,1+\tilde{\epsilon})\backslash\overline{B}(0,1))} \leq \epsilon. \tag{3.19}$$

Indeed, for any n, there exists $\tilde{\epsilon}>0$ such that no point situated in $B(0,1+\tilde{\epsilon})\backslash\overline{B}(0,1)$ at the end of the flow of $-y^n$ corresponds to a point coming from Γ_0. (Of course, in that case, the vorticity of the point is constant when following the flow.)

But this is clearly impossible for ϵ small for

$$\|(\nabla\operatorname{curl} y_0)(x)\| = 1 \text{ on } \partial B(0,1),$$

whereas

$$\nabla\operatorname{curl} y_1 \equiv 0 \text{ on } \partial B(0,1).$$

3.1.2 Notations

Let us introduce a few notations. We recall that g is the number of connected components of $\partial\Omega$, and k the number of connected components of $\partial\Omega$ which meet Σ. For i between $k+1$ and g, we will consider the curves Γ_ϵ^i obtained by regrouping the points situated at distance ϵ from Γ^i. These curves are regular and do not intersect themselves nor each other if ϵ is small enough, which we will systematically suppose. We will denote by Ω^ϵ the part of Ω which is on the oustide of all the curves Γ_ϵ^i for all i between $k+1$ anf g; precisely

$$\Omega^\epsilon :\ = \left\{x \in \Omega \ / \ \mathrm{dist}(x, \cup_{i=k+1}^g \Gamma^i) \geq \epsilon\right\}. \tag{3.20}$$

We will denote by Ω_ϵ^i the part of Ω situated on the inside of Γ_ϵ^i. Let us finally denote by $\gamma^\epsilon :\ = \cup_{i=k+1}^g \Gamma_i^\epsilon$.

We also introduce a real $R > 0$ big enough in order that $\overline{\Omega}$ is included in the ball in $\mathbb{R}^2$centered in 0 with radius R.

We will also be given a continuous operator π, which extends functions on Ω of regularity $C^\infty(\overline{\Omega})$ to functions on B_R of regularity $C^\infty(\overline{B_R})$, with compact support in B_R.

For $V \in C^\infty(\overline{B_R} \times \mathbb{R}, \mathbb{R}^2)$ such that $V.\tilde{\nu} = 0$ on $\partial B_R \times \mathbb{R}$ (where we denote by $\tilde{\nu}$ the unit normal exterior vector field on ∂B_R), we define the application $\phi^V : \mathbb{R}^2 \times B_R \longrightarrow B_R$ as the flow of the vector field V, that is the application satisfying

$$\frac{\partial \phi}{\partial t_2}(t_1, t_2, x) = V(\phi(t_1, t_2, x), t_2), \quad \phi(x, t_1, t_1) = x, \qquad \forall (t_1, t_2, x) \in \mathbb{R}^2 \times B_R. \tag{3.21}$$

The functions $\tau^i \in C^\infty(\overline{\Omega}; \mathbb{R})$, defined for all $i \in \{1, \ldots, g\}$ by

$$\Delta \tau^i = 0 \text{ in } \Omega, \tag{3.22}$$

$$\tau^i = 1 \text{ on } \Gamma^i, \tag{3.23}$$

$$\tau^i = 0 \text{ on } \partial\Omega \backslash \Gamma^i, \tag{3.24}$$

will also be useful.

Let b be a $C^\infty(\mathbb{R}; [0,1])$ function such that

$$b = 1 \text{ on } (-\infty, 1/2], \ \ b = 0 \text{ on } [3/4; +\infty), \ \ |b'| \leq 5 \text{ on } \mathbb{R}. \tag{3.25}$$

For $x \in \overline{B}_R$ and $d > 0$, we set $b^d(x) :\ = b(\mathrm{dist}\ (x, \overline{\Omega})/d)$

Finally, we will use the following notations: $|\cdot|_{W^{2,p}}$ is the sum of all L^p norms of second derivatives of a function; $|\cdot|_\delta$ is the Hölder norm with index $\delta \in (0,1)$.

3.1.3 The control

Let us remark that in the previous presentation, the control is not explicited, and we study an under-determined system. As a control, one may consider the normal local velocity of the fluid on Σ, and the vorticity of the fluid on the points of Σ which enter the domain Ω, that is on the set

$$\{x \in \Sigma \ / \ y(x).\nu(x) < 0\}.$$

When given these supplementary boundary conditions, the system (3.3)-(3.4) is uniquely determined.

3.1.4 Structure of the paper

In section 3.2, we introduce some tools necessary in the construction.

In section 3.3, we give, for a fixed time-dependent velocity field, the construction of a vector field which will be "reachable" for the linearized equation around W with initial value y_0 (under some assumptions on W).

In section 3.4, we reproduce the construction of [6], in the hope of making the paper clearer. Precisely, we show that this velocity field is actually reachable, if W is close enough to a given solution of the Euler system denoted by $\overline{y}$.

In section 3.5, we prove Theorem 3.1, by deducing a non-linear solution from a sequence of solutions of linear problems, and by showing that the obtained solution solves the $W^{1,p}$ controllability problem.

In section 3.6, we prove Theorem 3.2, by using a construction slightly modified with respect to section 3.3, and also a proposition which allows to reduce the problem to the case when $\operatorname{curl} y_1$ has "the good shape" (by modifying the function $\overline{y}$).

Section 3.7 is devoted to the proof of the proposition of section 3.6.

3.2 Some preliminary results

We will use the following lemma, due to J.-L. Lions (see [25, Théorème 5.1]):

Lemma 3.1 *Consider Ω a nonempty bounded regular open set in $\mathbb{R}^n$, with boundary $S_1 \cup S_2$ (where S_1 and S_2 are two nonempty disjoint open sets of* *the boundary $\partial\Omega$). Let $p > 1$. We consider the mapping from $W^{1-\frac{1}{p},p}(S_2)$ into $W^{1,p}(\Omega)$ defined by*

$$u \mapsto y(u) \text{ such that } \begin{cases} \Delta y(u) = 0 \text{ in } \Omega, \\ y(u) = 0 \text{ on } S_1, \\ y(u) = u \text{ on } S_2. \end{cases} \tag{3.26}$$

Then $\partial_\nu y(u)_{|S_1}$ describes a dense subspace of $W^{-\frac{1}{p},p}(S_1)$ when u describes the space $W^{1-\frac{1}{p},p}(S_2)$.

Proof of Lemma 3.1:

We reproduce the demonstration of [26], which is placed in the "$H^1(S_2)$" framework, instead of the "$W^{1-\frac{1}{p},p}(S_2)$" one as here, in order to make sure that the proof is still valid.

We argue by contradiction, and suppose that there exists a certain non-zero distribution ψ in the dual of $W^{-\frac{1}{p},p}(S_1)$, that is in $W^{1-\frac{1}{q},q}(S_1)$ ($q \in (1,+\infty)$ being defined by $\frac{1}{p}+\frac{1}{q}=1$), such that for all $u \in W^{1-\frac{1}{p},p}(S_2)$, one has:

$$< \psi, \partial_\nu y(u) >_{W^{1-\frac{1}{q},q}(S_1)\times W^{-\frac{1}{p},p}(S_1)} = 0. \tag{3.27}$$

Then one may define the function $\phi \in W^{1,q}(\Omega)$ by

$$\begin{cases} \Delta \phi = 0 \text{ in } \Omega, \\ \phi = \psi \text{ on } S_1, \\ \phi = 0 \text{ on } S_2. \end{cases} \tag{3.28}$$

Then by computing $\int_\Omega \nabla\phi . \nabla y(u)$, one obtains that for all $u \in W^{1-\frac{1}{p},p}(S_2)$

$$< \partial_\nu \phi, y(u) >_{W^{-\frac{1}{q},q}(\partial\Omega)\times W^{1-\frac{1}{p},p}(\partial\Omega)} = \\ < \partial_\nu y(u), \phi >_{W^{-\frac{1}{p},p}(\partial\Omega)\times W^{1-\frac{1}{q},q}(\partial\Omega)} . \tag{3.29}$$

With (3.27), one gets that for all $u \in W^{1-\frac{1}{p},p}(S_2)$, one has

$$< \partial_\nu \phi; y(u) >_{W^{-\frac{1}{q},q}(\partial\Omega)\times W^{1-\frac{1}{p},p}(\partial\Omega)} = 0,$$

which involves, with (3.26), $\partial_\nu \phi = 0$ on S_2, which with (3.28) implies $\phi = 0$ in Ω, and consequently $\psi = 0$, which is contradictory.

We add here two classical results to which we will refer in the next sections . The first one is a particular extension theorem:

Lemma 3.2 *For all $k \in \mathbb{N}$, for all $p \in (1, +\infty)$, there exists a constant C depending only on Ω, k and p such that: for all f in $C^\infty(\overline{\Omega}; \mathbb{R})$, there exists another function $g \in C^\infty(\overline{\Omega};\ R)$ satisfying the two following properties:*

$$\exists \epsilon > 0, \quad f \equiv g \ in\ \cup_{i=k+1}^{g} \overline{\Omega}_i^\epsilon, \tag{3.30}$$

$$\|g\|_{W^{k,p}(\Omega)} \leq C\|f\|_{W^{k-1/p,p}(\cup_{i=k+1}^{g}(\Gamma_i))}. \tag{3.31}$$

The proof is clear and let to the reader: let us just remark, that, as a constant C, one can take for example the best constant in the trace formula $W^{k,p}(\Omega) \to W^{k-1/p,p}(\partial\Omega)$ plus 1.

The second following lemma is a kind of Poincaré inequality. Recall that we denote by Γ^b the union of the connected components of $\partial\Omega$ which do not intersect Σ.

Lemma 3.3 *Given $k \in \mathbb{N}$, there exists some constant $C > 0$ such that for all $\epsilon > 0$, for all $f \in W^{k+1,p}(\Omega \backslash \overline{\Omega^\epsilon})$, one has the following relation*

$$|f|_{W^{k,p}(\Omega\backslash\overline{\Omega^\epsilon})} \leq C(\|f\|_{W^{k-\frac{1}{p},p}(\Gamma^b)} + \epsilon|f|_{W^{k+1,p}(\Omega\backslash\overline{\Omega^\epsilon})}). \tag{3.32}$$

Proof of Lemma 3.3:

This is a classical Poincaré's lemma. To obtain this, one first reduces to the case when f has a null trace on Γ^b by finding $\overline{f} \in W^{k+1,p}(\Omega)$ such that

$$f - \overline{f} = 0 \text{ on } \Gamma^b, \tag{3.33}$$

$$\|\overline{f}\|_{W^{k,p}(\Omega)} \leq C\|f\|_{W^{k-\frac{1}{p},p}(\Gamma^b)}. \tag{3.34}$$

Then, for f with 0 trace on Γ^b, relation (3.32) follows from the L^p Poincaré's inequality for a band with width ϵ, which one can adapt, by means of pasting, to an open set with "width" ϵ, such as $\Omega\backslash\overline{\Omega^\epsilon}$.

3.3 The construction of a particular accessible function

In this section, our goal is mainly to construct, for a given W, a solenoidal vector field which is a target for the controlled linearized system around W.

Let us precise here that by linearized system around W we mean in the whole paper the following system in z ("in vorticity"):

$$\begin{cases} \partial_t \omega + (W.\nabla)\omega = 0 \text{ in } B_R \times [0,T], \\ \operatorname{div} z = 0 \text{ in } \Omega \times [0,T], \\ \operatorname{curl} z = \omega \text{ in } \Omega \times [0,T], \\ z.\nu = 0 \text{ on } \partial\Omega \backslash \Sigma. \end{cases}$$

In the next section, we will actually prove that for proper W, the vector field presented here can be achieved as the final value of a solution of the linearized equation around W. Of course, this vector field is intended to be "close" to y_1 and particularily to satisfy a coincidence property such as (3.15). More precisely, we construct a family of targets y^β indexed by a positive number β, which will satisfy

$$y^\beta = y^1 \text{ in } \Omega^\beta.$$

In this whole part, p is a fixed real number, in $(1+\infty)$. (In fact, we construct a reachable vector field which will give (3.14) for a fixed p; we will later on prove that one can require (3.14) "for all p".)

The construction makes use of a potential solution of the Euler equation, that we call $\overline{y}$. It is not explicited here, but it will be in section 3.4.

The first step consists in constructing a solution "without control". Precisely, we construct a function $y^w \in C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2)$ as a fixed point of the following process.

