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Méthodes de volumes finis et multiniveaux pour les équations de Navier-Stokes, de Burgers et de la chaleur

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METHODES DE VOLUMES FINIS ET MULTINIVEAUX POUR LES EQUATIONS DE NAVIER-STOKES, DE BURGERS ET DE LA CHALEUR

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à mes parents...

à Muriel...

METHODES DE VOLUMES FINIS ET MULTINIVEAUX POUR LES EQUATIONS DE NAVIER-STOKES, DE BURGERS ET DE LA CHALEUR

Résumé

Cette thèse est composée de quatre chapitres traitant des méthodes de volumes finis et multiniveaux appliquées aux équations de la mécanique des fluides.

Dans un premier chapitre, nous considérons l'équation de la chaleur non linéaire avec des conditions aux bords de type Robin, les termes non linéaires vérifiant une hypothèse de monotonie. Nous proposons un schéma de volumes finis discrétisant ce problème. Nous prouvons l'existence d'une solution approchée satisfaisant des propriétés de stabilité et convergeant vers la solution du problème continu.

Dans les deux chapitres suivants, nous étudions la résolution des équations de Navier-Stokes incompressibles à l'aide de schémas de volumes finis à variables colocalisées i.e. les vitesses et la pression sont calculées au même endroit. Ces schémas permettent des simulations dans des domaines à géométries complexes et rendent plus aisée la mise en oeuvre de méthodes multiniveaux. Le chapitre deux est dédié à l'analyse de la stabilité d'un de ces schémas utilisant une méthode de projection comme discrétisation temporelle. Le chapitre trois décrit les techniques d'implémentation de ces schémas, la principale difficulté étant d'assurer un couplage correct entre la vitesse et la pression.

Enfin, dans un quatrième chapitre, nous présentons une nouvelle méthode multiniveaux qui résulte d'une adaptation aux volumes finis des Inconnues Incrémentales originellement définies dans un contexte de différences finies. Nous validons cette méthode à l'aide du problème de Burgers vu comme un problème non linéaire d'évolution modèle. Cette méthode a été construite pour fonctionner avec les schémas de volumes finis à variables colocalisées étudiés précédemment.

Mots clés : Méthodes volumes finis - Méthodes multiniveaux - Inconnues incrémentales - Schémas à variables colocalisées - Convergence - Stabilité - Equations de Navier-Stokes - Equation de Burgers - Equation de la chaleur

AMS Classification (2000) : 65M12, 76M12, 76F65, 65M55, 65N22, 65M06, 76D05, 35K05, 35Q30, 35Q53

FINITE VOLUME AND MULTILEVEL METHODS FOR THE NAVIER-STOKES, BURGERS AND HEAT EQUATIONS

Abstract

This thesis is composed of four chapters which deal with finite volume methods and multi-level methods applied to the fluid mechanic equations.

In the first chapter, we consider the nonlinear heat equation with Robin-type boundary conditions, the nonlinear terms satisfying a monotony hypothesis. We describe the finite volume scheme used to discretize this problem. We prove the existence of an approximate solution and we state a stability result. We then show that this approximate solution converges to the solution of the continuous problem.

In the next two chapters, we study the solution of the incompressible Navier-Stokes equations obtained with collocated finite volume schemes i.e. the velocity and the pressure are computed at the same location. These schemes facilitate the use of complex geometries and hierarchical space discretizations. The second chapter is dedicated to the stability analysis of one of these schemes using a projection method as time discretization. The third chapter describes the implementation of these schemes, the main difficulty being to obtain an appropriate coupling between the velocity and the pressure.

Finally, in the fourth chapter, we introduce a new multilevel method which consists of a finite volume adaptation of the incremental unknowns originally defined in the finite difference context. We validate this method by solving the Burgers equations which may be regarded as a simplified model of the Navier-Stokes equations in the context of the collocated finite volume schemes previously studied.

Key words : Finite volume methods - Multilevel methods - Incremental unknowns - Collocated/colocated schemes - Convergence - Stability - Navier-Stokes equations - Burgers equation - Heat equation

AMS Classification (2000) : 65M12, 76M12, 76F65, 65M55, 65N22, 65M06, 76D05, 35K05, 35Q30, 35Q53

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Introduction

L'objet de ces travaux est l'étude de la résolution numérique d'équations aux dérivées partielles à l'aide de méthodes volumes finis. Nous travaillerons également à la mise au point d'une méthode multi-niveaux adaptée aux volumes finis.

Les méthodes volumes finis, basées sur une intégration des équations considérées, sont très utilisées par les ingénieurs. De nombreux et variés types de problèmes peuvent ainsi être simulés, on peut entre autres citer :

- Les écoulements peu profonds à surface libre (équations de Saint-Venant) [BGH98], [GP01] tels que les rivières, les lacs, les régions côtières, pouvant être liés à des problèmes d'environnement : inondations, ruptures de barrage, transport et dispersion de polluants, réseaux d'irrigation, d'avalanches...
- Les écoulements turbulents (équations de Navier-Stokes) [BH96], [PKP01].
- Les problèmes de propagation des ondes, en electromagnétisme [PRF02] ou en aérocoustique [BP03] (propagation de bruits générés par des sources : sources mécaniques, moteurs d'avions ou de voitures...).

Les techniques de volumes finis ont été développées pour la qualité physique des approximations quelles procurent (conservativité) ainsi que pour leurs performances numériques (robustesse). On leur reproche cependant parfois de n'être que peu précises.

Les méthodes multi-niveaux sont intéressantes lors de l'étude de phénomènes multi-échelles dans les équations aux dérivées partielles. Ces phénomènes apparaissent typiquement lors de la simulation d'un écoulement turbulent. Un tel écoulement étant caractérisé par son irrégularité à la fois en temps et en espace, sa simulation est très délicate. Il est très sensible aux perturbations, même faibles, celles-ci ayant tendance à s'amplifier du fait de la faible viscosité.
La présence de structures de tailles très différentes interagissant entre elles caractérise également un écoulement turbulent. En effet la turbulence est engendrée près des obstacles ou aux bords du domaine, de là, elle se propage dans tout le fluide. Plus le nombre de Reynolds est grand (i.e. plus la viscosité est petite), plus la différence de taille entre les plus grandes et les plus petites structures présentes dans l'écoulement est importante. Les méthodes multi-niveaux permettent de mieux capter toutes ces structures et ainsi de mieux tenir compte du caractère local de la turbulence. De plus, on évite ainsi l'utilisation de grilles très fines aux zones "calmes" ce qui permet de diminuer sensiblement le temps de calcul.

Premier chapitre : Un schéma volumes finis pour l'équation de la chaleur non linéaire

Le problème que l'on étudiera dans ce chapitre est celui de la chaleur non linéaire dans un domaine borné Ω :

$$\frac{\partial u}{\partial t} - \nu \Delta u + \alpha(u) = f \text{ dans } \Omega, \tag{1}$$

où, $t \geq 0$, $f = f(x,y,t)$ est donnée, et en dimension deux, $u = u(x,y,t)$ représente par exemple la température dans un conducteur.
Les conditions aux bords considérées sont des conditions de type Robin :

$$\frac{\partial u}{\partial n} + \beta(u) = g \text{ sur } \partial\Omega, \tag{2}$$

où $g = g(x,y,t)$ est donnée.
Les fonctions α et β intervenant dans les termes non linéaires de (1) et (2) satisfont les hypothèses suivantes :

- $\alpha(u)$ est supposée monotone :

$$\big(\alpha(u) - \alpha(v)\big)\big(u - v\big) \geq 0, \tag{3a}$$

  et

$$c_1|u|^p - c_1' \leq \alpha(u)u \leq c_2|u|^p + c_2', \tag{3b}$$

  où c_1, c_1', c_2, c_2' sont des constantes et $p > 1$.

- $\beta(u)$ est également supposée monotone :

$$c_3|u|^q - c_3' \leq \beta(u)u \leq c_4|u|^q + c_4', \tag{4}$$

  où c_3, c_3', c_4, c_4' sont des constantes et $q > 1$.

L'étude théorique dans le cas continu de ce problème à été effectuée par Lions [Lio69]. L'existence et l'unicité d'une solution ont ainsi été démontrés.
Pour faire une étude dans le cas discret, nous construirons un schémas volumes finis puis nous montrerons que la solution approchée obtenue grâce à ce schéma converge vers la solution du problème (1), (2).
Concernant la discrétisation spatiale du problème, nous découperons le domaine Ω en plusieurs volumes de contrôle. L'obtention du schéma volumes finis se fera en intégrant l'équation (1) sur chaque volume de contrôle. La discrétisation temporelle sera assurée par un schéma différences finies de Crank-Nicholson.
Après avoir prouvé l'existence d'une solution approchée obtenue grâce au schéma, nous démontrerons la convergence de cette approximation vers la solution du problème continu (1), (2). Cette

démonstration de convergence suivra la démarche utilisée par Lions [Lio69]. Le passage à la limite se fera grâce à la monotonie et à l'hémicontinuité, cela nous permettra de n'utiliser qu'un minimum d'estimations a priori. Un passage à la limite par une méthode de compacité nécessiterait des estimations a priori supplémentaires.