First, we define the functionnal space in which this fixed point is to be found. For this, we introduce functions r and $\overline{q}$ the following way: we introduce the following function from $\mathbb{R}^{+*}$ into $\mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{cases} r(s) = s + s \log \frac{1}{s} \text{ for } s \in (0,1), \\ r(s) = s \text{ for } s \geq 1, \end{cases} \tag{3.35}$$

and the function $\overline{q}$ from $C^0(\overline{\Omega} \times [0,T])$ in $\mathbb{R} \cup \{+\infty\}$:

$$\overline{q}(y) : = \sup \Big\{ |y(\cdot,t)|_0 + \sup\{ \frac{|y(x_2,t) - y(x_1,t)|}{r(|x_2 - x_1|)} , \\ x_1,\ x_2 \in \overline{\Omega},\ x_1 \neq x_2 \},\ t \in [0,T] \Big\}. \tag{3.36}$$

Now, to any y in

$$S : = \Big\{ y \in C^0(\overline{\Omega} \times [0,T]),\ \overline{q}(y) < +\infty, y.\nu = b(\frac{2t}{T}) y_0.\nu \Big\}, \tag{3.37}$$

one associates $P(y)$ by

$$\left\{\begin{array}{l} \operatorname{div} P(y) = 0 \text{ in } \overline{\Omega} \times [0,T], \\ \operatorname{curl} P(y) = \omega^* \text{ in } \overline{\Omega} \times [0,T], \\ P(y).\nu = b(\frac{2t}{T}) y_0.\nu \text{ on } \partial\Omega \times [0,T], \\ \int_\Omega \left[\partial_t P(y) + (P(y).\nabla)P(y)\right].\nabla^\perp \tau_i = 0, \ \forall i \in \{1,\dots,g\}, \\ \int_\Omega P(y)(\cdot,0).\nabla^\perp \tau_i = \int_\Omega y_0.\nabla^\perp \tau_i, \end{array}\right. \tag{3.38}$$

where ω^* is a function in $C^\infty(\overline{B_R};\mathbb{R})$ defined by

$$\left\{\begin{array}{l} \omega^*(0,\cdot) = \operatorname{curl}(\pi y_0) \text{ in } B_R, \\ \partial_t \omega^* + (\pi(y).\nabla)\omega^* = 0 \text{ in } B_R \times [0,T]. \end{array}\right. \tag{3.39}$$

One can find a unique fixed point of P (this follows from the classical method of [23] and [34], except that here we impose non-homogenous boundary conditions; we refer to these articles for more precisions) which gives us a regular solution of the Euler system. Let us denote this fixed point by y^w. (Note that by "without control" we do not mean that the control described in section 3.1.3 is 0, but that we do not make the decisive control here.)

Let us now introduce the functions ψ^w in $C^\infty(\overline{\Omega} \times [0,T])$ and ψ_1 in $C^\infty(\overline{\Omega})$ the following way

$$\left\{\begin{array}{l} \Delta\psi^w = \operatorname{curl} y^w \text{ in } \Omega \times [0,T], \\ \psi^w = 0 \text{ on } \partial\Omega \times [0,T], \end{array}\right. \tag{3.40}$$

$$\left\{\begin{array}{l} \Delta\psi_0 = \operatorname{curl} y_0 \text{ in } \Omega, \\ \psi_0 = 0 \text{ on } \partial\Omega, \end{array}\right. \tag{3.41}$$

and

$$\left\{\begin{array}{l} \Delta\psi_1 = \operatorname{curl} y_1 \text{ in } \Omega, \\ \psi_1 = 0 \text{ on } \partial\Omega. \end{array}\right. \tag{3.42}$$

We obtain this way the following unique decomposition of y_0 and y_1

$$y_i = \nabla^\perp \psi_i + \nabla\theta_i + \sum_{j=1}^{j=g} l_i^j \nabla^\perp \tau_j, \tag{3.43}$$

for $i \in \{0,1\}$, where θ_i is a function defined up to a constant by

$$\left\{\begin{array}{l} \Delta\theta_i = 0 \text{ in } \Omega, \\ \partial_\nu \theta_i = y_i.\nu \text{ on } \partial\Omega. \end{array}\right. \tag{3.44}$$

We consider $\beta > 0$ a fixed number. For this β, according to Lemma 3.1 (for which we choose $S_1 = \Gamma^b$ and $S_2 = \partial\Omega\backslash\Gamma^b$), there exists a function u defined in $W^{1-\frac{1}{p}}(\cup_{i=1}^k \Gamma_i)$ such that

$$\|\partial_\nu y(u) + \partial_\nu \psi_1 - \partial_\nu \psi^w(T)\|_{W^{-\frac{1}{p},p}(\Gamma^b)} < \beta/2.$$

Regularizing u if needed, one can require that u should satisfy

$$u \in C^\infty(\cup_{i=1}^k \Gamma_i), \tag{3.45}$$

$$\|\partial_\nu y(u) + \partial_\nu \psi_1 - \partial_\nu \psi^w(T)\|_{W^{-\frac{1}{p},p}(\Gamma^b)} < \beta. \tag{3.46}$$

For this u, one applies Lemma 3.2 to $y(u)$. For β chosen small enough (in terms of Ω and p), one obtains a function $H \in C^\infty(\overline{\Omega})$ such that

$$H = y(u) \text{ in } \overline{\Omega}\backslash\Omega^{r(\beta)}, \tag{3.47}$$

(in such a way that relation (3.46) occurs when we replace $y(u)$ by H), and such that

$$|H|_{W^{2,p}(\Omega\backslash\overline{\Omega^\beta})} \leq 2. \tag{3.48}$$

Besides, we may choose $r(\beta)$ in order that it satisfies $r(\beta) < \beta/2$.
Note that by construction, one has

$$\Delta H = 0 \text{ in } \Omega\backslash\overline{\Omega^{r(\beta)}}. \tag{3.49}$$

Now for our considered W, we introduce the regular vector field $\tilde{W}$ defined in $C^\infty(\overline{B_R \times [0,T]}, \mathbb{R}^2)$ by

$$\tilde{W} : = \overline{y} + \pi(W - \overline{y}). \tag{3.50}$$

Note that this implies in particular that

$$\tilde{W}.\tilde{\nu} = 0 \text{ on } \partial B_R \times [0,T].$$

For this fixed $\tilde{W}$, we define ω^* as the function in $C^\infty([0,T] \times \overline{B_R}, \mathbb{R})$ satisfying

$$\begin{cases} \omega^*(\cdot, 0) = \operatorname{curl}(\pi y_0) \text{ in } B_R, \\ \partial_t \omega^* + (\tilde{W}.\nabla)\omega^* = 0 \text{ in } B_R \times [0,T]. \end{cases} \tag{3.51}$$

We deduce from it the function $\psi^* \in C^\infty([0,T] \times \overline{\Omega}, \mathbb{R})$ as follows

$$\begin{cases} \Delta\psi^* = \omega^* \text{ in } \Omega \times [0,T], \\ \psi^* = 0 \text{ on } \partial\Omega \times [0,T]. \end{cases} \tag{3.52}$$

Here, as in [6], the point is to "glue" $\psi^*(T)$ and ψ_1 in order to obtain a $\tilde{\psi}$ in such a way that its second derivatives are not "too big".

We define the following family of functions indexed by $\alpha > 0$ small:

$$\begin{cases} \rho_\alpha = 0 \text{ in } \Omega^\alpha, \\ \rho_\alpha = 1 \text{ in } \Omega \backslash \Omega^{\frac{\alpha}{2}}, \\ \|\rho_\alpha\|_{C^0} = 1, \\ \|\nabla \rho_\alpha\|_{C^0} < K/\alpha, \\ \|\nabla^2 \rho_\alpha\|_{C^0} < K/\alpha^2. \end{cases} \tag{3.53}$$

We then define $\tilde{\psi}$ on Ω by

$$\tilde{\psi} = (1 - \rho_\beta)\psi_1 + \rho_\beta(\psi^*(T) + H). \tag{3.54}$$

But we still have to modify once again this $\tilde{\psi}$.

For $i \in \{k+1, \dots, g\}$ and $\alpha > 0$ small, we introduce the function η_α^i from $\overline{\Omega}$ into $\mathbb{R}$, C^∞-regular, such that

$$\begin{cases} 0 \le \eta_\alpha^i \le 1, \\ |\nabla \eta_\alpha^i| \le \frac{C}{\alpha} \\ |\nabla\nabla \eta_\alpha^i| \le \frac{C}{\alpha^2} \\ \text{Supp } \eta_\alpha^i \subset \Omega \backslash \Omega^\alpha, \\ \text{Supp } (1 - \eta_\alpha^i) \subset \Omega^{\alpha/2}. \end{cases} \tag{3.55}$$

One may consider then the following function for $i \in \{k+1, \dots, g\}$

$$\tilde{\psi}_i = \mu_i \eta_\beta^i (1 - \tau_i), \tag{3.56}$$

where we have ruled $\mu_i \in \mathbb{R}$ in order that

$$\int_\Omega (\Delta \tilde{\psi} - \operatorname{curl} y_1).\tau_j = \mu_i \int_\Omega |\nabla \tau_i|^2. \tag{3.57}$$

(Note that this expression is different from the one of [6]; it is equivalent only because we have the supplementary assumption (3.9).)

The "$\nabla^\perp \psi$" part of the searched accessible function (in a decomposition such as (3.43)) is then given by

$$\hat{\psi} : = \tilde{\psi} + \sum_{i=k+1}^{g} \tilde{\psi}_i. \tag{3.58}$$

Finally, one defines the searched solenoidal vector field by

$$y^\beta = \nabla^\perp \hat{\psi} + \nabla \theta_1 + \sum_{j=1}^{j=g} l_1^j \nabla^\perp \tau_j. \tag{3.59}$$

3.4 The reachability of the presented velocity field

In this section, we recall the construction due to J.-M. Coron. The general idea is that the linearized equation around 0 is not controllable, but nevertheless one can hope to control the one around a particular solution $\overline{y}$ of the Euler system which begins and ends at 0.

First, we describe this $\overline{y}$. Then, we describe a solution of the linearized Euler system around W (which is regular for W close enough to $\overline{y}$). Finally, we show that for W close enough to $\overline{y}$, this solution actually reaches the vector field given by (3.59).

3.4.1 The function $\overline{y}$

The function $\overline{y}$ is chosen as a potential solution of the Euler system (i.e $\overline{y} = \nabla\theta$) with local support in time. In fact we choose (for the moment) two types of "$\nabla\theta$".

The first type of θ is given by the following lemma. To state it, we introduce a function $a \in C^\infty([0,1],[0,1])$, different from 0, and with support in $(0,1)$. We also fix R such that B_R is included in the open ball with center 0 and radius R, that we denote by B_R. Then one has

Lemma 3.4 *([6], Proposition 2.1) For any i in $\{2,\dots,k\}$, there exist $\theta^i \in C_0^\infty(B_R;\mathbb{R})$ and $\omega_0^i \in C^\infty(\overline{B}_R;\mathbb{R})$ such that*

$$\Delta\theta^i = 0 \textit{ in } \overline{\Omega}, \tag{3.60}$$

$$\partial_n\theta^i = 0 \textit{ on } \partial\Omega\backslash[(\Gamma_1\cup\Gamma_i)\cap\Sigma], \tag{3.61}$$

$$\textit{Supp } \omega_0^i \subset B_R\backslash\overline{\Omega}, \tag{3.62}$$

and if we define the function $\overline{\omega}^i : \overline{B_R}\times[0,1]\to\mathbb{R}$ by

$$\begin{cases} \overline{\omega}^i(\cdot,0) = \omega_0^i \textit{ in } \overline{B_R}, \\ \partial_t\overline{\omega}^i(x,t) + [(a(t)\nabla\theta^i(x)).\nabla]\overline{\omega}^i(x,t) = 0 \textit{ in } \overline{B_R}\times[0,1], \end{cases} \tag{3.63}$$

then one has

$$\textit{Supp } \overline{\omega}^i(\cdot,1) \subset B_R\backslash\overline{\Omega}, \tag{3.64}$$

$$\int_{\Gamma_i\times[0,1]} a(t)\partial_n\theta^i(x)\overline{\omega}^i(x,t)dxdt = 1. \tag{3.65}$$

The second type of θ is given by the following lemma

Lemma 3.5 *([6], Proposition 2.2) For any x in $\overline{\Omega}\backslash \cup_{i=k+1}^{g} \Gamma_i$, there exists θ in $C_0^\infty(B_R;\mathbb{R})$ such that*

$$\Delta\theta = 0 \ in \ \overline{\Omega}, \tag{3.66}$$

$$\partial_n \theta = 0 \ on \ \partial\Omega\backslash\Sigma, \tag{3.67}$$

$$\phi^{a\nabla\theta}(x,0,1) \notin \overline{\Omega}. \tag{3.68}$$

Now "the" function $\overline{y}$ is constructed the following way. We consider $\epsilon > 0$. For this ϵ, by Lemma 3.5 and using the compacity of $\overline{\Omega}$, there exists $l \in \mathbb{N}$ such that $l > k$, there exist $l-k$ functions $\theta^{k+1}, \dots, \theta^l$ such that for all x in $\overline{\Omega}$ such that dist $(x, \Gamma^b) > \epsilon$, one has

$$\text{dist } (\phi^{a\nabla\theta^i}(x,0,1), \overline{\Omega}) \geq 2d, \tag{3.69}$$

for a certain $i \in \{k+1, \dots, l\}$, the real number $d > 0$ being fixed, satisfying for $i \in \{2, \dots, k\}$:

$$\overline{\omega}^i(x,1) = \omega_0^i(x) = 0 \text{ if dist } (x, \overline{\Omega}) \leq 2d.$$

Now for a $T > 0$ and $\eta > 0$ (which will be small) one defines

$$t_{i/2} : = T - \eta(l + 1 - \frac{i}{2}) \text{ for } i \text{ in } \{0, \dots, 2(l+1)\}. \tag{3.70}$$

Then one defines $\overline{y} \in C^\infty(\overline{B_R} \times [0,T])$ (slightly differently from [6]) by the following formulas

$$\begin{cases} \overline{y}(x,t) = 0 \text{ for } t \text{ in } [0, t_0], \\ \overline{y}(x,t) = \frac{2}{\eta} a(\frac{2(t-t_{i-1})}{\eta}) \nabla\theta^i(x), \quad \forall i \in \{1, \dots, l\}, \ \forall t \in [t_{i-1}, t_{i-1/2}] \\ \overline{y}(x,t) = -\frac{2}{\eta} a(\frac{2(t_i - t)}{\eta}) \nabla\theta^i(x), \quad \forall i \in \{1, \dots, l\}, \ \forall t \in [t_{i-1/2}, t_i] \\ \overline{y}(x,t) = 0 \text{ for } t \text{ in } [t_l, T]. \end{cases} \tag{3.71}$$

Reducing η if necessary, we can demand that $t_0 > 1/2$ and $\eta < \beta$ in section 3.3. We underline here that $\overline{y}$ depends on two constants ϵ and η.

Note that $\overline{y}$ is actually a solution of the Euler system (3.3)-(3.4), satisfying moreover (3.5).

3.4.2 The construction of a solution of the linearized system around W

Here we describe how the solution of the linearized control problem around W is constructed.

We limit the study to the W satisfying

$$W.\nu = \overline{y}.\nu + b(\frac{2t}{T})y_0.\nu + b(\frac{T-t}{\eta})y_1.\nu. \tag{3.72}$$

For such a W, one defines $\tilde{W}$ by (3.50). To this $\tilde{W}$, one associates $\hat{\psi}$ by section 3.3. Then one defines the function $\hat{\omega}$ by

$$\begin{cases} \partial_t\hat{\omega} + (\tilde{W}.\nabla)\hat{\omega} = 0 \text{ in } \overline{B_R} \times [0,T], \\ \hat{\omega}(\cdot,T) = \pi(\Delta\hat{\psi}) \text{ in } \overline{B_R}. \end{cases} \tag{3.73}$$

We consider the functions ω^i for i in $\{k+1,\dots,l\}$ respectively defined on $\overline{B_R} \times [t_{i-1/2}, t_{i+1/2}]$, given by induction by the formulas

$$\omega^{k+1}(\cdot, t_{k+1/2}) = \omega^*(\cdot, t_{k+1/2}) \text{ on } \overline{B_R}, \tag{3.74}$$

$$\partial_t\omega^i + (\tilde{W}.\nabla)\omega^i = 0 \text{ in } \overline{B_R} \times [t_{i-1/2}, t_{i+1/2}], \tag{3.75}$$

$$\begin{aligned} \omega^i(x, t_{i-1/2}) = {} & b^d(x)\omega^{i-1}(x, t_{i-1/2}) \\ & + (1 - b^d(x))\hat{\omega}(x, t_{i-1/2}), \quad \forall i \in \{k+2,\dots,l\}. \end{aligned} \tag{3.76}$$

(ω^* is defined by (3.51).)

Now we consider the functions ω^i for $i \in \{1,\dots,k\}$ defined respectively in $\overline{B_R} \times [t_{i-1}, t_i]$ by the following formulas

$$\omega^i(\cdot, t_{i-1}) = \omega^*(\cdot, t_{i-1}) + \mu_i\omega_0^i, \tag{3.77}$$

(μ_i is a real number to be precised) and

$$\omega^i(\cdot, t_{i-1/2}) = \omega^*(\cdot, t_{i-1/2}). \tag{3.78}$$

Between times t_{i-1} and $t_{i-1/2}$ and between $t_{i-1/2}$ and t_i, one requires that ω^i should satisfy

$$\partial_t\omega^i + (\tilde{W}.\nabla)\omega^i = 0 \text{ in } \overline{B_R} \times [t_{i-1}, t_{i-1/2}) \text{ and in } \overline{B_R} \times [t_{i-1/2}, t_i). \tag{3.79}$$

Finally the μ_i are defined for $i \in \{1, \dots, k\}$ by the following equation

$$\begin{aligned}\int_{\Gamma_i \times [t_{i-1}, t_{i-1/2}]} (W.\nu)\omega^i = &\int_\Omega (y_0 - y_1).\nabla^\perp \tau_i + (\omega_0 - \hat{\omega}(\cdot, T))\tau_i \\ &- \int_{[0,t_0]\times\Gamma_i} (W.\nu)\omega^* - \sum_{j=k+1}^{l} \int_{[t_{j-1/2}, t_{j+1/2}]\times\Gamma_i} (W.\nu)\omega^j \\ &- \int_{\Gamma_i \times [t_k, t_{k+1/2}]} (y.n)\omega^* - \sum_{j=1}^{k} \int_{\Gamma_i \times [t_{j-1/2}, t_j]} (y.n)\omega^*. \quad (3.80)\end{aligned}$$

(We will show this equation actually has a solution in the next section.)

Then one finally defines

$$\begin{cases} \omega(x,t) = \omega^*(x,t) \text{ in } \overline{B_R} \times ([0,t_0] \cup [t_k, t_{k+1/2}]), \\ \omega(x,t) = \omega^i(x,t) \text{ in } \overline{B_R} \times [t_{i-1}, t_i], \ \forall i \in \{1, \dots, k\}, \\ \omega(x,t) = \omega^i(x,t) \text{ in } \overline{B_R} \times [t_{i-1/2}, t_{i+1/2}], \ \forall i \in \{k+1, \dots, l\}, \\ \partial_t \omega + (\tilde{W}.\nabla)\omega = 0 \text{ in } \overline{B_R} \times [t_{l+1/2}, T]. \end{cases} \quad (3.81)$$

The searched solution of the linear system is $z = F(W)$ defined in $\overline{\Omega} \times [0,T]$ in the following way:

$$\text{div } z = 0 \text{ in } \overline{\Omega} \times [0,T], \quad (3.82)$$

$$\text{curl } z = \omega \text{ in } \overline{\Omega} \times [0,T], \quad (3.83)$$

$$z.\nu = y.\nu \text{ on } \partial\Omega \times [0,T], \quad (3.84)$$

$$\int_\Omega (\partial_t z + (z.\nabla)z).\nabla^\perp \tau_i = 0, \ \forall i \in \{1, \dots, g\}, \quad (3.85)$$

$$\int_\Omega z(\cdot, 0).\nabla^\perp \tau_i = \int_\Omega y_0.\nabla^\perp \tau_i. \quad (3.86)$$

3.4.3 Why the previous solution of the linear system is correctly defined

In this section, we show that F is correctly defined if W satisfies some assumptions.

One defines (recall $\overline{y}$ depends on η)

$$\begin{aligned} S_{M,\eta} : = \Big\{ &W \in C^0(\overline{\Omega} \times [0,T]), \ \overline{q}(W) < +\infty, \ |W - \overline{y}^\eta|_0 < M, \\ &W.\nu = \overline{y}^\eta.\nu + b(\frac{2t}{T})y_0.\nu + b(\frac{T-t}{\eta})y_1.\nu \text{ on } \partial\Omega \times [0,T] \Big\}, \quad (3.87)\end{aligned}$$

where $\overline{q}$ is defined by (3.35) and (3.36).

For M large enough, $S_{M,\eta} \neq \emptyset$ for all η. We fix such a M, and show that ω (and hence F) is correctly defined for $\eta(M)$ small enough and $W \in S_{M,\eta}$.

The problem for the correct definition of ω in $C^\infty(\overline{\Omega} \times [0,T])$ is the definition of the μ_i and the continuity at times $t_{i/2}$ for $i \in \{0,\dots,2k\}$. (Indeed, for other times ω is given by the composition of a regular function by a regular flow.)

We deduce from Gronwall's lemma the following formula

$$|\phi^{\tilde{W}}(x,s,t) - \phi^{\overline{y}}(x,s,t)| \leq \eta[e^{C|t-s|/\eta} - 1]\|\tilde{W} - \overline{y}\|_{C^0(\overline{B_R}\times[0,T])}. \tag{3.88}$$

Consequently, for fixed M, for η small enough, one has

$$\omega^i(\cdot,t_{i-1/2}) = \omega^*(\cdot,t_{i-1/2}) \text{ in } \overline{\Omega}, \text{ for } i \in \{1,\dots,k\}. \tag{3.89}$$

Hence, with the construction of ω^i for $i \in \{1,\dots,l\}$, one gets

$$\omega \in C^\infty(\overline{\Omega} \times [0,T]). \tag{3.90}$$

Let us now specify why the μ_i are well defined. For η small enough, one has for any $W \in S_{M,\eta}$,

$$\Big|\int_{\Gamma_i\times[t_{i-1},t_{i-1/2}]} \tilde{\omega}^i(W.\nu) - 1\Big| \text{ is small}, \tag{3.91}$$

as a consequence of (3.88), where $\tilde{\omega}^i$ is defined for $i \in \{1,\dots,k\}$, on $B_R \times [t_{i-1},t_i]$ by

$$\begin{cases} \tilde{\omega}^i(\cdot,t_{i-1}) = \omega_0^i \text{ in } \overline{B_R}, \\ \partial_t\tilde{\omega}^i + (\tilde{W}.\nabla)\tilde{\omega}^i = 0 \text{ in } B_R \times [t_{i-1},t_{i-1/2}), \\ \tilde{\omega}^i = 0 \text{ in } B_R \times [t_{i-1/2},t_i), \end{cases} \tag{3.92}$$

(in such a way that $\mu_i\tilde{\omega}^i = \omega^i - \omega^*$ in $\mathcal{B}_R \times [t_{i-1},t_i)$). Hence (3.80) is a solvable equation.

3.4.4 The final value of the solution of the linear system

Let us now explain why the function $F(W)$ actually reaches the desired vector field, at least if ϵ and η are chosen small enough in the definition of $\overline{y}$ (this ϵ is chosen as a function of β). For this, we use the following decomposition of $z := F(W)(T)$

$$z = \nabla^\perp\psi_z + \nabla\theta_z + \sum_{j=1}^{j=g}\lambda_z^j\nabla^\perp\tau_j, \tag{3.93}$$

where ψ_z equals 0 on $\partial\Omega$, and where θ_z is a harmonic function on Ω. We want to precise the different terms in this decomposition, and then to compare them with those of y_1 in (3.43). (Let us remark that in this decomposition, all of the three terms are L^2-orthogonal one to another.)

First, for the "gradient" part, one has

$$\nabla\theta_z = \nabla\theta_1, \tag{3.94}$$

as a consequence of the normal velocity imposed on the boundary by (3.84).