Deuxième et troisième chapitres : Schémas volumes finis à variables colocalisées pour les équations de Navier-Stokes

Dans ces deux chapitres, nous considérerons les équations de Navier-Stokes dans leur formulation vitesses-pression et l'équation de continuité écrite pour un fluide visqueux incompressible. Ω est ici un domaine borné de $\mathbb{R}^2$. Pour une force $\mathbf{f}$ donnée, nous résolverons le problème suivant :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \text{ dans } \Omega \times [0, T], \tag{5}$$

$$div\ \mathbf{u} = 0, \tag{6}$$

où ν est la viscosité et, en dimension deux, $\mathbf{u} = \big(u(x,y,t), v(x,y,t)\big)$, $t \geq 0$.
Les conditions aux bords considérées sont de type Dirichlet :

$$\mathbf{u}\Big|_{\partial\Omega} = 0. \tag{7}$$

La résolution numérique d'équations aux dérivées partielles telles que les équations de Navier-Stokes pour des fluides visqueux incompressibles a motivé de nombreux auteurs. Du point de vue de la discrétisation spatiale, on trouve deux principaux types de discrétisations : les méthodes à variables colocalisées (i.e. les vitesses et la pression sont calculées aux centres des mailles) et les méthodes à variables décalées (i.e. la pression est calculée aux centres des mailles et les vitesses aux interfaces). Traditionnellement, les méthodes à variables décalées, comme par exemple les schémas de type MAC en différences finies [HW65], ont été le plus utilisées. Ce n'est qu'en 1981 que la première méthode volumes finis à variables colocalisées fût élaborée par Hsu [Hsu81], Prakash [Pra81] et Rhie [Rhi81]. La principale raison du développement tardif de ces méthodes vient du fait qu'elles étaient considérées comme peu réalistes [Pat80] car elles donnaient de mauvais résultats numériques. La cause de tels résultats venant souvent d'un mauvais couplage des vitesses et de la pression. Nous verrons que dans certains cas cette difficulté peut être résolue. Une comparaison intéressante de ces deux types de méthodes peut être trouvée dans [PKS88]. Concernant les avantages des méthodes à variables colocalisées, on peut citer d'une part la facilité avec laquelle elles permettent de faire des simulations dans des domaines à géométrie complexe, et d'autre part, le fait qu'il est plus aisé de construire une méthode multi-niveaux lorsque les variables sont toutes situées au même endroit. Ce dernier point sera étudié plus en détail dans le quatrième chapitre lorsque que nous travaillerons au développement d'une méthode multi-niveaux adaptée aux volumes finis colocalisés.

Dans le deuxième chapitre, après avoir expliqué comment est construit un schéma volumes finis à variables colocalisées dont la discrétisation temporelle est assurée par une méthode de projection [Cho68], [Tem69], nous démontrerons sa stabilité.

Le troisième chapitre illustrera l'importance du choix de la discrétisation temporelle en étudiant deux schémas volumes finis à variables colocalisées ayant une discrétisation temporelle différente : le premier utilise une méthode de projection et le second une récente méthode de "splitting" mise au point par Guermond et Shen [GS03]. Notre étude portera bien entendu sur les techniques utilisées afin d'assurer un couplage des vitesses et de la pression satisfaisant. Nous présenterons et commenterons ensuite les résultats numériques obtenus à l'aide de ces deux schémas dans le cas de la simulation d'une cavité entraînée.

Quatrième chapitre : Discrétisations volumes finis et méthodes multiniveaux pour simuler des écoulements

Ce chapitre doit être vu comme une première étape vers la mise au point d'une nouvelle méthode multi-niveaux conçue pour des schémas volumes finis à variables colocalisées résolvant les équations de Navier-Stokes incompressible. Nous établirons donc dans un premier temps les concepts d'une telle méthode. Pour la mettre au point, nous nous inspirerons de la décomposition de l'inconnue en Inconnues Incrémentales utilisée en différences finies [Tem90], [Che98] et [Pou96]. La décomposition obtenue sera donc mathématique car basée sur les propriétés du développement de Taylor.

Pour la validation de cette nouvelle méthode, afin de ne pas cumuler les difficultés inhérentes à la méthode multi-niveaux que nous essayerons de développer avec les difficultés propres à la résolution des équations de Navier-Stokes incompressible, nous nous intéresserons à la résolution du problème de Burgers 2D. Nous considérons donc le problème de Burgers 2D comme un problème non linéaire d'évolution modèle. En effet, ces équations peuvent être interprêtées comme une formulation simplifiée des équations de Navier-Stokes incompressible pour lesquelles le gradient de pression et la condition d'incompressibilité sont supprimés.

Soit Ω un domaine borné de $\mathbb{R}^2$ dont la frontière est notée $\partial\Omega$, le problème étudié est donc le suivant :

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \mathbf{f} \text{ dans } \Omega \times [0,T], \tag{8}$$

$$\mathbf{u} = \mathbf{g} \text{ sur } \partial\Omega, \tag{9}$$

$$\mathbf{u}\Big|_{t=0} = \mathbf{u}_0 \text{ dans } \Omega, \tag{10}$$

où $\mathbf{u} = \big(u(x,y,t), v(x,y,t)\big)$ est le champ des vitesses que l'on cherche à calculer, et $\mathbf{f}(x,y,t)$, $\mathbf{g}(x,y,t)$, $\mathbf{u}_0(x,y)$ sont données.

Après avoir décrit comment est faite la hiérarchisation spatiale pour un volume de contrôle à l'intérieur du domaine ainsi que pour un volume de contrôle en contact avec la frontière, nous donnerons les détails de l'algorithme multi-niveaux considéré. Nous ferons également une analyse de la stabilité de l'algorithme. Enfin, nous reprendrons un cas test de nombreuses fois utilisé, voir [Pou96], [CLT97], [DJT99] et [LZ03], afin de valider notre méthode.

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Chapitre 1

A finite volume scheme for the nonlinear heat equation

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A finite volume scheme for the nonlinear heat equation

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Abstract : The aim of this article is to study the finite volume approximation of a nonlinear heat equation displaying a monotone nonlinear term. In a first time, we describe the finite volume scheme used to solve this equation and show the existence of the discrete solution. Then, we prove that the approximate solution is bounded in suitable spaces and that it converges to the solution of the continuous problem.

1 Introduction

In this article, we consider the nonlinear heat equation in a bounded domain Ω :

$$\frac{\partial u}{\partial t} - \nu \Delta u + \alpha(u) = f \text{ in } \Omega, \tag{1.1}$$

where, in space dimension 2, $u = u(x, y, t)$, $t \geq 0$.

On $\partial\Omega$, we consider a nonlinear boundary condition of Robin's type :

$$\frac{\partial u}{\partial n} + \beta(u) = g \text{ on } \partial\Omega, \tag{1.2}$$

where $g = g(x, y, t)$ is known.

The functions α and β in (1.1) and (1.2) satisfy the following hypotheses :

- About the nonlinear term $\alpha(u)$ of (1.1), we assume that it is monotone :

$$\big(\alpha(u) - \alpha(v)\big)\big(u - v\big) \geq 0, \tag{1.3a}$$

and

$$c_1|u|^p - c_1' \le \alpha(u)u \le c_2|u|^p + c_2', \tag{1.3b}$$

where c_1, c_1', c_2, c_2' are constants and $p > 1$.

About the boundary term $\beta(u)$ of (1.2), we assume that it is monotone and :

$$c_3|u|^q - c_3' \le \beta(u)u \le c_4|u|^q + c_4', \tag{1.4}$$

where c_3, c_3', c_4, c_4' are constants and $q > 1$.
More generally the c_i, c_i' and c_i'' will denote various constants.

The mathematical theory of such nonlinear equations was developped e. g. in Minty [Min62], [Min63] and in Lions [Lio69]. Following [Lio69], we now introduce the appropriate function spaces and recall the main result of existence and uniqueness of solution for this problem. The appropriate function spaces for this problem are :

$$\mathbf{V} = \Big\{v \in H^1(\Omega) \cap L^p(\Omega), \gamma_0 v \in L^q(\partial\Omega)\Big\},$$
$$\mathbf{W} = \Big\{v \in L^2(0,T; H^1(\Omega)) \cap L^p(0,T; L^p(\Omega)), \gamma_0 v \in L^q(0,T; L^q(\partial\Omega))\Big\},$$

where $\gamma_0 \in \mathcal{L}\big(H^1(\Omega), L^2(\partial\Omega)\big)$ is the trace operator.

We obtain the weak formulation of this problem by multiplying (1.1) by a test function $\varphi \in \mathbf{V}$, integrating over Ω and integrating by parts using (1.2). We arrive at the relation valid for all $\varphi \in \mathbf{V}$ and $t \in (0,T)$:

$$\frac{d}{dt}(u, \varphi) + a\big(u, \varphi\big) + b\big(u, \varphi\big) = \big\langle l, \varphi\big\rangle; \tag{1.5}$$

here :

$$\begin{aligned}
(u, \varphi) &= \int_\Omega u\varphi \, d\Omega, \\
a\big(u, \varphi\big) &= \nu \int_\Omega \nabla u \nabla \varphi \, d\Omega, \\
b\big(u, \varphi\big) &= \int_\Omega \alpha(u)\varphi \, d\Omega + \nu \int_{\partial\Omega} \beta(u)\varphi \, d\Gamma, \\
\big\langle l, \varphi\big\rangle &= \int_\Omega f\varphi \, d\Omega + \nu \int_{\partial\Omega} g\varphi \, d\Gamma.
\end{aligned}$$

We infer from [Lio69], [Min63] the following result :

Theorem 1.1 (Existence and uniqueness for the exact problem). *We assume that α and β satisfy (1.3a), (1.3b) and (1.4). Moreover, we assume that $u\big|_{t=0} = u_0$, f and g are given* *satisfying :*

$$u_0 \in L^2(\Omega),$$
$$f \in L^{p'}(0,T;L^{p'}(\Omega)) \quad \textit{with} \quad \frac{1}{p'}+\frac{1}{p}=1,$$
$$g \in L^{q'}(0,T;L^{q'}(\partial\Omega)) \quad \textit{with} \quad \frac{1}{q'}+\frac{1}{q}=1.$$

Then, there exists a unique solution $u : t \mapsto u(t)$, $u \in \boldsymbol{W}$, of (1.1) and (1.2) (or (1.5)) such that $u(0) = u_0$.

The aim of this paper is to describe the implementation of a finite volume discretization of this problem and to study the existence, stability and convergence of the scheme adapted to this problem. The results are presented for a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$ discretized by a rectangular grid in order to simplify notations and readings of some proofs, but they can be extended to more general situations using the notations and concepts of [EGH00], on which we rely for the finite volume framework.

Remark 1.1. For a better legibility, we have chosen a simpler boundary condition but another possible boundary condition for this problem is :

$$\frac{\partial u}{\partial n} + a\beta(u) = g \text{ on } \partial\Omega, \tag{1.6}$$

where $g = g(x, y, t)$ and $a = a(x, y, t)$ are known.
All the proofs in this paper can be easily extended to this case if we assume that :

$$a(x, y) \geq a_0 > 0 \text{ for all } (x, y) \in \Omega, \tag{1.7}$$

where a_0 is a constant.