Now we are interested in the "$\nabla^\perp\tau_i$" terms. Thanks to (3.85), we can affirm that for all i in $\{1,\ldots,g\}$, one has

$$\sum_{j=1}^{g}\left(\int_\Omega \nabla\tau_i.\nabla\tau_j\right)\frac{d\lambda^j}{dt}+\frac{d}{dt}\int_\Omega \omega\tau_i+\int_{\Gamma_i}(W.n)\omega=0. \tag{3.95}$$

With (3.80) and (3.81), this leads to the fact that for all i in $\{1,\ldots,k\}$ one has

$$\sum_{j=1}^{g}\left(\int_\Omega \nabla\tau_i.\nabla\tau_j\right)(\lambda_z^j-l_1^j)=0. \tag{3.96}$$

For $i\in\{k+1,\ldots,g\}$, equation (3.95) becomes

$$\sum_{j=1}^{g}\left(\int_\Omega \nabla\tau_i.\nabla\tau_j\right)\lambda^j+\int_\Omega \omega\tau_i \text{ is constant for } t\in[0,T]. \tag{3.97}$$

But on the other hand, by the further assumption (3.9), and by (3.43) one deduces that

$$\int_{\partial\Omega}\tau_i(\nabla^\perp\psi_1+\sum_{j=1}^{g}l_i^1\nabla^\perp\tau_j).d\tau=\int_{\partial\Omega}\tau_i(\nabla^\perp\psi_0+\sum_{j=1}^{g}l_i^0\nabla^\perp\tau_j).d\tau, \tag{3.98}$$

which leads to

$$\sum_{j=1}^{g}\left(\int_\Omega \nabla\tau_i.\nabla\tau_j\right)(l_1^j-l_0^j)=\int_\Omega \tau_i(\operatorname{curl}y_1-\operatorname{curl}y_0)+\int_{\partial\Omega}\partial_\nu(\psi_1-\partial_\nu\psi_0).$$

With (3.97), we deduce that for all $i\in\{k+1,\ldots,g\}$, one has

$$\sum_{j=1}^{j=g}(\int_\Omega \nabla\tau_i.\nabla\tau_j)(\lambda_j^z(T)-l_j^1)dx=\int_{\Gamma_i}(\partial_\nu\hat{\psi}-\partial_\nu\psi_1)d\tau. \tag{3.99}$$

Now (3.56) and (3.58) imply

$$\int_{\Gamma_i}\partial_\nu\hat{\psi}=\int_{\Gamma_i}\partial_\nu\tilde{\psi}-\mu_i\partial_n\tau_i.$$

Hence, using (3.57), one gets

$$\int_{\Gamma_i} \partial_\nu \hat{\psi} = \int_{\Gamma_i} \partial_\nu \psi_1. \tag{3.100}$$

Transferring this into (3.99), one gets

$$\sum_{j=1}^{j=g} \left(\int_\Omega \nabla \tau_i . \nabla \tau_j \right) (\lambda_j^z(T) - l_j^1) = 0. \tag{3.101}$$

for i in $\{k+1, \ldots, g\}$. We deduce together with (3.96) that

$$\lambda_z^j = l_1^j \text{ for all } j \in \{1, \ldots, g\}. \tag{3.102}$$

Now we show that the "$\nabla^\perp \psi_z$" part equals in fact $\nabla^\perp \hat{\psi}$, or equivalently (both ψ_z and $\hat{\psi}$satisfy the 0 Dirichlet boundary condition) that

$$\operatorname{curl} z = \Delta \hat{\psi} \text{ in } \Omega. \tag{3.103}$$

To obtain this result, we need to have chosen ϵ small enough, precisely here we take $\epsilon := r(\beta)$.

Then we distinguish points situated a distance superior or inferior of ϵ to Γ^b.

For points situated at distance at least ϵ from Γ^b, there is a number i for which one has dist $(\phi^{\bar{W}}(x, 0, t_i), \overline{\Omega}) \geq d$ (if one chooses η small enough). At time t_i, formula (3.76) gives "the good value" to the vorticity of the point (thanks to formula (3.73)). Then this value does not change any more, even if the point is sent again out of $\overline{\Omega}$, thanks to (3.76).

For points at distance at most ϵ, (3.103) is a consequence of the form of $\Delta \hat{\psi}$ (see in particular (3.53) to (3.58)): the vorticity allocated to these points was "the good one" from the beginning.

3.5 Proof of Theorem 3.1

3.5.1 Preliminary step

The main step of the proof will be to establish the following proposition:

Proposition 3.1 *Let $T > 0$ and y_0, y_1 be two functions of $C^\infty(\overline{\Omega}; \mathbb{R}^2)$ satisfying (3.1), (3.2) and (3.9). Then for every p in $[1, +\infty)$, there exists a sequence (y^n) of functions in $C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ which satisfy (3.3), (3.4), (3.5) and (3.6), and moreover*

$$y^n_{|t=T} \longrightarrow y_1 \text{ in norm } W^{1,q}(\Omega), \tag{3.104}$$

for all $q < p$. One can moreover make (y^n) satisfy (3.15).

Theorem 3.1 follows easily from Proposition 3.1 (using a diagonal extraction argument); the proof is hence left to the reader.

3.5.2 Passing to a non-linear solution

The goal of this section is to make sure that one can obtain, given a $\beta > 0$, a fixed point of the operator F, and to ensure that this answers the problem of approximate controllability for the $W^{1,q}$ topology, for any $1 < q < p$ (for proper ϵ and η computed in function of β).

One may observe that during the construction of section 3.3, if one defines

$$\begin{cases} \Delta\psi = \omega \text{ in } \Omega \times [0,T], \\ \psi_{|\partial\Omega} = 0, \end{cases} \tag{3.105}$$

then one obtains

$$\|\Delta\psi\|_0 \leq C,$$

for any W, because ω "made" from bounded functions (as the vorticity is transported by the flow of the velocity). Using techniques due to Wolibner in the reference [34], and following Kato (see [23]), one may deduce from this that for all W obtained as the image by F of a certain function, we get

$$|\operatorname{curl} F(W)|_\delta \leq C, \tag{3.106}$$

for a certain $\delta > 0$.

That involves that $\|F(W)\|$ is bounded C^δ in time, and $C^{1+\delta}$ in space. This involves compacity in $S_{M,\eta}$. Finally, with the fact that for proper η, $S_{M,\eta}$ is sent into itself (as a consequence of (3.88)), we get, using the Leray-Schauder Theorem, the existence of a fixed point to F, which is moreover in the class C^∞. For further precisions, we refer to the article [6, Part 4].

Proceeding this way, we obtain a solution of the Euler system y^β, such that $y^\beta_{|t=T}$ is the vector field described by (3.59), ψ being the function $\hat{\psi}^\beta$ of the previous section computed for $W = y^\beta$.

3.5.3 The $W^{1,q}$ convergence

We have left to prove that this implies

$$\|y^\beta - y^1\|_{W^{1,q}(\Omega)} \longrightarrow 0 \text{ when } \beta \to 0, \tag{3.107}$$

for any $q < p$ and moreover (3.15) for $1/n = \beta$.

First, we prove that, β being fixed, one has

$$\|\psi^*(T) - \psi^w(T)\|_{2,p} \leq C\eta. \tag{3.108}$$

Let us first remark that it is a consequence of the construction that

$$\psi^w(t_0) = \psi^*(t_0).$$

(We consider non-linear solutions here.) So (3.108) is a consequence of the form of $\overline{y}$ (note particularly that $\psi^{\nabla\overline{\theta}}(\cdot, t_0, T) = Id$) and of (3.88).

We now show that $\hat{\psi}^\beta$ converges to ψ_1 for the $W^{2,q}$ topology (for $q < p$). As these functions have a zero trace on the boundary, it suffices to show the convergence of their laplacian in the L^q sense, as a consequence of the classical elliptic estimate (in fact we will bound it in the L^p norm).

Now

$$\Delta(\hat{\psi}^\beta - \psi_1) = \Delta[\rho_\beta(\psi^*(T) + H - \psi_1)] + \sum_{i=k+1}^{g} \mu_i \Delta\tilde{\psi}_i. \tag{3.109}$$

Of course, this is null on Ω^β, and only what happens on $\Omega \backslash \Omega^\beta$ does interest us.

There are three terms to estimate in the development of the first term in the previous expression. We are going to bound them in L^p. These three terms are the following:

- $\rho_\beta \Delta(\psi^*(T) + H - \psi_1) = \rho_\beta(\Delta H + \omega^*(T) - \operatorname{curl} y_1)$ which is bounded in L^p when $\beta \to 0$ (because of (3.48)).
- $2\nabla\rho_\beta\nabla(\psi^*(T) + H - \psi_1)$. The first factor has a norm L^∞ which is a term of order $1/\beta$. The other factor has a norm L^p of order β, as a consequence of Lemma 3.3, (3.46) and (3.48).
- $\Delta\rho_\beta(\psi^*(T) + H - \psi_1)$. The first factor has a L^∞ norm of order $1/\beta^2$. The other factor has an L^p norm of order β^2, as a consequence all the same of Lemma 3.3, (3.46) and (3.48).

Now, let us deal with the second term in (3.109). Precisely, let us show that

$$\|\tilde{\psi}_i\|_{W^{2,p}(\Omega)} \text{ is bounded as } \beta \to 0. \tag{3.110}$$

The same way as for $\hat{\psi}$, one may reduce the problem, for each $i \in \{k + 1, \dots, g\}$, to the estimating of $\|\Delta\tilde{\psi}_i\|_{L^p(\Omega)}$. In the same way, there is three terms to deal with, in the domain Ω^i_β. These three terms are the following

- $\eta^i_\beta \Delta(1 - \tau_i) = 0$.

- $-2\nabla\eta^i_\beta\nabla\tau_i$. The second factor is bounded in the L^∞ norm, and the first factor has its L^∞ norm of order $1/\beta$.
- $\Delta\eta^i_\beta(1-\tau_i)$. The first factor has an L^∞ norm of order $1/\beta^2$. The other factor has an L^∞ of order β, since $1-\tau_i$ has a zero trace on Γ_i.

In order to prove (3.110), it is sufficient hence to prove that $\mu_i = O(\beta)$. But considering (3.57), one deduces

$$\mu_i \int_\Omega |\nabla\tau_i|^2 = \int_{\Gamma_i} \partial_\nu(\psi^*(T) + H - \psi_1), \tag{3.111}$$

hence

$$|\mu_i| \leq C\|\partial_\nu(\psi^*(T) + H - \psi_1)\|_{W^{-\frac{1}{p},p}(\Gamma_i)}. \tag{3.112}$$

We deduce that

$$\begin{aligned}|\mu_i| \leq C[\|\partial_\nu(\psi^w(T) + H - \psi_1)\|_{W^{-\frac{1}{p},p}(\Gamma_i)} \\ + \|\partial_\nu(\psi^w(T) - \psi^*(T))\|_{W^{-\frac{1}{p},p}(\Gamma_i)}],\end{aligned} \tag{3.113}$$

which implies

$$|\mu_i| \leq C(\beta + \|\psi^w(T) - \psi^*(T)\|_{W^{1,p}(\Omega)}).$$

But thanks to (3.108) and with $\eta < \beta$, one can deduce

$$|\mu_i| \leq C\beta,$$

that is, the researched estimate.

As the support of $\Delta(\hat{\psi}^\beta - \psi_1)$ "tends" to 0, one deduces that

$$\Delta(\hat{\psi}^\beta - \psi_1) \longrightarrow 0 \text{ for the } L^q \text{ topology, } \forall q < p. \tag{3.114}$$

Now with (3.94), (3.102) and (3.114), we obtain as claimed that $y^\beta \to y_1$ as $\beta \to 0$ in the $W^{1,q}$ sense for all $q < p$.

3.6 Proof of Theorem 3.2

The proof of Theorem 3.2 is approximately the same as the one of Theorem 3.1, but in that case, we have to "prepare" the solution before the beginning of the active control at time t_0 and to modify the construction of section 3.3 a little. If one has prepared the solution well, one can obtain better estimates in the study of section 3.5.3.

3.6.1 Supplementary Propositions

In this section, we use the direct orientation on the plane $\mathbb{C}$. This is useful only to ensure that a diffeomorphism of a Jordan curve which conserves orientation is homotopic to the identity of the Jordan curve.

To "prepare" the solution properly, we need the following proposition:

Proposition 3.2 *Consider $g-k$ diffeomorphisms $\psi_{k+1}, \dots, \psi_g$, respectively from Γ_i into itself, which each conserve orientation. Then for all $\epsilon > 0$, there exists a function $\breve{\theta}^\epsilon$ defined in $C^\infty(\overline{\Omega} \times [0,1]; \mathbb{R})$ satisfying the following properties:*

$$Supp\ \breve{\theta}^\epsilon \subset \overline{\Omega} \times (0,1), \tag{3.115}$$

$$\Delta\breve{\theta}^\epsilon = 0 \text{ in } \Omega \times [0,1], \tag{3.116}$$

$$\partial_n\breve{\theta}^\epsilon = 0 \text{ on } (\partial\Omega\backslash\Sigma) \times [0,1], \tag{3.117}$$

$$\|\phi^{\nabla\breve{\theta}^\epsilon}(0,1,\cdot) - \psi_i\|_{C^1(\Gamma_i)} < \epsilon, \quad \forall i \in \{k+1, \dots, g\}. \tag{3.118}$$

Proving this proposition is the goal of section 3.7.

Simultaneously with Proposition 3.2, we will prove the following result, which is an improvement of Proposition 3.1, but limited to the 2-dimensional case and to the assumption "Γ_1 connected".