In Section 2, we introduce the notations used in the following sections. Then, in Section 3, we give the finite volume scheme used to solve our problem, the discrete weak formulation, and we show that there exists a solution for our scheme. In Section 4, we prove the numerical stability of the scheme and in Section 5 we show that the discrete solution converges to the exact solution of the continuous problem. At last, Section 6 is a technical section which contains the proof of the lemmas used in the previous sections.

2 Notations

In order to discretize the domain $\Omega = (0, L_1) \times (0, L_2)$, we use rectangular finite volumes. We assume that all of the volumes have the same dimensions $\Delta x \times \Delta y$ with $M\Delta x = L_1$ and $N\Delta y = L_2$ where M, N are given integers. Due to the Robin boundary conditions, we also define the fictitious control volumes around the boundary : K_{0j}, K_{M+1j}, K_{i0} and K_{iN+1} for

$i = 1, ..., M$ and $j = 1, ..., N$. Hence, we have MN control volumes and $2M + 2N$ fictitious control volumes which are defined by :

$$\left(K_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\right)_{i=0,...M+1,\, j=0,...,N+1},$$

where $(i,j) \notin \left\{(M+1, N+1), (M+1, 0), (0, N+1), (0,0)\right\}$, and

$$x_{i+\frac{1}{2}} = i\Delta x \ \text{ for } i = -1, \ldots, M+1,$$
$$y_{j+\frac{1}{2}} = j\Delta y \ \text{ for } j = -1, \ldots, N+1.$$

The above definition means that we have an admissible finite volume mesh and we can define the space $\mathbf{V}_h$ as the set of functions from Ω to $\mathbf{R}$ which are constant over each control volume K_{ij} of the mesh for $i = 1, ..., M+1$, $j = 1, ..., N+1$ and $(i,j) \neq (M+1, N+1)$.
For a function $\varphi \in \mathbf{V}$, we define its approximate function $\varphi_h \in \mathbf{V}_h$ by :

$$\varphi_h(x,y,t) = \varphi_{ij}(t) \text{ if } (x,y) \in K_{ij}, \tag{2.1a}$$

where

$$\varphi_{ij}(t) = \frac{1}{\Delta x\, \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \varphi(x,y,t)\,dxdy. \tag{2.1b}$$

Morever, we also define the trace of φ_h by :

$$\gamma_0\varphi_h(x,y) = \begin{cases} \varphi_{1j} & \text{if } x = 0 \text{ and } y_{j-\frac{1}{2}} \le y \le y_{j+\frac{1}{2}}, \\ \varphi_{M+1j} & \text{if } x = L_1 \text{ and } y_{j-\frac{1}{2}} \le y \le y_{j+\frac{1}{2}}, \\ \varphi_{i1} & \text{if } x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}} \text{ and } y = 0, \\ \varphi_{iN+1} & \text{if } x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}} \text{ and } y = L_2, \end{cases} \tag{2.2}$$

and we introduce the following discrete operators for $i = 1, ..., M$ and $j = 1, ..., N$:

$$D_h\varphi_h(x,y) = \begin{pmatrix} \dfrac{\varphi_{i+1j} - \varphi_{ij}}{\Delta x} \\ \dfrac{\varphi_{ij+1} - \varphi_{ij}}{\Delta y} \end{pmatrix} \text{ if } (x,y) \in K_{ij}, \tag{2.3}$$

$$\Delta_h\varphi_h(x,y) = \frac{1}{\Delta x\, \Delta y}\left[\Delta y\, \frac{\varphi_{i+1j} - \varphi_{ij}}{\Delta x} + \Delta y\, \frac{\varphi_{i-1j} - \varphi_{ij}}{\Delta x} + \Delta x\, \frac{\varphi_{ij+1} - \varphi_{ij}}{\Delta y} + \Delta x\, \frac{\varphi_{ij-1} - \varphi_{ij}}{\Delta y}\right], \text{ if } (x,y) \in K_{ij}, \tag{2.4}$$

and

$$\frac{\partial\varphi_h}{\partial n_h}(x,y) = \begin{cases} \dfrac{\varphi_{0j} - \varphi_{1j}}{\Delta x} & \text{if } x = 0 \text{ and } y_{j-\frac{1}{2}} \le y \le y_{j+\frac{1}{2}}, \\ \dfrac{\varphi_{M+1j} - \varphi_{Mj}}{\Delta x} & \text{if } x = L_1 \text{ and } y_{j-\frac{1}{2}} \le y \le y_{j+\frac{1}{2}}, \\ \dfrac{\varphi_{i0} - \varphi_{i1}}{\Delta y} & \text{if } x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}} \text{ and } y = 0, \\ \dfrac{\varphi_{iN+1} - \varphi_{iN}}{\Delta y} & \text{if } x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}} \text{ and } y = L_2. \end{cases} \tag{2.5}$$

Therefore, let $T > 0$ be fixed, and let the time step be $\Delta t = T/N_t$ where N_t is an integer. We can define the space $\mathbf{W}_{hk}$ as the set of functions from $\Omega \times [0,T]$ to $\mathbf{R}$ which are constant over $K_{ij} \times \left[n\Delta t\,, (n+1)\Delta t\,\right)$ for $i = 1, ..., M+1$, $j = 1, ..., N+1$, $(i,j) \neq (M+1, N+1)$ and $n = 1, ..., N_t$. For a function $\varphi \in \mathbf{W}$, we define its approximate function $\varphi_{hk} \in \mathbf{W}_{hk}$ by :

$$\varphi_{hk}(x,y,t) = \varphi_{ij}^n \text{ if } t \in \left[nk, (n+1)k\right) \text{ and } (x,y) \in K_{ij}, \tag{2.6a}$$

where

$$\varphi_{ij}^n = \frac{1}{\Delta x\, \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \varphi(x, y, n\Delta t\,)\, dxdy. \tag{2.6b}$$

In the following sections, we also use the notations $k = \Delta t$ and $h = (\Delta x\,, \Delta y\,)$.

In particular, the meaning of our unknowns u_{ij}^n here, which will be computed with the finite volume scheme described in the Section 3, is to be an approximation of the cell space average at time $t_n = n\Delta t$:

$$u_{ij}^n \simeq \frac{1}{\Delta x\, \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x, y, n\Delta t\,)\, dxdy. \tag{2.7}$$

So we will look for the solution u_{hk} in the space $\mathbf{W}_{hk}$.

Then, we define the discrete dot-products and norms of the functional spaces used. For φ_h, ψ_h in $\mathbf{V}_h$, we write :

- Discrete dot-product and norm corresponding to the dot-product $(.,.)$ and norm $|.|$ of the space $L^2(\Omega)$ are written :

$$(\varphi_h,\, \psi_h)_h = \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y\, \varphi_{ij}\psi_{ij}, \tag{2.8}$$

$$|\varphi_h|_h^2 = (\varphi_h,\, \varphi_h)_h\,. \tag{2.9}$$

- Discrete norm corresponding to the norm $|.|_{L^p}$ of the space $L^p(\Omega)$ is written :

$$|\varphi_h|_{h,L^p}^p = \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y\, \varphi_{ij}^p. \tag{2.10}$$

- We also need to introduce the discrete $H^1(\Omega)$ seminorm and its positive symmetric bilinear form :

$$\begin{aligned}((\varphi_h,\, \psi_h))_h &= (D_h\varphi_h,\, D_h\psi_h)_h \\ &= \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y \frac{\varphi_{ij+1} - \varphi_{ij}}{\Delta y}\, \frac{\psi_{ij+1} - \psi_{ij}}{\Delta y} \\ &+ \sum_{j=1}^{N}\sum_{i=1}^{M} \Delta x\, \Delta y \frac{\varphi_{i+1j} - \varphi_{ij}}{\Delta x}\, \frac{\psi_{i+1j} - \psi_{ij}}{\Delta x},\end{aligned} \tag{2.11}$$

$$||\varphi_h||_h^2 = ((\varphi_h, \varphi_h))_h . \tag{2.12}$$

– Discrete dot-product and norm corresponding to the dot-product $(.,.)_{\partial\Omega}$ and norm $|.|_{\partial\Omega}$ of the space $L^2(\partial\Omega)$ are written :

$$(\varphi_h, \psi_h)_{h,\partial\Omega} = \sum_{j=1}^{N} \Delta y\ (\varphi_{1j}\psi_{1j} + \varphi_{M+1j}\psi_{M+1j}) + \sum_{i=1}^{M} \Delta x\ (\varphi_{i1}\psi_{i1} + \varphi_{iN+1}\psi_{iN+1}), \tag{2.13}$$

$$|\varphi_h|_{h,\partial\Omega}^2 = (\varphi_h, \varphi_h)_{h,\partial\Omega} . \tag{2.14}$$

– The discrete norm corresponding to the norm $|.|_{L^q(\partial\Omega)}$ of the space $L^q(\partial\Omega)$ is written :

$$|\varphi_h|_{h,L^q(\partial\Omega)}^q = \sum_{j=1}^{N} \Delta y\ (\varphi_{1j}^q + \varphi_{M+1j}^q) + \sum_{i=1}^{M} \Delta x\ (\varphi_{i1}^q + \varphi_{iN+1}^q). \tag{2.15}$$

3 Finite volume scheme, weak formulation and existence

In this section, we begin by the description of the scheme used. Then, we give the associated weak formulation and show the existence of the discrete solution obtained from the scheme.