Lemma 3.6 *Consider Ω a nonempty bounded regular open set in $\mathbb{R}^2$, with boundary $\Gamma_1 \cup \Gamma_2$, with Γ_1 connected. Let $k \in \mathbb{N}$. We consider the mapping y from $C^\infty(\Gamma_2)$ into $C^\infty(\overline{\Omega})$ defined by (3.26). Then $\partial_\nu y(u)_{|\Gamma_1}$ describes a dense subspace of $C^k(\Gamma_1)$ when u describes the space $C^\infty(\Gamma_2)$ (for any integer k).*

3.6.2 A new construction for the reachable velocity field

Our goal is principally to reduce to the case when

$$\|\omega^*(T) - \operatorname{curl} y_1\|_{C^1(\Gamma^b)} \text{ is small.} \tag{3.119}$$

This is possible as a consequence of Proposition 3.2. Indeed, by our assumption (3.16), one can write $\operatorname{curl} y_1$ on each Γ_i for $i \in \{k+1, \dots, g\}$ as the composition of $\operatorname{curl} y_0$ by a certain direct diffeomorphism $\mathcal{A}_i$ of Γ_i.

As during the movement of the perfect fluid, the vorticity is transported by the flow of the velocity, one can obtain all the same $\operatorname{curl} y_0$ on each Γ_i for $i \in \{k+1, \dots, g\}$ as the composition of $\operatorname{curl} y^w(T)$ by a certain direct diffeomorphism $\mathcal{B}_i$ of Γ_i.

Now for $\epsilon^r > 0$, we consider a function $\overline{\overline{\theta}}$ given by Proposition 3.2 for the diffeomorphisms $\mathcal{B}_i \circ \mathcal{A}_i : \Gamma_i \to \Gamma_i$.

We then define

$$\overline{\overline{y}}(x,t) : = \overline{y}(x,t) + \nabla\overline{\overline{\theta}}(x, \frac{t - t_0 - \eta}{\eta}), \tag{3.120}$$

where $\overline{\overline{\theta}}$ is extended to $\overline{\Omega} \times \mathbb{R}$ by 0 at the exterior of $\overline{\Omega} \times [0,1]$.

Let us now describe the construction of a new target for the velocity field at time T. Indeed, we need here to modify the construction of section 3.3 at two points: we consider a reference solution y^w which takes $\overline{\overline{y}}$ into account, and we use a stronger approximating function $y(u)$, and then H.

Here, y_2^w is defined as the fixed point of the following operator P_2

$$\begin{cases} \omega^*(0,\cdot) = \operatorname{curl}(\pi y_0) \text{ in } B_R, \\ \partial_t \omega^* + (\pi(W).\nabla)\omega^* = 0 \text{ in } B_R \times [0,T], \\ \operatorname{div} P_2(W) = 0 \text{ in } \Omega \times [0,T], \\ \operatorname{curl} P_2(W) = \omega^* \text{ in } \Omega \times [0,T], \\ P_2(W).\nu = b(\frac{2t}{T}) y_0.\nu + \overline{\overline{y}}.\nu \text{ on } \partial\Omega \times [0,T], \\ \int_\Omega [\partial_t P_2(W) + (P_2(W).\nabla)P_2(W)] .\nabla^\perp \tau_i = 0, \ \forall i \in \{1,\dots,g\}, \\ \int_\Omega P_2(W)(\cdot,0).\nabla^\perp \tau_i = \int_\Omega y_0.\nabla^\perp \tau_i. \end{cases} \tag{3.121}$$

We still consider $\beta > 0$, and now we define for each $i \in \{k+1, \dots, g\}$ a u_i by Lemma 3.6 (which one can take C^∞) in order that

$$\|\partial_\nu y(u) + \partial_\nu \psi_1 - \partial_\nu \psi^w(T)\|_{C^2(\Gamma_i)} < \beta^2. \tag{3.122}$$

And then, as in section 3.3, we define $H \in C^\infty(\overline{\Omega \backslash \Omega^\beta})$ and $r(\beta)$ in order to satisfy

$$\Delta H = 0 \text{ in } \Omega \backslash \overline{\Omega^{2r(\beta)}}, \tag{3.123}$$

$$|H|_{W^{3,p}(\Omega \backslash \overline{\Omega^\beta})} \leq 2, \tag{3.124}$$

and such that H coincides with $y(u_i)$ in a boundary of Γ_i, in such a way that

$$\|\partial_\nu H + \partial_\nu \psi_1 - \partial_\nu \psi^w(T)\|_{C^2(\Gamma^b)} < \beta^2. \tag{3.125}$$

In the same way as for H, one considers $r'(\beta)$ such that for all $x \in \Omega^{r'(\beta)}$, one has

$$| \operatorname{curl} y_0(\phi^{\overline{\overline{y}}}(t_0,0,x)) - \operatorname{curl} y_1(x)| \leq \\ \| \operatorname{curl} y_0(\phi^{\overline{\overline{y}}}(t_0,0,\cdot)) - \operatorname{curl} y_1\|_{C^1(\Gamma^b)} + \beta. \tag{3.126}$$

Now we constuct a function $G \in C^\infty(\overline{\Omega\backslash\Omega^\beta})$ such that

$$\left\{ \begin{array}{l} G(x) = \psi^* - \psi_1 \text{ in } \overline{\Omega^{r'(\beta)}}, \\ \|G\|_{C^3(\overline{\Omega\backslash\Omega^\beta})} = \|G\|_{C^3(\overline{\Omega^{r'(\beta)}})}. \end{array} \right. \tag{3.127}$$

Then we replace formula (3.54) of section 3.3 by

$$\tilde{\psi} = \psi_1 + \rho_\beta(G + H). \tag{3.128}$$

The rest of the construction, that is (3.55) to (3.59), is kept as in section 3.3.

Then the arguments for reachability of section 3.4 and the one for passing to a non-linear solution in section 3.5.2 are still valid, in such a way that we can consider a fixed point of the obtained process (with a new "$\hat{\psi}$" and a new "$\overline{y}$"), whose final value is given again by the corresponding formula (3.59). Indeed, for $\epsilon < \max(r(\beta), r'(\beta))$ fixed for the choice of $\overline{y}$, the same proof as before can be done for the reachability of the velocity field. (By the way, one may impose to η and ϵ to be inferior to β.)

For what concerns the fixed point of the process, the point is that here we want to find it in a different functional space, viz.

$$S'_{M,\eta} : = \Big\{ y \in C^1(\overline{\Omega} \times [0,T]),\ \overline{q}(y) < +\infty,\ |y - \overline{y}^\eta|_{C^0([0,T],C^1(\overline{\Omega};\mathbb{R}^2))} < M \\ y.\nu = \overline{y}^\eta.\nu + b(\frac{2t}{T})y_0.\nu + b(\frac{T-t}{\eta})y_1.\nu \text{ on } \partial\Omega \times [0,T] \Big\}. \tag{3.129}$$

One gets all the same some compacity of the operator in this new space. One has to verify that for proper η, $S'_{M,\eta}$ is sent by F into itself. This point follows from the fact that one can obtain the following Gronwall-type estimate for s and t in $[0,T]$:

$$\|\phi^{\tilde{W}}(t,s,\cdot) - \phi^{\overline{\overline{y}}}(t,s,\cdot)\|_{C^1(\overline{B_R})} \\ \leq \eta e^{|t-s|\|\overline{\overline{y}}\|_{C^0([0,T],C^2(\overline{B_R}))}} \|\tilde{W} - \overline{\overline{y}}\|_{C^0([0,T],C^1(\overline{B_R}))} \tag{3.130}$$

The problem is now reduced to check that with this new preparation, one gets the $W^{2,q}$ convergence for $q < p$.

3.6.3 The $W^{2,q}$ convergence

First, one easily deduces from the previous construction that, at the end of the process, one has in addition to above, the result

$$\| \operatorname{curl} y^\beta - \operatorname{curl} y^1 \|_{C^1(\Gamma^b)} < C(\epsilon^r + \eta). \tag{3.131}$$

where C is a constant depending on the domain, y_0 and y_1. This is indeed a consequence of (3.130).

Now, we reconsider section 3.5.3 in the light of the supplementary information (3.131).

Of course, the estimates that allowed the $W^{1,q}$ convergence ($q < p$) in section 3.5.3 are still valid, and we just deal with $\nabla(\operatorname{curl} y^\beta)$. As previously, we are concerned only with what happens on $\Omega \backslash \Omega^\beta$.

For this we consider first the "$\nabla^2 \hat{\psi}$" part. There are four terms to study (precisely, for which we want to prove the L^p boundedness on Ω^β): $\nabla^3 \rho_\beta (G + H - \psi_1)$, $\nabla^2 \rho_\beta \nabla (G + H - \psi_1)$, $\nabla \rho_\beta \nabla^2 (G + H - \psi_1)$ and $\rho_\beta \nabla^3 (G + H - \psi_1)$.

To that effect, let us first study the traces of $G + H$ and of its derivatives on Γ^b.

- 0-th order trace: $G + H = 0$ on Γ_b.
- 1st order normal trace: $\|\partial_\nu (G + H)\|_{W^{\frac{1}{p},p}} = \|\partial_\nu (\psi^* - \psi^1 + H)\|_{W^{\frac{1}{p},p}} < \beta^2$.
- 2nd order "normal" trace: by (3.131) one can bound the second derivatives on the boundary in the direction of the normal in terms of second derivatives in the tangent direction (which are null) plus first derivatives corresponding to curvature terms (and controlled with the help of the preceeding point) plus a term of order "$\epsilon^r + \eta$" (and hence of order β).

Now we can go back to our "four terms" that we want to bound in L^p.

- $\rho_\beta \nabla^3 (G + H - \psi_1)$. = The "$H$" part is actually bounded because of (3.124). Of course $\nabla^3 \psi_1$ does not change. Finally the term $\nabla^3 G$ is also bounded in L^p as a consequence of (3.126), (3.127) and (3.130).
- $\nabla \rho_\beta \nabla^2 (G + H - \psi_1)$. By the previous study of traces, (3.123), (3.124) and (3.127), one gets (with a proper Poincaré's inequality) that $\|\nabla^2 (G + H - \psi_1)\|_{L^p(\Omega^\beta)}$ is of order β. But $\|\nabla \rho_\beta\|_{L^\infty}$ is of order $1/\beta$.
- $\nabla^2 \rho_\beta \nabla (G + H - \psi_1)$. The same way as previously, one can get by a proper Poincaré 's inequality that $\|G + H - \psi_1\|_{L^p(\Omega^\beta)}$ is of order β^2.
- $\nabla^3 \rho_\beta (G + H - \psi_1)$. The same way, by the previous study of the traces and by Poincaré's inequality, one gets that the second part of this product is of order β^3.

Now we study again the second term in (3.58). Here, we have to prove that $\mu_i = O(\beta^2)$ to get the result. It follows the same way as in section 3.5.3 from (3.125).

Then, one can conclude as previously.

3.7 Proof of Proposition 3.2

The proof of Proposition 3.2 relies on the proofs of the two following propositions, that we are going to establish before coming back to the principal demonstration.

Proposition 3.3 *Let J be a C^∞-regular Jordan curve of the plane. Let ψ be a C^∞ diffeomorphism of J, which preserves orientation. For all $\epsilon > 0$ (small), there exists a time-dependent tangent vector field $v : J \times [0,1] \to TJ$ of class C^∞ satisfying the constraints*

$$Supp\ v \subset J \times (0,1), \tag{3.132}$$

$$\int_J v(x,t).\vec{dx} = 0, \quad \forall t \in [0,1], \tag{3.133}$$

and such that

$$\|\phi^v(1,0,\cdot) - \psi\|_{C^1(J)} \leq \epsilon. \tag{3.134}$$

The second Proposition is the following:

Proposition 3.4 *Let Ω be a regular, bounded, non empty open set of $\mathbb{C}$. Consider Σ an non empty open part of its boundary. Let $k \in \mathbb{N}$ and $f \in C^k(\partial\Omega\backslash\Sigma;\mathbb{C})$. For all $\epsilon > 0$, there exists a holomorphic function $\phi \in H(\Omega) \cap C^\infty(\overline{\Omega};\mathbb{C})$ such that*

$$\|f - \phi\|_{C^k(\partial\Omega\backslash\Sigma;\mathbb{C})} < \epsilon. \tag{3.135}$$

3.7.1 Proof of Proposition 3.3

Let us explain the general strategy. First, we consider a vector field whose flow between time 0 and time 1 gives ψ on J. We reproduce this vector field everywhere on J except on a small connected subset of J (which moves with the flow), and on which we impose a vector field in order to obtain (3.133). But at the end of this stage, the obtained $\phi^v(1,0,\cdot)$ is close to ψ in the C^0 norm, but its derivatives are certainly very different on the small subset. To obtain the C^1 approximation, we then "dilute" the irregularity of $\phi^v(1,0,\cdot)$ on the whole J, during a second stage.

From the fact that ψ preserves orientation, we deduce that it is in the same connected component as Id.

Let us be given a homotopy $\Phi : [0,1] \to \mathrm{Diff}_\infty(J)$, differentiable in time, for which $\Phi(0) = \mathrm{Id}_J$ and $\Phi(1) = \psi$. If needed, one can add to this the condition that $(\partial_t \Phi)$ has a compact support in time in $(0,1)$. (We will note also Φ_t for $\Phi(t)$.)

The homotopy Φ can be seen as the flow of the time-dependent tangent vector field

$$\tilde{v}(x,t) = (\partial_t \Phi)(\Phi_t^{-1}(x), t). \tag{3.136}$$

The problem is that in general, $\tilde{v}(\cdot,t)$ is not of null circulation on J.

We consider a positive number ϵ. We introduce a connected closed subset in J, say $\mathcal{I}_\epsilon$, of length at most ϵ, and such that its image by the flow of $\tilde{v}$, say $\mathcal{I}_\epsilon(t) : = \phi^{\tilde{v}}(0,t,\mathcal{I}_\epsilon)$ is of length at most ϵ. Then $\mathcal{I}_\epsilon(t)$ has a minimal length; let us denote it by ϵ_0. (It suffices, by Gronwall's lemma, to choose $\mathcal{I}_\epsilon$ sufficiently small.)

Now we consider a modification on $\tilde{v}$: we set

$$\hat{v}(x,t) = \tilde{v}(x,t) \ \ \forall t, \ \forall x \in J \backslash \mathcal{I}_\epsilon(t), \tag{3.137}$$

and $\hat{v}$ on $\{(x,t) \ / \ x \in J\backslash \mathcal{I}_\epsilon(t)\}$ is ruled in order that

$$\int_J \hat{v}(\cdot,t).\vec{dx} = 0, \ \ \forall t \in [0,1], \tag{3.138}$$

and in order that $\hat{v}$ is regular in space and in time (and still has a compact support in time).

We consider the flow of $\hat{v}$. For any x in $J\backslash\mathcal{I}_\epsilon$, one has $\phi^{\hat{v}}(0,t,x) \in J\backslash\mathcal{I}_\epsilon(t)$ and finally one has

$$\phi^{\hat{v}}(0,1,x) = \psi(x), \ \ \forall x \in J\backslash\mathcal{I}_\epsilon. \tag{3.139}$$

The problem is that we do not measure well the regularity of the flow for points originally situated in $\mathcal{I}_\epsilon$ (even if we know that $\mathcal{I}_\epsilon$ is sent into $\mathcal{I}_\epsilon(1)$). So to ensure that the researched diffeomorphism $\phi^v(0,1,\cdot)$ is not "too irregular" on $\mathcal{I}_\epsilon$, we use the following lemma:

Lemma 3.7 *Let J be a C^∞ Jordan curve of the plane. There exist two constants $K(J)$ and $\epsilon_0(J)$ depending only on J, such that if $\mathcal{D}$ is a C^∞-regular diffeomorphism $J \to J$, which conserves orientation and which moreover satisfies*

$$\forall x \in J\backslash I_\epsilon, \ \mathcal{D}(x) = x, \tag{3.140}$$

where I_ϵ is a connected subset in J of length at most ϵ, with $\epsilon < \epsilon_0(J)$, then there exists a C^∞ time-dependent tangent vector field $\overline{v} : J \times [0,1] \to TJ$ such that

$$\int_J \overline{v}(x,t).\vec{dx} = 0, \quad \forall t \in [0,1], \tag{3.141}$$

$$Supp\ \overline{v} \subset J \times (0,1), \tag{3.142}$$

and such that if we use the notation

$$T: = \phi^{\overline{v}}(0,1,\cdot), \tag{3.143}$$

then one has

$$\|T - Id\|_{C^1(J\backslash I_\epsilon)} \leq K(J)\epsilon, \tag{3.144}$$

and moreover

$$(T \circ \mathcal{D})_{|I_\epsilon} = Id_{I_\epsilon}. \tag{3.145}$$

This lemma will allow us to "smoothen" the transform of J given by the flow of $\hat{v}$. The proof of this lemma is delayed till the end of the proof of Proposition 3.2.

We apply lemma 3.7 with $\mathcal{D}: = \phi^{\hat{v}}(0,1,\cdot) \circ \psi^{-1}$ and $I_\epsilon: = \mathcal{I}_\epsilon(1)$.

Hence, one gets a certain operator T – which can be represented as the flow between time 0 and time 1 of a null-circulation (in space for each time) tangent vector field of J – such that

$$\|T - Id\|_{C^1(J\backslash I_\epsilon)} \leq K(J)\epsilon,$$

which implies

$$\|T \circ \phi^{\hat{v}}(0,1,\cdot) - \phi^{\hat{v}}(0,1,\cdot)\|_{C^1(J\backslash \mathcal{I}_\epsilon)} \leq K(J,\psi)\epsilon. \tag{3.146}$$

(For $x \in J\backslash\mathcal{I}_\epsilon$, on has $\phi^{\hat{v}}(0,1,\cdot) = \psi(x)$.)

By (3.145), one has on the interval $\mathcal{I}_\epsilon$

$$T \circ \phi^{\hat{v}}(0,1,\cdot) = \psi. \tag{3.147}$$

Consequently, by (3.146) and (3.147),

$$\|T \circ \phi^{\hat{v}}(0,1,\cdot) - \psi\|_{C^1(J)} \leq K(J,\psi)\epsilon.$$

But $T \circ \phi^{\hat{v}}(0,1,\cdot)$ consists of the flow of $\hat{v}$ during $[0,1]$ followed by the flow of $\overline{v}$ corresponding to T during $[1,2]$.

So finally, one considers

$$\left\{ \begin{array}{l} v(x,t) = 2\hat{v}(x,2t), \quad \forall (x,t) \in J \times [0,\frac{1}{2}], \\ v(x,t) = 2\overline{v}(x,2t-1), \quad \forall (x,t) \in J \times [\frac{1}{2},1]. \end{array} \right. \tag{3.148}$$

Hence taking ϵ small enough, one gets (3.134).

Proof of lemma 3.7:

Let us first introduce the time-dependent vector field $\overline{v}$, and then we will show that it satisfies the required properties.

We introduce a C^∞ function $m : [0,1] \to \mathbb{R}$ such that

$$\text{Supp } m \subset (0,1), \tag{3.149}$$

$$0 \leq m(t) \leq 2, \quad \forall t \in [0,1], \tag{3.150}$$

$$\int_{[0,1]} m = 1. \tag{3.151}$$

Let M be the primitive of m such that $M(0) = 0$.

Let us also introduce the interval $\tilde{I}_\epsilon$ obtained by extending I_ϵ of length $\epsilon/2$ on each side. For ϵ small, one obviously has $\tilde{I}_\epsilon \subset\subset J$.

We introduce a parametrisation of J, say $j : \frac{\mathbb{R}}{L\mathbb{Z}} \to J$ (where L is the total length of J) which is compatible with the arc length, that is if we denote by s the arc length on J starting from $j(0)$, $s : J \to \mathbb{R}$, the one has $j \circ S \circ s = Id_J$, with S the canonical surjection $\mathbb{R} \to \frac{\mathbb{R}}{L\mathbb{Z}}$.

We introduce the following time-dependent transform of J:

$$\left\{ \begin{array}{l} \varphi : J \times [0,1] \to J, \\ \varphi(x,t) : = j \circ S\{s(x) + M(t)[s(\mathcal{D}(x)) - s(x)]\}. \end{array} \right. \tag{3.152}$$

Let us remark that from (3.140) and (3.152) one deduces $\varphi(x,t) = x$ for $J \backslash I_\epsilon$. The transform is thus internal in $\tilde{I}_\epsilon$ and by the way, one has $|\overline{\mathcal{D}}(s) - s| < \epsilon$ on J.

From the fact that $\mathcal{D}$ is a direct diffeomorphism, one deduces together with (3.152) that the transform φ is an homotopy of (direct) diffeomorphisms. At each time, we note $\varphi_t : = \varphi(\cdot,t)$.

Then one chooses $\overline{v}$ in $C^\infty(J \times [0,1]; \mathbb{R}^2)$ in the set of all the tangent vector fields satisfying

$$\overline{v}(x,t) = (\partial_t \varphi)(\varphi_t^{-1}(x),t) \text{ in } \tilde{I}_\epsilon \times [0,1], \tag{3.153}$$

$$\int_J \overline{v}(\cdot,t).\vec{dx} = 0, \quad \forall t \in [0,1], \tag{3.154}$$

$$\|\overline{v}\|_{C^1(J\backslash\tilde{I}_\epsilon)} \leq 10\frac{1+L}{L^2}\left|\int_{\tilde{I}_\epsilon} v.\vec{dx}\right|, \tag{3.155}$$

for ϵ small enough with respect to L (say $\epsilon < L/10$).

Let us remark that such a $\overline{v}$ satisfies $\overline{v}(x,t) = 0$ for all x in $\tilde{I}_\epsilon\backslash I_\epsilon$. Then to obtain (3.153)-(3.155), the work consists in finding a regular function with support in $J\backslash\overline{\tilde{I}_\epsilon}$ with prescribed integral on $J\backslash\tilde{I}_\epsilon$ (which is an interval of length at least $L - 2\epsilon$), precisely

$$-\int_{\tilde{I}_\epsilon} m(t)(\overline{\mathcal{D}}(s) - s)ds.$$

This can clearly be done regularly in time and such that (3.155) holds (taking for example, C^∞ approximations of piecewise affine functions).

Let us prove that the $\overline{v}$ constructed this way is convenient. The point (3.145) is a trivial consequence of the form of φ, and of the choice of m.

Let us verify the point (3.144). By (3.150), (3.152) and (3.153), one deduces that $\|\overline{v}\|_{C^0(I_\epsilon)} \leq 2\epsilon$, and consequently with (3.155) that

$$\|\overline{v}\|_{C^1(J\backslash\tilde{I}_\epsilon)} \leq C(J)\epsilon. \tag{3.156}$$

So one has for any $x \in J\backslash\tilde{I}_\epsilon$

$$|T(x) - x| \leq C(J)\epsilon. \tag{3.157}$$

So we have left to study $\|\partial_x T - 1\|_{C^0}$ in $J\backslash\tilde{I}_\epsilon$. But (3.144) can be easily obtained for ϵ small by a classical Gronwall's inequality and (3.156).

3.7.2 Proof of Proposition 3.4

First, it is easy to see that one can suppose f of class C^∞.

We shall cut the proof in two parts. During a first step, we prove that one can approximate f on $\partial\Omega\backslash\Sigma$ by a holomorphic function defined in a neighbourhood of $\partial\Omega\backslash\Sigma$. In a second step we give a holomorphic function approximating f on $\partial\Omega\backslash\Sigma$ and defined globally on Ω.

Part I: The local problem

Step 1: Let us first treat the case when $\partial\Omega\backslash\Sigma = S^1$ is the unit circle.

As the function f is C^∞, the Fourier series

$$P_N^f(\theta) : = \sum_{n=-N}^{n=+N} c_n(f)e^{in\theta}, \tag{3.158}$$

converges to f in the C^k sense on S^1.

Hence, we choose N so that

$$\|f - P_N^f\|_{C^k(S^1)} < \epsilon. \tag{3.159}$$

We now consider the rational function on $\mathbb{C}$

$$Q_N^f(z) : = \sum_{n=-N}^{n=+N} c_n(f) z^n. \tag{3.160}$$

Then the function $z \mapsto Q_N^f(z)$, holomorphic on a neighbourhood of S^1 (in fact, in the whole $\mathbb{C}^*$) is such that

$$\|\psi - f\|_{C^k(S^1)} < \epsilon. \tag{3.161}$$

Step 2: We treat the case when $\partial\Omega\backslash\Sigma$ is a real-analytic Jordan curve.

We consider a conformal mapping M from the interior of J into the unit disc. Then by real-analyticity of J, this mapping can be enhanced slightly across J, as a consequence of the Schwarz reflexion principle. For this, we refer for example to [32, p. 41, Proposition 3.1].

So if we can solve the problem in a neighbourhood of the unit circle (what was done in step 1), we can solve it in a neighbourhood of any real-analytic Jordan curve.

Let us remark here that the local property holds *a fortiori* when instead of a Jordan curve, one considers only an interval in a Jordan curve.

Part II: The global problem

Step 3: Let us treat the case when $\partial\Omega\backslash\Sigma$ is the union of g real-analytic Jordan curves and interval of real-analytic Jordan curves.

Let us call $J_1, \dots, J_g$ these curves.

Let us fix $f \in C^\infty(\partial\Omega\backslash\Sigma)$.