3.1 Finite volume, time and space discretization

The finite volume scheme is suggested by integration of equation (1.1) over each volume K_{ij} for $i = 1, ..., M$ and $j = 1, ..., N$. For the time discretization, we use a Crank-Nicholson finite difference scheme. We arrive at :

$$\begin{aligned}\Delta x\, \Delta y \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = \frac{\nu}{2}\Big(&\Delta y\ \frac{u_{i+1j}^{n+1} - u_{ij}^{n+1}}{\Delta x} + \Delta y\ \frac{u_{i-1j}^{n+1} - u_{ij}^{n+1}}{\Delta x} + \Delta x\ \frac{u_{ij+1}^{n+1} - u_{ij}^{n+1}}{\Delta y} \\ &+ \Delta x\ \frac{u_{ij-1}^{n+1} - u_{ij}^{n+1}}{\Delta y} + \Delta y\ \frac{u_{i+1j}^{n} - u_{ij}^{n}}{\Delta x} + \Delta y\ \frac{u_{i-1j}^{n} - u_{ij}^{n}}{\Delta x} + \Delta x\ \frac{u_{ij+1}^{n} - u_{ij}^{n}}{\Delta y} \\ &+ \Delta x\ \frac{u_{ij-1}^{n} - u_{ij}^{n}}{\Delta y}\Big) - \Delta x\, \Delta y\, \alpha\Big(\frac{u_{ij}^{n+1} + u_{ij}^n}{2}\Big) + \Delta x\, \Delta y\ \frac{f_{ij}^{n+1} + f_{ij}^n}{2},\end{aligned} \tag{3.1}$$

and, in agreement with the boundary conditions (1.2) :

$$\begin{aligned}\frac{\nu}{2}\Big(\Delta y\ \frac{u_{M+1j}^{n+1} - u_{Mj}^{n+1}}{\Delta x} + \Delta y\ \frac{u_{M+1j}^{n} - u_{Mj}^{n}}{\Delta x}\Big) &= \nu\Delta y\ \Big(\frac{g_{M+1j}^{n+1} + g_{M+1j}^n}{2} - \beta\big(\frac{u_{M+1j}^{n+1} + u_{M+1j}^n}{2}\big)\Big), \\ \frac{\nu}{2}\Big(\Delta y\ \frac{u_{0j}^{n+1} - u_{1j}^{n+1}}{\Delta x} + \Delta y\ \frac{u_{0j}^{n} - u_{1j}^{n}}{\Delta x}\Big) &= \nu\Delta y\ \Big(\frac{g_{1j}^{n+1} + g_{1j}^n}{2} - \beta\big(\frac{u_{1j}^{n+1} + u_{1j}^n}{2}\big)\Big), \\ \frac{\nu}{2}\Big(\Delta x\ \frac{u_{iN+1}^{n+1} - u_{iN}^{n+1}}{\Delta y} + \Delta x\ \frac{u_{iN+1}^{n} - u_{iN}^{n}}{\Delta y}\Big) &= \nu\Delta y\ \Big(\frac{g_{iN+1}^{n+1} + g_{iN+1}^n}{2} - \beta\big(\frac{u_{iN+1}^{n+1} + u_{iN+1}^n}{2}\big)\Big), \\ \frac{\nu}{2}\Big(\Delta x\ \frac{u_{i0}^{n+1} - u_{i1}^{n+1}}{\Delta y} + \Delta x\ \frac{u_{i0}^{n} - u_{i1}^{n}}{\Delta y}\Big) &= \nu\Delta y\ \Big(\frac{g_{i1}^{n+1} + g_{i1}^n}{2} - \beta\big(\frac{u_{i1}^{n+1} + u_{i1}^n}{2}\big)\Big),\end{aligned} \tag{3.2}$$

where, for $i = 1, ..., M$, $j = 1, ..., N$, and $n = 0, ..., N_t$:

$$\begin{aligned} g_{i1}^n &= \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(x, 0, n\Delta t\ )\ dx \\ g_{iN+1}^n &= \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(x, L_2, n\Delta t\ )\ dx \\ g_{1j}^n &= \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} g(0, y, n\Delta t\ )\ dy \\ g_{M+1j}^n &= \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} g(L_1, y, n\Delta t\ )\ dy \end{aligned}$$

Remark 3.1. Note that the spatial accuracy order of these boundary conditions is equal to one and not two like the scheme at the interior of the domain. We will study in a separate work the choice of second order boundary conditions.

Remark 3.2. Using the boundary conditions (3.2) and a Newton method, we can compute the values u_{M+1j}^{n+1} and u_{iN+1}^{n+1} (for $i = 1, ..., M$ and $j = 1, ..., N$).

To prove that we are able to compute a finite volume approximation of the exact solution u from this scheme, we have to show the existence of u_{ij}^{n+1} for $i = 1, ..., M$ and $j = 1, ..., N$. This is the aim of the next subsections where we give the weak formulation of (3.1),(3.2), and then prove the existence of $u_h^{n+1} \in \mathbf{V}_h$ for each n.

About the finite volume method for different classes of equations, the reader is refered to e.g. [EGH00] which contains also the study and description of triangular finite volume methods. About the nonlinear heat equation, other computational approaches are described in [EEK94], [Rép92] and [Rou90].

3.2 Discrete weak formulation

We consider $\mathbf{V}_h$ as defined above and look for $u_h \in \mathbf{V}_h$. We have :

Lemma 3.1 (Weak formulation). *With the notations above, equations (3.1), (3.2) can be reinterpreted in the following way :* $u_h^{n+1} \in \mathbf{V}_h$ *and*

$$\frac{1}{\Delta t}\left(u_h^{n+1} - u_h^n, \varphi_h\right)_h + \nu\left(D_h\frac{u_h^{n+1} + u_h^n}{2}, D_h\varphi_h\right)_h + \nu\left(\beta(\frac{u_h^{n+1} + u_h^n}{2}), \varphi_h\right)_{h,\partial\Omega}$$
$$+ \left(\alpha(\frac{u_h^{n+1} + u_h^n}{2}), \varphi_h\right)_h = \nu\left(\frac{g^{n+1} + g^n}{2}, \varphi_h\right)_{h,\partial\Omega} + \left(\frac{f^{n+1} + f^n}{2}, \varphi_h\right)_h, \tag{3.3}$$

for all $\varphi_h \in \mathbf{V}_h$.

Proof :

To obtain the discrete weak formulation, we multiply each side of (3.1) by $(\varphi_{ij})_{i=1,...,M,\, j=1,...,N}$ and sum for $i = 1, ..., M$, $j = 1, ..., N$:

$$
\begin{aligned}
\sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y \frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t}\varphi_{ij} &= \frac{\nu}{2}\sum_{i=1}^{M}\sum_{j=1}^{N}\Big(\Delta y\, \frac{u_{i+1j}^{n+1} - u_{ij}^{n+1}}{\Delta x} + \Delta y\, \frac{u_{i-1j}^{n+1} - u_{ij}^{n+1}}{\Delta x} \\
&+ \Delta x\, \frac{u_{ij+1}^{n+1} - u_{ij}^{n+1}}{\Delta y} + \Delta x\, \frac{u_{ij-1}^{n+1} - u_{ij}^{n+1}}{\Delta y} + \Delta y\, \frac{u_{i+1j}^{n} - u_{ij}^{n}}{\Delta x} + \Delta y\, \frac{u_{i-1j}^{n} - u_{ij}^{n}}{\Delta x} \\
&+ \Delta x\, \frac{u_{ij+1}^{n} - u_{ij}^{n}}{\Delta y} + \Delta x\, \frac{u_{ij-1}^{n} - u_{ij}^{n}}{\Delta y}\Big)\varphi_{ij} - \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y\, \alpha\Big(\frac{u_{ij}^{n+1} + u_{ij}^{n}}{2}\Big)\varphi_{ij} \\
&+ \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y \frac{f_{ij}^{n+1} + f_{ij}^{n}}{2}\varphi_{ij}.
\end{aligned}
$$

This can be rewritten in the following form :

$$
\begin{aligned}
\sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Delta y \frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t}\varphi_{ij} &= \Big(\frac{f^{n+1} + f^{n}}{2}, \varphi_h\Big)_h - \Big(\alpha\big(\frac{u_h^{n+1} + u_h^{n}}{2}\big), \varphi_h\Big)_h \\
&- \frac{\nu}{2}\sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x\, \Big[\frac{u_{ij+1}^{n+1} - u_{ij}^{n+1}}{\Delta y} + \frac{u_{ij+1}^{n} - u_{ij}^{n}}{\Delta y}\Big]\Big[\varphi_{ij+1} - \varphi_{ij}\Big] \\
&+ \frac{\nu}{2}\sum_{i=1}^{M} \Delta x\, \Big[\frac{u_{i0}^{n+1} - u_{i1}^{n+1}}{\Delta y} + \frac{u_{i0}^{n} - u_{i1}^{n}}{\Delta y}\Big]\varphi_{i1} + \frac{\nu}{2}\sum_{i=1}^{M} \Delta x\, \Big[\frac{u_{iN+1}^{n+1} - u_{iN}^{n+1}}{\Delta y} + \frac{u_{iN+1}^{n} - u_{iN}^{n}}{\Delta y}\Big]\varphi_{iN+1} \\
&- \frac{\nu}{2}\sum_{i=1}^{M}\sum_{j=1}^{N} \Delta y\, \Big[\frac{u_{i+1j}^{n+1} - u_{ij}^{n+1}}{\Delta x} + \frac{u_{i+1j}^{n} - u_{ij}^{n}}{\Delta x}\Big]\Big[\varphi_{i+1j} - \varphi_{ij}\Big] \\
&+ \frac{\nu}{2}\sum_{j=1}^{N} \Delta y\, \Big[\frac{u_{0j}^{n+1} - u_{1j}^{n+1}}{\Delta x} + \frac{u_{0j}^{n} - u_{1j}^{n}}{\Delta x}\Big]\varphi_{1j} + \frac{\nu}{2}\sum_{j=1}^{N} \Delta y\, \Big[\frac{u_{M+1j}^{n+1} - u_{Mj}^{n+1}}{\Delta x} + \frac{u_{M+1j}^{n} - u_{Mj}^{n}}{\Delta x}\Big]\varphi_{M+1j}.
\end{aligned}
$$