There are g neighbourhoods $\mathcal{O}_1, \dots, \mathcal{O}_g$ (we can suppose these neighbourhoods do not intersect each other, by reducing them if necessary) of respectively $J_1, \dots, J_g$, and one can find g holomorphic functions $\psi_1, \dots, \psi_g$ defined respectively on $\mathcal{O}_i$ such that one has

$$\|f - \psi_i\|_{C^k(J_i;\mathbb{C})} < \epsilon/2, \ \forall i \in \{1, \dots, g\}. \tag{3.162}$$

Then the problem reduces to extracting from the ψ_i a global holomorphic function ψ.

As $\Sigma \neq \emptyset$, one gets that one of the connected components of $\mathbb{C}\backslash\overline{\Omega}$ in the topological space $\mathbb{C}\backslash(\partial\Omega\backslash\Sigma)$ contains Ω.

Hence, let us consider g points $x_1, \dots, x_g$ in $\mathbb{C}\backslash\overline{\Omega}$, such that any connected component of $\mathbb{C}\backslash(\partial\Omega\backslash\Sigma)$ contains at least one point x_i.

Then one obtains the global holomorphic function ϕ by Runge's theorem: it gives us a sequence of rational functions with poles in $\{x_1, \dots, x_g\}$ (and hence, holomorphic on Ω), and which converge to ψ_i uniformly on any compact of $\mathcal{O}_i$. But for holomorphic functions, the uniform convergence on compacts determines the C^k convergence on compacts. Consequently, one can find the solution by getting a element of the sequence sufficiently far.

Step 4: We treat the general case.

The general case is a consequence of the step 3, because any (bounded regular) domain Ω is conformaly equivalent to a domain whose boundary is composed with analytic Jordan curves.

This point is rather classical (see e.g. [1, p. 244]): it suffices to compose conformal mappings obtained by the Riemann's theorem for either exterior (in the Riemann sphere) or interior domain of the Jordan curves composing the boundary (and its iterated transformations), computed one after another. The important fact that we would like to underline is that during this process each conformal mapping is C^∞ up to the boundary by the Kellogg-Warschawski theorem (see e.g. [32, Theorem 3.6, p. 46]), because the Jordan curves are all C^∞. The resulting conformal mapping to a domain bounded by real-analytic curves is hence also C^∞ up to the boundary.

Hence, it is sufficient to have the step 3 solved to solve the general case.

Remark 3.2 *The local result is a very particular case of the result of R. Nirenberg and R.O. Wells (see [30]), which gives approximating holomorphic functions around (instead of a Jordan curve in $\mathbb{C}$) C^∞ totally real submanifolds of n dimensional complex manifolds.*

3.7.3 Back to the proof of Proposition 3.2

In this whole part, we identify $\mathbb{R}^2$ to $\mathbb{C}$ and hence, points in Ω will sometimes be considered as complex numbers.

Let us fix $\epsilon > 0$. For this ϵ, and for the ψ_i, one can find by Proposition 3.3, $g - k$ time-dependent tangent vector fields v_i defined respectively on $\Gamma_i \times [0, 1]$ for i in $\{k+1, \dots, g\}$ and such that (3.132), (3.133) and (3.134) hold respectively on Γ_i. For $i \in \{2, \dots, k\}$, we fix $v_i := 0$ on Γ_i, and v_1 is fixed to 0 on $\Gamma_1\backslash\Sigma$ and "free" on $\Sigma \cap \Gamma_1$ (that is we use $\Sigma \cap \Gamma_1$ as the only control region).

The main work is now to extend these v_i inside Ω in a form $v = \nabla\theta$ in order that (3.116) and (3.117) occur.

In a first step, we show that we can limit ourselves to the case when $v_i(x, t)$ is of the form $\sum_j \lambda_j(t) w_j(x)$. For that, we fix $\epsilon_2 > 0$, to be ruled later (in function of ϵ).

For this ϵ_2, we consider $\kappa \in \mathbb{N}$, $\kappa \geq 3$, such that for $t_1,\ t_2$ in $[0,1]$,

$$|t_1 - t_2| < \frac{2}{\kappa} \Rightarrow \|v_i(t_1,\cdot) - v_i(t_2,\cdot)\|_{C^2(\Gamma_i)} < \epsilon_2, \qquad \forall i \in \{k+1,\dots,g\}, \tag{3.163}$$

$$\text{Supp } v_i \subset (\frac{1}{\kappa}, \frac{\kappa-1}{\kappa}), \quad \forall i \in \{k+1,\dots,g\}. \tag{3.164}$$

Then we consider a partition of unity adapted to $[0,3/2\kappa] \cup [1/\kappa, 5/2\kappa] \cup \dots \cup [(\kappa-2)/\kappa, (2\kappa-1)/2\kappa] \cup [(\kappa-1)/\kappa, 1]$; that is, we consider κ functions $\rho_1,\dots,\rho_\kappa$ in $C^\infty([0,1];\mathbb{R})$ such that:

$$\begin{cases} \text{Supp } \rho_j \subset [0,1] \cap [\frac{j-1}{\kappa}, \frac{j+1/2}{\kappa}), \\ 0 \leq \rho_j \leq 1, \\ \sum_{j=1}^{\kappa} \rho_j = 1 \text{ on } [0,1]. \end{cases} \tag{3.165}$$

Now we consider the function

$$w_i(t,x) = \sum_{j=1}^{\kappa} \rho_j(t) v_i(\frac{j}{\kappa}, x) \text{ on } [0,1] \times \Gamma_i. \tag{3.166}$$

Then the difference between w_i and

$$v_i(t,x) = \sum_{j=1}^{\kappa} \rho_j(t) v_i(t,x), \tag{3.167}$$

in the $C^2(\Gamma_i)$ norm is majored by ϵ_2 for all t (note that for a given t, there are at most two non null terms in the previous sums).

Now we consider, for each $j \in \{1,\dots,\kappa\}$, a holomorphic C^2 approximation of the $v_i(\frac{j}{\kappa},\cdot)$ on the Γ_i for $i \in \{2,\dots,g\}$ and of 0 on $\Gamma_1 \backslash \Sigma$, with error at most ϵ_2. This is given by Proposition 3.4. We obtain κ holomorphic functions defined on Ω, viz. $H_1,\dots,H_\kappa$, such that

$$\begin{cases} \|H_j - v_i(\frac{j}{\kappa},\cdot)\|_{C^2(\Gamma_i)} < \epsilon_2, \quad \forall i \in \{2,\dots,g\}, \\ \|H_j\|_{C^2(\Gamma_1 \backslash \Sigma)} < \epsilon_2. \end{cases} \tag{3.168}$$

We add as a condition that if for a given j, the $v_i(\frac{j}{\kappa},\cdot)$ are all null for all i, then one chooses as function H_j the function 0.

Remark that by (3.133) and by the choice of v_i for $i \in \{2,\dots,g\}$, this implies in particular

$$|\int_{\Gamma_i} H_j . \vec{dx}| \leq C(\Omega)\epsilon_2, \quad \forall i \in \{2,\dots,g\}. \tag{3.169}$$

Now we consider $g-1$ points $x_2, \dots, x_g$ in $\mathbb{R}^2 \backslash \overline{\Omega}$ respectively in the interior of Γ_i for $i \in \{2, \dots, g\}$ (if Γ_i is the external curve, we fix x_i more precisely in the interior of Γ_1).

Then we consider the modified functions:

$$\tilde{H}_j(x) = H_j(x) - \sum_{i=2}^{g} \frac{\int_{\Gamma_i} H_j . dx}{2i\pi(x - x_i)}. \tag{3.170}$$

(We consider here x as a complex number.)

Then $\tilde{H}_j$ is a holomorphic function on Ω, regular up to the boundary. Its circulation around any Γ_m for $m \in \{2, \dots, g\}$ is null; hence the circulation of $\tilde{H}_j$ is also null around Γ_1.

Consequently, the circulation of $\tilde{H}_j$ around any inner connected component of the boundary is 0. Hence, the holomorphic function $\tilde{H}_j$ is the gradient of a harmonic function:

$$\tilde{H}_j = \nabla \tilde{\theta}_j = \frac{\partial \tilde{\theta}_j}{\partial x_1} + i \frac{\partial \tilde{\theta}_j}{\partial x_2}, \tag{3.171}$$

for $\tilde{\theta}_j \in C^\infty(\overline{\Omega}; \mathbb{R})$, with

$$\Delta \tilde{\theta}_j = 0 \text{ in } \Omega. \tag{3.172}$$

We have left to modify a little $\tilde{\theta}_j$ in order to obtain (3.117). For that, for $j \in \{1, \dots, \kappa\}$, one defines a function $\mathcal{G}_j$ of class C^∞ on $\partial\Omega$ such that

$$\left\{ \begin{array}{l} \mathcal{G}_j = \partial_n \tilde{\theta}_j \text{ on } \partial\Omega \backslash \Sigma, \\ \|G_j\|_{C^{1,\alpha}(\partial\Omega)} \leq C(\Omega, \Sigma) \|\partial_n \tilde{\theta}_j\|_{C^{1,\alpha}(\partial\Omega\backslash\Sigma)}, \\ \int_{\partial\Omega} \mathcal{G}_j = 0, \end{array} \right. \tag{3.173}$$

for a given $\alpha \in (0,1)$.

Then one introduces the function $\overline{\theta}_j$ in $C^{2,\alpha}$ such that

$$\left\{ \begin{array}{l} \Delta \overline{\theta}_j = 0 \text{ in } \Omega, \\ \partial_n \overline{\theta}_j = \mathcal{G}_j \text{ on } \partial\Omega, \\ \int_\Omega \overline{\theta}_j = 0. \end{array} \right. \tag{3.174}$$

Then one finally defines

$$\check{\theta}(x,t) := \sum_{j=1}^{\kappa} \rho_j(t) (\tilde{\theta}_j(x) - \overline{\theta}_j(x)). \tag{3.175}$$

Then (3.115) follows from the fact that the v_i are of compact support in $(0,1)$. Relation (3.116) follows from (3.172), (3.174) and (3.175). One easily deduces from (3.173), (3.174) and (3.175) that (3.117) holds.

There remains to prove (3.118).

From (3.169), one deduces that the correcting term $\tilde{H}_j - H_j$ satisfies

$$\|\tilde{H}_j - H_j\|_{C^1(\overline{\Omega})} < C(\Omega, \Sigma, x_i)\epsilon_2. \tag{3.176}$$

Furthermore, as the v_i are tangent on $\partial\Omega\backslash\Sigma$, the normal part of $\tilde{H}_j$ on $\partial\Omega\backslash\Sigma$ is less (in norm $C^{1,\alpha}$) than a factor of ϵ_2. Consequently with (3.173) and (3.174), one obtains

$$\|\overline{\theta}_j\|_{C^{2,\alpha}(\overline{\Omega})} < C(\Omega, \Sigma, x_i)\epsilon_2. \tag{3.177}$$

Finally, (3.163), (3.165), (3.168), (3.175), (3.176) and (3.177) lead to

$$\|v - \nabla\check{\theta}\|_{C^0([0,1],C^1(\overline{\Omega}))} < 2\epsilon_2 + C(\Omega, \Sigma, x_i)\epsilon_2. \tag{3.178}$$

It follows from Gronwall's lemma that

$$\|\phi^v(0,1,\cdot) - \phi^{\nabla\check{\theta}}(0,1,\cdot)\|_{C^1(\Gamma_i)} < K\|v - \nabla\check{\theta}\|_{C^0([0,1],C^1(\overline{\Omega}))}, \tag{3.179}$$

where K is a constant depending on the second derivatives of v (and of the domain, of Σ and of the choice of the x_i).

So finally, by setting ϵ_2 small enough (v being fixed for a given ϵ), one deduces (3.118).

3.7.4 Proof of Lemma 3.6

As Γ_1 is connected in the boundary of a regular domain, it is either a Jordan curve, or a connected part of a Jordan curve.

First, we can restrict ourselves to the case when Γ_2 intersects exactly one connected component of the boundary (we consider a greater domain if needed).

Then, one can observe that it is equivalent to prove this density in the higher-derivatives Hölder spaces $C^{k,\alpha}$. Let us hence prove this later density.

Consider g in $C^\infty(\Gamma_1)$ (we can trivially restrict ourselves to the case when g has this regularity). We want to approximate it by a $\partial_n y(u)$. For that, we consider the vector field $\hat{g} : \Gamma_1 \to \mathbb{R}^2$ such that $\hat{g}.n = g$ and $\hat{g}.\tau = 0$ on Γ_1.