Then, taking into account the boundary conditions (3.2), we can write :

$$
\begin{aligned}
\frac{1}{\Delta t}\left(u_h^{n+1}-u_h^n,\varphi_h\right)_h &= -\nu\left(D_h\left(\frac{u_h^{n+1}+u_h^n}{2}\right), D_h\varphi_h\right)_h - \left(\alpha\left(\frac{u_h^{n+1}+u_h^n}{2}\right),\varphi_h\right)_h \\
&+ \nu\sum_{j=1}^{N}\Delta y\left(\frac{g_{M+1j}^{n+1}+g_{M+1j}^n}{2}\varphi_{M+1j}+\frac{g_{1j}^{n+1}+g_{1j}^n}{2}\varphi_{1j}\right) \\
&+ \nu\sum_{i=1}^{M}\Delta x\left(\frac{g_{iN+1}^{n+1}+g_{iN+1}^n}{2}\varphi_{iN+1}+\frac{g_{i1}^{n+1}+g_{i1}^n}{2}\varphi_{i1}\right) \\
&- \nu\sum_{j=1}^{N}\Delta y\left(\beta\left(\frac{u_{M+1j}^{n+1}+u_{M+1j}^n}{2}\right)\varphi_{M+1j}+\beta\left(\frac{u_{1j}^{n+1}+u_{1j}^n}{2}\right)\varphi_{1j}\right) \\
&- \nu\sum_{i=1}^{M}\Delta x\left(\beta\left(\frac{u_{iN+1}^{n+1}+u_{iN+1}^n}{2}\right)\varphi_{iN+1}+\beta\left(\frac{u_{i1}^{n+1}+u_{i1}^n}{2}\right)\varphi_{i1}\right) + \left(\frac{f^{n+1}+f^n}{2},\varphi_h\right)_h .
\end{aligned}
$$

Finally, we obtain the discrete weak formulation (3.3) :

$$
\begin{aligned}
&\frac{1}{\Delta t}\left(u_h^{n+1}-u_h^n,\varphi_h\right)_h + \nu\left(D_h\left(\frac{u_h^{n+1}+u_h^n}{2}\right), D_h\varphi_h\right)_h + \nu\left(\beta\left(\frac{u_h^{n+1}+u_h^n}{2}\right),\varphi_h\right)_{h,\partial\Omega} \\
&+ \left(\alpha\left(\frac{u_h^{n+1}+u_h^n}{2}\right),\varphi_h\right)_h = \nu\left(\frac{g^{n+1}+g^n}{2},\varphi_h\right)_{h,\partial\Omega} + \left(\frac{f^{n+1}+f^n}{2},\varphi_h\right)_h .
\end{aligned}
$$

The proof of Lemma 3.1 is complete.

Remark 3.3. For the functions f and g of (1.1) and (1.2), we only need their time approximate functions f_k and g_k defined by :

$$
\begin{aligned}
g_k(x,y,t) &= g^n(x,y) \text{ if } t\in[nk,(n+1)k) \text{ and } (x,y)\in\partial\Omega, \\
f_k(x,y,t) &= f^n(x,y) \text{ if } t\in[nk,(n+1)k) \text{ and } (x,y)\in\Omega.
\end{aligned}
$$

Indeed, we notice that :

$$
\begin{aligned}
(u_{hk},f_k)_h &= \sum_{i=1}^{M}\sum_{j=1}^{N}\int_{K_{ij}} u_{hk}f_k d\Omega \\
&= \sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\,\Delta y\,u_{ij}^n\frac{1}{\Delta x\,\Delta y}\int_{K_{ij}} f^n(x,y)d\Omega \text{ if } t\in[nk,(n+1)k[\\
&\leq |u_{hk}|_h\,|f_k| .
\end{aligned}
$$

So, we only use the continuous space norms for f_k and g_k.

3.3 Existence of the discrete solution

Since in the scheme (3.1), (3.2) and in the weak formulation (3.3) the term $\dfrac{u_{ij}^n + u_{ij}^{n+1}}{2}$ often appears, we introduce the notation :

$$\tau_k\varphi(t) = \varphi(t+\Delta t\ )\ \text{(Translation)}\ ,$$
$$\tilde{u}_{hk}(x,y,t) = \frac{u_{hk}(x,y,t) + \tau_k u_{hk}(x,y,t)}{2} = \tilde{u}_{ij}^{n+1} \text{ if } t \in \big[nk,(n+1)k\big) \text{ and } (x,y) \in K_{ij},$$

where $\tilde{u}_{ij}^{n+1} = \dfrac{u_{ij}^n + u_{ij}^{n+1}}{2}$.

Moreover, before stating the existence theorem, we introduce the discrete forms corresponding to the forms $a(.,.)$, $b(.,.)$ and $< l,. >$ used for the continuous case in Section 1 :

$$\begin{aligned}
a_h\Big(\tilde{u}_h^{n+1},\varphi_h\Big) &= \nu\left(D_h\tilde{u}_h^{n+1},\, D_h\varphi_h\right)_h,\\
b_h\Big(\tilde{u}_h^{n+1},\varphi_h\Big) &= \left(\alpha(\tilde{u}_h^{n+1}),\, \varphi_h\right)_h + \nu\left(\beta(\tilde{u}_h^{n+1}),\, \varphi_h\right)_{h,\partial\Omega},\\
< l_h^{n+1},\varphi_h > &= \nu\left(\frac{g^{n+1}+g^n}{2},\, \varphi_h\right)_{h,\partial\Omega} + \left(\frac{f^{n+1}+f^n}{2},\, \varphi_h\right)_h .
\end{aligned}$$

Theorem 3.1 (Existence of solution for the approximate problem). *We assume that α and β satisfy (1.3a), (1.3b) and (1.4). Moreover, we assume that $u_h\big|_{t=0} = u_{0_h}$, f_k and g_k are given satisfying :*

$$\begin{aligned}
&u_{0_h} \in \boldsymbol{V}_h,\\
&f_k \in L^{p'}(0,T;L^{p'}(\Omega)) \quad \text{with} \quad \frac{1}{p'}+\frac{1}{p} = 1,\\
&g_k \in L^{q'}(0,T;L^{q'}(\partial\Omega)) \quad \text{with} \quad \frac{1}{q'}+\frac{1}{q} = 1.
\end{aligned}$$

Then, for every $n = 0,...,N_t - 1$, there exists a unique solution $\tilde{u}_h^{n+1} = \dfrac{u_h^{n+1}+u_h^n}{2} \in \boldsymbol{V}_h$ such that, for all $\varphi_h \in \boldsymbol{V}_h$:

$$\frac{2}{\Delta t}\left(\tilde{u}_h^{n+1},\, \varphi_h\right)_h + a_h\Big(\tilde{u}_h^{n+1},\varphi_h\Big) + b_h\Big(\tilde{u}_h^{n+1},\varphi_h\Big) = \frac{2}{\Delta t}\left(u_h^n,\, \varphi_h\right)_h + < l_h^{n+1},\varphi_h > .$$

So, for a given u_h^n, we are able to compute u_h^{n+1} which verifies the discrete weak formulation (3.3).

Proof :

To show the existence, we use the following lemma proved in [Lio69], [Tem01] :

Lemma 3.2. *Let $\boldsymbol{X}$ be a finite dimensional Hilbert space with scalar product* $[.,.]$ *and norm* $[.]$ *and let P be a continuous mapping from $\boldsymbol{X}$ into itself such that :*

$$\left[P(\xi),\xi\right] > 0 \quad for \quad \left[\xi\right] = r > 0.$$

Then there exists $\xi \in \boldsymbol{X}$, $[\xi] \leq r$, such that :

$$P(\xi) = 0.$$

Here, we apply this lemma for proving the existence of $\tilde{u}_h^{n+1}$. We define, for all φ_h, $\tilde{u}_h^{n+1} \in \mathbf{V}_h$, the continuous mapping P from $\mathbf{V}_h$ into itself :

$$\begin{aligned}\left[P(\tilde{u}_h^{n+1}),\varphi_h\right] &= \frac{2}{\Delta t}\left(\tilde{u}_h^{n+1},\, \varphi_h\right)_h + a_h\left(\tilde{u}_h^{n+1},\varphi_h\right) + b_h\left(\tilde{u}_h^{n+1},\varphi_h\right) \\ &- \frac{2}{\Delta t}\left(u_h^n,\, \varphi_h\right)_h - < l_h^{n+1},\varphi_h > .\end{aligned}$$

The continuity of the mapping P is obvious because $a_h(.,.)$ is a continuous bilinear form, $< l_h^{n+1},. >$ is a continuous linear form and as α, β are continuous, $b_h(.,.)$ is continuous.

So we just have to show that $[P(\tilde{u}_h^{n+1}),\tilde{u}_h^{n+1}] > 0$ for $\left|\tilde{u}_h^{n+1}\right|_h = r > 0$ sufficiently large :

$$\begin{aligned}\left[P(\tilde{u}_h^{n+1}),\tilde{u}_h^{n+1}\right] &= \frac{2}{\Delta t}\left|\tilde{u}_h^{n+1}\right|_h^2 + a_h\left(\tilde{u}_h^{n+1},\tilde{u}_h^{n+1}\right) + b_h\left(\tilde{u}_h^{n+1},\tilde{u}_h^{n+1}\right) \\ &- \frac{2}{\Delta t}\left(u_h^n,\, \tilde{u}_h^{n+1}\right)_h - < l_h^{n+1},\tilde{u}_h^{n+1} >, \\ &= \frac{2}{\Delta t}\left|\tilde{u}_h^{n+1}\right|_h^2 + \nu\left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \left(\alpha(\tilde{u}_h^{n+1}),\, \tilde{u}_h^{n+1}\right)_h + \nu\left(\beta(\tilde{u}_h^{n+1}),\, \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} \\ &- \frac{2}{\Delta t}\left(u_h^n,\, \tilde{u}_h^{n+1}\right)_h - \nu\left(\frac{g^{n+1}+g^n}{2},\, \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} - \left(\frac{f^{n+1}+f^n}{2},\, \tilde{u}_h^{n+1}\right)_h .\end{aligned}$$

$$\begin{aligned}\left[P(\tilde{u}_h^{n+1}),\tilde{u}_h^{n+1}\right] &\geq \frac{2}{\Delta t}\left|\tilde{u}_h^{n+1}\right|_h^2 + \nu\left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \left(c_1\left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p - c_1'\right) \\ &+ \nu\left(c_3\left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q - c_3'\right) - \frac{2}{\Delta t}\left|u_h^n\right|_h\left|\tilde{u}_h^{n+1}\right|_h \\ &- \nu\left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}\left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)} - \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}\left|\tilde{u}_h^{n+1}\right|_{h,L^p}, \\ &\geq I + II,\end{aligned}$$

where :

$$\begin{aligned}I &= \frac{2}{\Delta t}\left|\tilde{u}_h^{n+1}\right|_h^2 + \nu\left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \left(c_1\left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p - c_1'\right) \\ &- \frac{2}{\Delta t}\left|u_h^n\right|_h\left|\tilde{u}_h^{n+1}\right|_h - \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}\left|\tilde{u}_h^{n+1}\right|_{h,L^p}, \\ II &= \nu\left[\left(c_3\left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q - c_3'\right) - \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}\left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}\right].\end{aligned}$$

Now, we use the Young inequality $ab \le \epsilon a^s + \dfrac{1}{s'(s\epsilon)^{s'/s}} b^{s'}$ with $\dfrac{1}{s'} + \dfrac{1}{s} = 1$ and $\epsilon > 0$:

$$
\begin{aligned}
I \;\ge\; & \frac{2}{\Delta t} \left|\tilde{u}_h^{n+1}\right|_h^2 + \nu \left|D_h \tilde{u}_h^{n+1}\right|_h^2 - \frac{1}{\Delta t} \left|u_h^n\right|_h^2 - \frac{1}{\Delta t} \left|\tilde{u}_h^{n+1}\right|_h^2 - \frac{1}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} \\
- & \frac{c_1}{p} \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p + c_1 \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p - c_1', \\
\ge\; & \frac{1}{\Delta t} \left|\tilde{u}_h^{n+1}\right|_h^2 + \nu \left|D_h \tilde{u}_h^{n+1}\right|_h^2 + c_1\Big(1 - \frac{1}{p}\Big) \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p \\
- & \left(\frac{1}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} + c_1' + \frac{1}{\Delta t} \left|u_h^n\right|_h^2 \right),
\end{aligned}
$$

$$
\begin{aligned}
II \;\ge\; & \nu \left[\left(c_3 \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q - c_3' \right) - \frac{1}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}^{q'} - \frac{c_3}{q} \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q \right], \\
\ge\; & \nu \left[c_3\Big(1 - \frac{1}{q}\Big) \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q - c_3' - \frac{1}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}^{q'} \right].
\end{aligned}
$$

Hence :

$$
\begin{aligned}
\left[P(\tilde{u}_h^{n+1}), \tilde{u}_h^{n+1}\right] \;\ge\; & \frac{1}{\Delta t} \left|\tilde{u}_h^{n+1}\right|_h^2 + \nu \left|D_h \tilde{u}_h^{n+1}\right|_h^2 + c_1\Big(1 - \frac{1}{p}\Big) \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p \\
+ & \nu c_3\Big(1 - \frac{1}{q}\Big) \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q - \left(\frac{\nu}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}^{q'} \right. \\
+ & \left. \frac{1}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} + c_1' + \nu c_3' + \frac{1}{\Delta t} \left|u_h^n\right|_h^2 \right).
\end{aligned}
$$

Thus, since $q > 1$ and $p > 1$, we have $[P(\tilde{u}_h^{n+1}), \tilde{u}_h^{n+1}] > 0$ for $\left|\tilde{u}_h^{n+1}\right|_h = r$ large enough. We can apply the lemma and the existence of $\tilde{u}_h^{n+1} \in \mathbf{V}_h$ is proved.

4 Stability result

Our aim in this section is to perform the stability analysis of this scheme. As $u_h^0 \in L^2(\Omega)$, $f_k \in L^{p'}(0,T;L^{p'}(\Omega))$ and $g_k \in L^{q'}(0,T;L^{q'}(\partial\Omega))$, we know that :

$$
\left|u_h^0\right|_h^2 \le K_1, \tag{4.1}
$$

$$
\Delta t \sum_{n=0}^{N_t - 1} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} \le K_2, \tag{4.2}
$$

$$\Delta t \sum_{n=0}^{N_t-1} \left| \frac{g^{n+1}+g^n}{2} \right|^{q'}_{L^{q'}(\partial\Omega)} \leq K_3. \tag{4.3}$$

Theorem 4.1 (Stability). *We assume that (4.1)-(4.3) hold. Then there exits K_4 independent of Δt and h such that :*

$$|u_h^n|_h^2 \leq K_4 = K_1 + \frac{2\nu}{q'}K_3 + \frac{2}{p'}K_2 + 4\nu T(L_1+L_2)c_3' + 2TL_1L_2c_1',$$

for all $n = 0, ..., N_t$.

Proof :

The first step of this proof consists in proving the following lemma in order to obtain a priori estimates.

Lemma 4.1. *A priori estimates for the discret problem are :*

$$\left|u_h^{n+1}\right|_h^2 \leq |u_h^n|_h^2 + r^n, \tag{4.4}$$

where :

$$\begin{aligned} r^n &= \frac{2\nu\Delta t}{q'c_3^{q'/q}} \left| \frac{g^{n+1}+g^n}{2} \right|^{q'}_{L^{q'}(\partial\Omega)} + \frac{2\Delta t}{p'c_1^{p'/p}} \left| \frac{f^{n+1}+f^n}{2} \right|^{p'}_{L^{p'}} \\ &+ 4\nu\Delta t\,(L_1+L_2)c_3' + 2\Delta t\, L_1L_2c_1'. \end{aligned}$$

Proof :

We replace φ_h by $\tilde{u}_h^{n+1} = \dfrac{u_h^{n+1}+u_h^n}{2}$ in (3.3), we find :

$$\begin{aligned} &\left|u_h^{n+1}\right|_h^2 - |u_h^n|_h^2 + 2\nu\Delta t\,\left|D_h\tilde{u}_h^{n+1}\right|_h^2 + 2\nu\Delta t\,\left(\beta(\tilde{u}_h^{n+1}),\,\tilde{u}_h^{n+1}\right)_{h,\partial\Omega} + 2\Delta t\,\left(\alpha(\tilde{u}_h^{n+1}),\,\tilde{u}_h^{n+1}\right)_h \\ &= 2\nu\Delta t\,\left(\frac{g^{n+1}+g^n}{2},\,\tilde{u}_h^{n+1}\right)_{h,\partial\Omega} + 2\Delta t\,\left(\frac{f^{n+1}+f^n}{2},\,\tilde{u}_h^{n+1}\right)_h. \end{aligned} \tag{4.5}$$

Assumption (1.3b) gives :

$$c_1\left|\tilde{u}_h^{n+1}\right|^p_{h,L^p} - |\Omega|c_1' \leq \left(\alpha(\tilde{u}_h^{n+1}),\,\tilde{u}_h^{n+1}\right)_h \leq c_2\left|\tilde{u}_h^{n+1}\right|^p_{h,L^p} + |\Omega|c_2', \tag{4.6}$$

where $|\Omega| = L_1L_2$ is the area of Ω.

Assumption (1.4) then implies :

$$c_3\left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)} - |\partial\Omega|c_3' \leq \left(\beta(\tilde{u}_h^{n+1}),\,\tilde{u}_h^{n+1}\right)_{h,\partial\Omega} \leq c_4\left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)} + |\partial\Omega|c_4', \tag{4.7}$$

where $|\partial\Omega| = 2L_1 + 2L_2$ is the length of $\partial\Omega$.

We can majorize as follows the terms involving g and f :

$$\begin{aligned}\left(\frac{g^{n+1}+g^n}{2}, \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} &\leq \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)} \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)} \\ &\leq \frac{1}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|^{q'}_{L^{q'}(\partial\Omega)} + \frac{c_3}{q} \left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)}, \end{aligned} \tag{4.8}$$

$$\begin{aligned}\left(\frac{f^{n+1}+f^n}{2}, \tilde{u}_h^{n+1}\right)_{h} &\leq \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}} \left|\tilde{u}_h^{n+1}\right|_{h,L^p} \\ &\leq \frac{1}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|^{p'}_{L^{p'}} + \frac{c_1}{p} \left|\tilde{u}_h^{n+1}\right|^p_{h,L^p}. \end{aligned} \tag{4.9}$$

With (4.6)-(4.9), we arrive at :

$$\begin{aligned} & \left|u_h^{n+1}\right|_h^2 - |u_h^n|_h^2 + 2\nu\Delta t \, \left|D_h\tilde{u}_h^{n+1}\right|_h^2 + 2\nu\Delta t\left(c_3 \left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)} - |\partial\Omega| c_3'\right) \\ + \; & 2\Delta t\left(c_1 \left|\tilde{u}_h^{n+1}\right|^p_{h,L^p} - |\Omega| c_1'\right) \leq \frac{2\nu\Delta t}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|^{q'}_{L^{q'}(\partial\Omega)} + \frac{2\nu\Delta t \, c_3}{q} \left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)} \\ + \; & \frac{2\Delta t}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|^{p'}_{L^{p'}} + \frac{2\Delta t \, c_1}{p} \left|\tilde{u}_h^{n+1}\right|^p_{h,L^p}, \end{aligned}$$

which can be rewritten as :

$$\begin{aligned} & \left|u_h^{n+1}\right|_h^2 - |u_h^n|_h^2 + 2\nu\Delta t \, \left|D_h\tilde{u}_h^{n+1}\right|_h^2 + 2\nu\Delta t \, c_3\big(1-\frac{1}{q}\big) \left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)} \\ + \; & 2\Delta t \, c_1\big(1-\frac{1}{p}\big) \left|\tilde{u}_h^{n+1}\right|^p_{h,L^p} \leq \frac{2\nu\Delta t}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|^{q'}_{L^{q'}(\partial\Omega)} + \frac{2\Delta t}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|^{p'}_{L^{p'}} \\ + \; & 2\nu\Delta t \, |\partial\Omega| c_3' + 2\Delta t \, |\Omega| c_1'. \end{aligned} \tag{4.10}$$

Moreover, since $p > 1$ and $q > 1$, $\dfrac{1}{p'} = 1 - \dfrac{1}{p} > 0$ and $\dfrac{1}{q'} = 1 - \dfrac{1}{q} > 0$, this yields :

$$2\nu\Delta t \, \left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \frac{2\nu\Delta t \, c_3}{q'} \left|\tilde{u}_h^{n+1}\right|^q_{h,L^q(\partial\Omega)} + \frac{2\Delta t \, c_1}{p'} \left|\tilde{u}_h^{n+1}\right|^p_{h,L^p} \geq 0.$$

Finally, we obtain the inequality :

$$\begin{aligned}\left|u_h^{n+1}\right|_h^2 - |u_h^n|_h^2 &\leq \frac{2\nu\Delta t}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|^{q'}_{L^{q'}(\partial\Omega)} + \frac{2\Delta t}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|^{p'}_{L^{p'}} \\ &+ \; 4\nu\Delta t \, (L_1+L_2)c_3' + 2\Delta t \, L_1 L_2 c_1'. \end{aligned}$$

This last inequality is the same as (4.4). So, the proof of Lemma 4.1 is achieved.