By Proposition 3.4, one can find $\phi \in H(\Omega) \cap C^\infty(\overline{\Omega}; \mathbb{C})$ such that

$$\|\hat{g} - \phi\|_{C^{k+1}(\Gamma_1)} < \epsilon. \tag{3.180}$$

As for (3.170) we consider a modified function ϕ_2 in order that the circulation of the new function around any connected component of the boundary is 0. As the circulation of $\hat{g}$ around the connected components of Γ_1 is of order ϵ, one finally gets that ϕ_2 is the gradient of a harmonic function.

Now we have to modify ϕ_2 in order to have a corresponding ϕ_3 satisfying moreover $\phi_{3|\Gamma_1} = 0$. If we add a constant to ϕ_2, that does not change $\partial_n \phi_2$, so one can require

$$\int_{\Gamma_2} \phi_2 . d\tau = 0. \tag{3.181}$$

Now we construct $\overline{\phi}_2$ by

$$\begin{cases} \Delta \overline{\phi}_2 = 0 \text{ in } \Omega, \\ \overline{\phi}_2 = \phi_2 \text{ on } \Gamma_1, \\ \overline{\phi}_2 = 0 \text{ on } \Gamma_2, \end{cases}$$

With (3.180), (3.181) and $\hat{g}.\tau = 0$, one gets that $\|\overline{\phi}_2\|_{C^{k,\alpha}}$ is of order ϵ. So $\phi_3 : = \phi_2 - \overline{\phi}_2$ solves the problem raised by Lemma 3.6.

Chapitre 4

Conclusion et perspectives

Dans ce travail, nous nous sommes efforcés de donner de nouveaux résultats en matière de contrôlabilité des fluides parfaits incompressibles. L'intérêt nous en semble avant tout mathématique : en comprenant comment une équation peut se contrôler, il s'agit bien sûr également de mieux comprendre comment elle «fonctionne». Par exemple, en ce qui concerne le cas tridimensionnel, un contrôle localisé au bord permet de donner la forme voulue au système et ce en un temps donné ; cela implique en particulier que l'on peut reporter le temps d'existence arbitrairement (y compris le rendre infini). Or pour ce système, on n'a pas de résultat d'existence pour tout temps ; le contrôle permet donc d'éviter une éventuelle explosion du système en temps fini. Dans la même optique, il serait intéressant de trouver d'autres systèmes dont on sait qu'ils explosent en temps fini, ou du moins dont on ignore s'ils ont cette possibilité ou non, et pour lesquels un contrôle permettrait de reporter arbitrairement le temps d'existence d'une solution forte.

Par la suite, l'exemple de [7] illustre la possibilité de partir de l'équation d'Euler pour obtenir des résultats de contrôle approché sur l'équation de Navier-Stokes, ce qui est plus intéressant du point de vue des éventuelles applications. L'idée pour cela est que si l'on agit assez vite et assez fort, le terme «$(v.\nabla)v$» est prépondérant sur le terme «Δv». Il est à noter également que les fonctions «$\overline{y}$» considérées sont des solutions potentielles de l'équation de Navier-Stokes incompressible. Le résultat de [7] concerne les conditions au bord un peu particulières de Navier, et il serait intéressant de savoir s'il est possible de l'étendre au cas de non-glissement (c'est à dire avec une condition de Dirichlet nulle au bord). En effet, ce résultat est approximativement – mis à part pour le cas où y_1 est nul, voir [14] – le seul résultat de contrôle approché global dont on dispose pour l'équation de Navier-Stokes. Un travail sur lequel pourra déboucher ce mémoire sera d'examiner si ce résultat de J.-M. Coron peut s'étendre à la dimension 3 (dans un premier temps, avec des conditions de glissement au bord analogues, même si en dimension 3 celles-ci se révèlent un peu plus compliquées), et également de savoir si les résultats de la troisième partie peuvent apporter des améliorations au résultat de [7] soit du point de vue de la qualité de la convergence, soit du point de vue de la condition au bord.

Par ailleurs, les méthodes considérées dans ce travail sont «semi-constructives» au sens suivant : une fois connue la fonction $\overline{y}$, la solution du problème de contrôlabilité s'obtient comme solution d'un problème de point fixe explicite ; malheureusement, les fonctions $\overline{y}$ sont pour leur part obtenues de façon non constructive. Le premier travail en vue d'un éventuel traitement numérique serait de chercher à rendre constructive la méthode d'obtention de cette fonction, comme cela est déjà le cas pour les domaines simplement connexes de dimension 2.

Nous noterons enfin que le domaine de recherche regorge de problèmes ouverts : à ceux mentionnés précédemment s'ajoutent ceux des problèmes «fluides-structure» où l'on ajoute contre le fluide un corps (flexible ou non). Ces sujets offrent, en plus d'une multitude de problèmes mathématiques, de nombreuses possibilités d'applications. Pour ce genre de problèmes, au sujet desquels la littérature physique abonde, nous renvoyons par exemple à la bibliographie de [21] ; pour des études mathématiques, on se référera par exemple à [29] et [31].

Ajoutons également les problèmes de stabilisation par retour d'état (le résultat de [8] ne concerne que les fluides parfaits incompressibles bidimensionnels dans un domaine simplement connexe). Dans la perspective d'une éventuelle mise en application pratique de ce genre de travaux mathématiques, celle-ci offre l'avantage de prendre en compte l'état du système à chaque instant, ce dernier pouvant s'écarter de son itinéraire prévu soit du fait de l'imprécision avec laquelle on a mesuré son état initial, soit de l'imperfection avec laquelle est appliqué le contrôle, soit de celle (inévitable) du modèle utilisé, etc. Nous renvoyons à [8] et [9] pour de plus amples informations à ce sujet, et où au demeurant quelques conjectures sont proposées.

Une autre direction de recherche à terme est également de toute évidence la considération de fluides compressibles, et de manière générale de modélisations plus complexes, qui par exemple prennent en compte des contrôles mieux adaptés aux applications. Des résultats sur les problèmes directs (sans contrôle) se sont en effet multipliés ces dernières années, qui laissent espérer l'accès à de nouveaux résultats de contrôlabilité.

Bibliographie

[1] L. V. Ahlfors, *Complex analysis: An introduction of the theory of analytic functions of one complex variable*, Second edition, McGraw-Hill Book Co., New York-Toronto-London, 1966.

[2] C. Bardos & U. Frisch, Finite-time regularity for bounded and unbounded ideal incompressible fluids using Hölder estimates, in *Proceedings of the conference held at the university of Paris-Sud Orsay*, France, June 12-13, 1975, Springer-Verlag, *Lectures Notes in Mathematics* 565, pp. 1-13.

[3] J.-M. Coron, Global Asymptotic Stabilization for controllable systems without drift, *Math. Control Signal Systems*, 5, 1992, pp. 295-312.

[4] J.-M. Coron, Return method: Application to controllability, *Séminaire Équations aux Dérivées Partielles*, 1992-1993, École polytechnique, Centre de Mathématiques, exposé XIV.

[5] J.-M. Coron, Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels, *C. R. Acad. Sci. Paris, Sér. I Math.* 317 (1993), no. 3, pp. 271-276.

[6] J.-M. Coron, On the controllability of 2-D incompressible perfect fluids, *J. Math. Pures Appl.*, 75, 1996, pp. 155-188.

[7] J.-M. Coron, On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions, *ESAIM: Control Optimisation and Calculus of Variations*, 1, 1996, pp. 35-75. (http://www.emath.fr/cocv/)

[8] J.-M. Coron, On null asymptotic stabilisation of the 2-D Euler equation of incompressible fluids on simply connected domains, *Prépublication d'Orsay*, no. 59 (1998).

[9] J.-M. Coron, Sur la stabilisation des fluides parfaits incompressibles bidimensionnels, *Séminaire sur les Équations aux Dérivées Partielles*, 1998-1999, École polytechnique, Centre de Mathématiques, Exposé VII.

[10] J.-M. Coron & A.V. Fursikov, Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.* 4 (1996), no. 4, pp. 429-448.

[11] C. Fabre, Uniqueness results for Stokes equations and their consequences in linear and nonlinear control problems. *ESAIM: Control Optimisation and Calculus of Variations* 1 (1996), pp. 267-302. (http://www.emath.fr/cocv/)

[12] C. Fabre, J.-P. Puel & E. Zuazua, Approximate controllability for semilinear heat equation, *Proc. Royal Soc. Edinbourgh*, 125A (1995), pp. 31-61.

[13] E. Fernàndez-Cara & J. Real, On a conjecture due to J.-L. Lions. *Nonlinear Anal.* 21 (1993), no. 11, pp. 835-847.

[14] A.V. Fursikov, On boundary zero controllability of the three-dimensional Navier-Stokes equations. *Theory of the Navier-Stokes equations*, pp. 31-45, *Ser. Adv. Math. Appl. Sci.*, 47 (1998).

[15] A.V. Fursikov & O.Yu. Imanuvilov, Local exact boundary controllability of the Navier-Stokes system. *Optimization methods in partial differential equations*, pp. 115-129, *Contemp. Math.*, 209 (1997).

[16] A.V. Fursikov & O.Yu. Imanuvilov, Local exact controllability of the Navier-Stokes equations. *C. R. Acad. Sci. Paris Sér. I Math.* 323 (1996), no.3, pp. 275-280.

[17] A.V. Fursikov & O.Yu. Imanuvilov, On exact boundary zero-controllability of two-dimensional Navier-Stokes equations. *Mathematical problems for Navier-Stokes equations. Acta Appl. Math.* 37 (1994), no. 1-2, pp. 67-76.

[18] O. Glass, Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles en dimension 3, *C. R. Acad. Sci. Paris Sér. I Math.* 325 (1997), no. 9, pp. 987-992.

[19] O. Glass, Contrôlabilité de l'équation d'Euler tridimensionnelle pour les fluides parfaits incompressibles, *Séminaire sur les Équations aux Dérivées Partielles*, 1997-1998, École Polytechnique, Centre de Mathématiques, Exposé XV.

[20] P. Hermann & H. Kersten, Über die stetige Abhängigkeit der Lösung des Neumann-Problems für die Prae-Maxwellschen Gleichungen von ihren Randdaten. *Arch. Math.* (Basel) 36 (1981), no. 1, pp. 79-82.

[21] G. Hernandez, Contrôle actif des instabilités hydrodynamiques des écoulements subsoniques compressibles, Thèse de doctorat de l'Institut National Polytechnique de Toulouse, 1996.

[22] O. Yu. Imanuvilov, On exact controllability for Navier-Stokes equations, *ESAIM: Control Optimisation and Calculus of Variations* 3 (1998), pp. 97-131. (http://www.emath.fr/cocv/)

[23] T. Kato, On classical solutions of the two-dimensional nonstationary Euler equation, *Arch. Rational Mech. Anal.*, 25, 1967, pp. 188-200.

[24] A. V. Kazhikov, Note on the formulation of the problem of flow through a bounded region using equations of perfect fluid, *PMM USSR*, 44, 1981, pp. 672-674.

[25] J.-L. Lions, Are there connections between turbulence and controllability?, 9th INRIA International Conference, Antibes, June 12-15, 1990.

[26] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Paris, Gauthier-Villars, 1968.

[27] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Tomes 1 et 2, Recherches en Mathématiques Appliquées 8 et 9, Masson, Paris, 1988.

[28] J.-L. Lions & E. Zuazua, Contrôlabilité exacte des approximations de Galerkin des équations de Navier-Stokes. *C. R. Acad. Sci. Paris Sér. I Math.* 324 (1997), no. 9, pp. 1015-1021.

[29] J.-L. Lions & E. Zuazua, Approximate controllability of a hydro-elastic coupled system. *ESAIM: Control Optimisation and Calculus of Variations* 1 (1996), pp. 1-15. (http://www.emath.fr/cocv/)

[30] R. Nirenberg & R. O. Wells Jr., Approximation theorems on diff erentiable submanifolds of a complex manifold, *Trans. Amer. Math. Soc.* 142 (1969), pp. 15–35.

[31] A. Osses & J.-P. Puel, Approximate controllability for a linear model of fluid structure interaction, *ESAIM: Control Optimisation and Calculus of Variations* 4 (1999), pp. 497-514. (http://www.emath.fr/cocv/)

[32] C. Pommerenke, *Boundary behaviour of conformal maps*, *Grundlehren der mathematischen wissenschaften* 299, Spinger Verlag, Berlin, 1992.

[33] R. Temam, *Navier-Stokes equations. Theory and numerical analysis.* Third Edition. *Studies in Mathematics and its Applications*, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1984.

[34] W. Wolibner, Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long, *Math. Z.* 37, 1933, pp. 698-726.

[35] E. Zuazua, Exact controllability for semilinear wave equatios in one space dimension, *Ann. Inst. Henri Poincaré, Nonlinear Analysis*, 10 (1993), pp. 109-129.