To complete Proof of Theorem 4.1, we deduce from (4.4) the following inequality :

$$\left|u_h^{n+1}\right|_h^2 \leq \left|u_h^0\right|_h^2 + \sum_{j=0}^{n} r^j,$$

where :

$$\begin{aligned} r^n &= \frac{2\nu\Delta t}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}^{q'} + \frac{2\Delta t}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} \\ &+ 4\nu\Delta t\,(L_1+L_2)c_3' + 2\Delta t\, L_1 L_2 c_1'. \end{aligned}$$

Thanks to (4.1)-(4.3) :

$$\sum_{j=0}^{n} r^j \leq \frac{2\nu}{q'}K_3 + \frac{2}{p'}K_2 + 4\nu T(L_1+L_2)c_3' + 2TL_1L_2c_1',$$

$$\left|u_h^0\right|_h^2 \leq K_1.$$

Hence :

$$\left|u_h^{n+1}\right|_h^2 \leq K_4 = K_1 + \frac{2\nu}{q'}K_3 + \frac{2}{p'}K_2 + 4\nu T(L_1+L_2)c_3' + 2TL_1L_2c_1'.$$

The stability is proved.

To prove the convergence in the next section, we need some properties which are easy consequences of the stability theorem. We include them in the following corollary :

Corollary 4.1. *We assume that the stability theorem hypotheses (4.1) to (4.3) are satisfied. Then :*

$u_h^{N_t}$ *is bounded in* $L^2(\Omega)$ *independently of* h, (4.11)

u_{hk} *is bounded in* $L^2(0,T;L^2(\Omega))$ *independently of* h *and* k, (4.12)

$D_h\tilde{u}_{hk}$ *is bounded in* $L^2(0,T;L^2(\Omega))$ *independently of* h *and* k, (4.13)

$\gamma_0\tilde{u}_{hk}$ *is bounded in* $L^q(0,T;L^q(\partial\Omega))$ *independently of* h *and* k, (4.14)

$\tilde{u}_{hk}$ *is bounded in* $L^p(0,T;L^p(\Omega))$ *independently of* h *and* k. (4.15)

Proof :

We recall the inequality (4.10) :

$$\begin{aligned} &\left|u_h^{n+1}\right|_h^2 - \left|u_h^n\right|_h^2 + 2\nu\Delta t\,\left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \frac{2\nu\Delta t\, c_3}{q'}\left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q + \frac{2\Delta t\, c_1}{p'}\left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p \\ \leq\; &\frac{2\nu\Delta t}{q' c_3^{q'/q}} \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}^{q'} + \frac{2\Delta t}{p' c_1^{p'/p}} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} + 4\nu\Delta t\,(L_1+L_2)c_3' + 2\Delta t\,(L_1L_2)c_1', \end{aligned}$$

we sum it for $n = 0, ..., N_t - 1$, and we find :

$$\left|u_h^{N_t}\right|_h^2 + 2\nu\Delta t \sum_{n=0}^{N_t-1} \left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \frac{2\nu\Delta t\, c_3}{q'} \sum_{n=0}^{N_t-1} \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q + \frac{2\Delta t\, c_1}{p'} \sum_{n=0}^{N_t-1} \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p$$
$$\leq \frac{2\nu\Delta t}{q' c_3^{q'/q}} \sum_{n=0}^{N_t-1} \left|\frac{g^{n+1}+g^n}{2}\right|_{L^{q'}(\partial\Omega)}^{q'} + \frac{2\Delta t}{p' c_1^{p'/p}} \sum_{n=0}^{N_t-1} \left|\frac{f^{n+1}+f^n}{2}\right|_{L^{p'}}^{p'} + 4\nu \sum_{n=0}^{N_t-1} \Delta t\, (L_1 + L_2) c_3'$$
$$+ \ 2\sum_{n=0}^{N_t-1} \Delta t\, (L_1 L_2) c_1' + \left|u_h^0\right|_h^2 .$$

According to (4.1),(4.2) and (4.3), we have :

$$\left|u_h^{N_t}\right|_h^2 + 2\nu\Delta t \sum_{n=0}^{N_t-1} \left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \frac{2\nu c_3\Delta t}{q'} \sum_{n=0}^{N_t-1} \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q + \frac{2\Delta t\, c_1}{p} \sum_{n=0}^{N_t-1} \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p \leq K_4.$$

Finally, as $p > 1$ and $q > 1$, the terms :

$$\left|u_h^{N_t}\right|_h^2, \Delta t \sum_{n=0}^{N_t-1} \left|D_h\tilde{u}_h^{n+1}\right|_h^2, \Delta t \sum_{n=0}^{N_t-1} \left|\tilde{u}_h^{n+1}\right|_{h,L^q(\partial\Omega)}^q \text{ and } \Delta t \sum_{n=0}^{N_t-1} \left|\tilde{u}_h^{n+1}\right|_{h,L^p}^p$$

are bounded. Moreover, $\Delta t \sum_{n=0}^{N_t-1} |u_h^n|_h^2 \leq TK_4$. The stability corollary is proved.

5 Convergence result

In this section we give a convergence theorem. Let us consider the Banach space $\mathbf{W}$ defined in the introduction, and let u_{hk} be the solution of (3.1)-(3.2) . Then u_{hk} verifies the following convergence properties :

Theorem 5.1. *When $h \to 0$ and $k \to 0$:*

$$\begin{aligned} &u_{hk} \to u \text{ in } L^2(0,T;L^2(\Omega)),\\ &D_h\tilde{u}_{hk} \to \nabla u \text{ in } L^2(0,T;L^2(\Omega)),\\ &\gamma_0\tilde{u}_{hk} \to \gamma_0 u \text{ in } L^q(0,T;L^q(\partial\Omega)),\\ &\tilde{u}_{hk} \to u \text{ in } L^p(0,T;L^p(\Omega)), \end{aligned} \tag{5.1}$$

where u is the weak solution of the exact problem (1.1)-(1.2) (i.e. it verifies (1.5)).

Proof :

First, according to Corollary 4.1, there exist subsequences still denoted u_{hk} such that, when $h \to 0$ and $k \to 0$:

$$u_{hk} \rightharpoonup u \text{ in } L^2(0,T;L^2(\Omega)), \tag{5.2}$$
$$D_h\tilde{u}_{hk} \rightharpoonup \nabla u \text{ in } L^2(0,T;L^2(\Omega)), \tag{5.3}$$
$$\gamma_0\tilde{u}_{hk} \rightharpoonup \gamma_0 u \text{ in } L^q(0,T;L^q(\partial\Omega)) \text{ (due to Lemma 6.2)}, \tag{5.4}$$
$$\tilde{u}_{hk} \rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)). \tag{5.5}$$

Moreover there exist ξ, χ and θ such that, when $h \to 0$, $k \to 0$:

$$u_h^{N_t} \rightharpoonup \xi \text{ in } L^2(\Omega), \tag{5.6}$$
$$\alpha(\tilde{u}_{hk}) \rightharpoonup \chi \text{ in } L^{p'}(0,T;L^{p'}(\Omega)), \tag{5.7}$$
$$\beta(\tilde{u}_{hk}) \rightharpoonup \theta \text{ in } L^{q'}(0,T;L^{q'}(\partial\Omega)). \tag{5.8}$$

We also know (see e.g. [EGH00]) that when $h \to 0$, $k \to 0$:

$$u_h^0 \to u_0 \text{ in } L^2(\Omega), \tag{5.9}$$
$$\tilde{f}_k \to f \text{ in } L^{p'}(0,T;L^{p'}(\Omega)), \tag{5.10}$$
$$\tilde{g}_k \to g \text{ in } L^{q'}(0,T;L^{q'}(\partial\Omega)). \tag{5.11}$$

For φ fixed in $\mathbf{V}$, we define φ_h as in (2.1a), (2.1b). More precisely we choose φ in a dense subspace of $\mathbf{V}$ made of smooth functions, say $\varphi \in \mathbf{V} \cap \mathcal{C}^3(\bar{\Omega})$. In Section 6, it is easy to simplify Lemma 6.1 in order to prove that, for such a smooth φ, we have :

$$\varphi_h \to \varphi \text{ in } L^p(\Omega) \text{ for all } p > 1, \tag{5.12}$$
$$\gamma_0\varphi_h \to \gamma_0\varphi \text{ in } L^q(\partial\Omega) \text{ for all } q > 1, \tag{5.13}$$
$$D_h\varphi_h \to \nabla\varphi \text{ in } L^2(\Omega)^2. \tag{5.14}$$

Moreover, we consider a function ψ which is regular and only time dependent, $\psi \in \mathcal{C}^2([0,T])$, and such that $\psi(T) = 0$. We define ψ_k by :

$$\psi_k(t) = \psi^n \quad \text{if } t \in [t_n, t_{n+1}).$$

Thus, we have the following convergences :

$$\psi_k \to \psi \quad \text{in } L^2(0,T), \tag{5.15}$$
$$\partial_t\psi_k \to \partial_t\psi \quad \text{in } L^2(0,T), \tag{5.16}$$

where $\partial_t\psi_k(t) = (\psi^{n+1} - \psi^n)/\Delta t$ if $t \in \big[t_n, t_{n+1}\big)$.

The first step of this proof consists in passing to the limit in (3.3). For that purpose, we replace φ_h by $\varphi_h\psi^n$ in (3.3), we sum for $n = 0, ..., N_t$, and we obtain :

$$\begin{aligned} &\sum_{n=0}^{N_t-1} \left(u_h^{n+1} - u_h^n, \varphi_h\psi^n\right)_h + \nu\Delta t \sum_{n=0}^{N_t-1} \left(D_h\tilde{u}_h^{n+1}, D_h\varphi_h\psi^n\right)_h \\ + &\ \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}), \varphi_h\psi^n\right)_{h,\partial\Omega} + \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}), \varphi_h\psi^n\right)_h \\ = &\ \nu\Delta t \sum_{n=0}^{N_t-1} \left(\frac{g^{n+1}+g^n}{2}, \varphi_h\psi^n\right)_{h,\partial\Omega} + \Delta t \sum_{n=0}^{N_t-1} \left(\frac{f^{n+1}+f^n}{2}, \varphi_h\psi^n\right)_h . \end{aligned} \tag{5.17}$$

Now, we pass to the limit in the above equation when $h \to 0$, $k \to 0$:

– For the first term, using the regularity of ψ, we find :

$$\begin{aligned}
&\sum_{n=0}^{N_t-1} \left(u_h^{n+1} - u_h^n,\, \varphi_h \psi^n\right)_h \\
&= \sum_{n=0}^{N_t-1} \left(u_h^{n+1},\, \varphi_h \psi^n\right)_h - \sum_{n=0}^{N_t-1} \left(u_h^n,\, \varphi_h \psi^n\right)_h , \\
&= \sum_{n=1}^{N_t} \left(u_h^n,\, \varphi_h \psi^{n-1}\right)_h - \sum_{n=0}^{N_t-1} \left(u_h^n,\, \varphi_h \psi^n\right)_h , \\
&= \left(u_h^{N_t},\, \varphi_h \psi^{N_t}\right)_h - \left(u_h^0,\, \varphi_h \psi^0\right)_h - \sum_{n=1}^{N_t} \left(u_h^n,\, \varphi_h\right)_h (\psi^n - \psi^{n-1}), \\
&= \left(u_h^{N_t},\, \varphi_h \psi^{N_t}\right)_h - \left(u_h^0,\, \varphi_h \psi^0\right)_h - \sum_{n=1}^{N_t} \left(u_h^n,\, \varphi_h\right)_h \Delta t \left(\frac{d}{dt}\psi\right)^n + \mathcal{O}(\Delta t\,), \\
&= \left(u_h^{N_t},\, \varphi_h \psi^{N_t}\right)_h - \left(u_h^0,\, \varphi_h \psi^0\right)_h - \Delta t \sum_{n=0}^{N_t} \left(u_h^n,\, \varphi_h\right)_h \left(\frac{d}{dt}\psi\right)^n + \mathcal{O}(\Delta t\,).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=0}^{N_t-1} \left(u_h^{n+1} - u_h^n,\, \varphi_h \psi^n\right)_h \;&\to\; (\xi,\, \varphi)\, \psi\Big|_{t=T} - (u_0,\, \varphi)_h\, \psi\Big|_{t=0} - \int_0^T (u,\, \varphi)\, \frac{d}{dt}\psi \, dt, \\
&= \; - (u_0,\, \varphi)_h\, \psi\Big|_{t=0} - \int_0^T (u,\, \varphi)\, \frac{d}{dt}\psi \, dt, \qquad (5.18)
\end{aligned}$$

due to the convergences (5.6), (5.9), (5.12), (5.15) and (5.16).

– Due to the convergences (5.3), (5.14) and (5.15), we have :

$$\nu \Delta t \sum_{n=0}^{N_t-1} \left(D_h \tilde{u}_h^{n+1},\, D_h \varphi_h \psi^n\right)_h \to \nu \int_0^T (\nabla u,\, \nabla \varphi)\, \psi dt. \qquad (5.19)$$

– Due to the convergences (5.8), (5.13) and (5.15), we have :

$$\nu \Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}),\, \varphi_h \psi^n\right)_{h,\partial\Omega} \to \nu \int_0^T (\theta,\, \varphi)_{\partial\Omega}\, \psi dt. \qquad (5.20)$$

– Due to the convergences (5.7), (5.12) and (5.15), we have :

$$\Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}),\, \varphi_h \psi^n\right)_h \to \int_0^T (\chi,\, \varphi)\, \psi dt. \qquad (5.21)$$

- Due to the convergences (5.11), (5.13) and (5.15), we have :

$$\nu\Delta t\sum_{n=0}^{N_t-1}\left(\frac{g^{n+1}+g^n}{2},\,\varphi_h\psi^n\right)_{h,\partial\Omega}\to\nu\int_0^T(g,\,\varphi)_{\partial\Omega}\,\psi dt. \tag{5.22}$$

- Due to the convergences (5.10), (5.12) and (5.15), we have :

$$\Delta t\sum_{n=0}^{N_t-1}\left(\frac{f^{n+1}+f^n}{2},\,\varphi_h\psi^n\right)_h\to\int_0^T(f,\,\varphi)\,\psi dt. \tag{5.23}$$

According to (5.18)-(5.23), we obtain :

$$\begin{aligned}&-(u_0,\,\varphi)_h\,\psi\Big|_{t=0}-\int_0^T(u,\,\varphi)\,\frac{d}{dt}\psi\,dt+\nu\int_0^T(\nabla u,\,\nabla\varphi)\,\psi\,dt+\nu\int_0^T(\theta,\,\varphi)_{\partial\Omega}\,\psi\,dt\\&+\int_0^T(\chi,\,\varphi)\,\psi\,dt=\nu\int_0^T(g,\,\varphi)_{\partial\Omega}\,\psi\,dt+\int_0^T(f,\,\varphi)\,\psi\,dt.\end{aligned} \tag{5.24}$$

Now, writing in particular (5.24) with $\psi\in\mathcal{D}(0,T)$, we find the following equality which is valid in the distribution sense on $(0,T)$:

$$\frac{d}{dt}(u,\,\varphi)+\nu\,(\nabla u,\,\nabla\varphi)+\nu\,(\theta,\,\varphi)_{\partial\Omega}+(\chi,\,\varphi)=\nu\,(g,\,\varphi)_{\partial\Omega}+(f,\,\varphi), \tag{5.25}$$

for all $\varphi\in\mathbf{V}$.

Therefore, as f is in $L^{p'}(0,T;L^{p'}(\Omega))$, $u|_{t=0}$ and $u|_{t=T}$ make sense. Multiplying (5.25) by ψ (the same ψ as before), integrating with respect to t and integrating by parts, we find :

$$\int_0^T\frac{d}{dt}(u,\,\varphi)\,\psi\,dt=-\int_0^T(u,\,\varphi)\,\frac{d}{dt}\psi\,dt+(u|_{t=0},\,\varphi)\,\psi|_{t=0}, \tag{5.26}$$

and thus :

$$\begin{aligned}&-\left(u\Big|_{t=0},\,\varphi\right)_h\psi\Big|_{t=0}-\int_0^T(u,\,\varphi)\,\frac{d}{dt}\psi\,dt+\nu\int_0^T(\nabla u,\,\nabla\varphi)\,\psi\,dt+\nu\int_0^T(\theta,\,\varphi)_{\partial\Omega}\,\psi\,dt\\&+\int_0^T(\chi,\,\varphi)\,\psi\,dt=\nu\int_0^T(g,\,\varphi)_{\partial\Omega}\,\psi\,dt+\int_0^T(f,\,\varphi)\,\psi\,dt.\end{aligned} \tag{5.27}$$

By comparison with (5.24), we see that :

$$\left(u\Big|_{t=0}-u_0,\,\varphi\right)\psi\Big|_{t=0}=0, \tag{5.28}$$

for each $\varphi\in\mathbf{V}$, and for each function ψ of the type considered. We can choose ψ such that $\psi\Big|_{t=0}\neq 0$, and therefore

$$\left(u\Big|_{t=0}-u_0,\,\varphi\right)=0,\qquad\text{for all }\varphi\in\mathbf{V}. \tag{5.29}$$

This equality implies that

$$u\Big|_{t=0} = u_0. \tag{5.30}$$

Considering then arbitrary functions $\psi \in \mathcal{C}^2([0,T])$ (i.e. $\psi\Big|_{t=T}$ is not necessary equal to 0), using the same method and (5.30), we find that

$$\xi = u\Big|_{t=T}. \tag{5.31}$$

Thus, we have shown that :

$$u_h^{N_t} \rightharpoonup u\big|_{t=T} \text{ in } L^2(\Omega), \tag{5.32}$$

and that, u is the solution of the problem :

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + \chi = f \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial n} + \theta = g \quad \text{on } \partial\Omega, \end{aligned}$$

i.e. u verifies :

$$\int_0^T (\partial_t u,\, \varphi)\; dt + \nu \int_0^T (\nabla u,\, \nabla\varphi) + \nu\int_0^T (\theta,\, \varphi)_{\partial\Omega} + \int_0^T (\chi,\, \varphi) = \nu\int_0^T (g,\, \varphi)_{\partial\Omega} + \int_0^T (f,\, \varphi). \tag{5.33}$$

for all $\varphi \in \mathbf{W}$.
Now, in order to show that $\chi = \alpha(u)$ and $\theta = \beta(u)$, we consider the following expression :

$$\begin{aligned} \mathbf{X}_{hk}(\varphi_{hk}) &= \frac{1}{2}\left|u_h^{N_t} - \varphi_h^{N_t}\right|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} \left|D_h(\tilde{u}_h^{n+1} - \varphi_h^n)\right|_h^2 \\ &+ \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}) - \alpha(\varphi_h^n),\, \tilde{u}_h^{n+1} - \varphi_h^n\right)_h \\ &+ \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}) - \beta(\varphi_h^n),\, \tilde{u}_h^{n+1} - \varphi_h^n\right)_{h,\partial\Omega}, \end{aligned} \tag{5.34}$$

for $\varphi_{hk} \in \mathbf{W}_{hk}$. We refer to [Lio69] for the motivation for considering this expression. An important property of $\mathbf{X}_{hk}$, due to the monotony of the nonlinear terms α and β, is :

$$\mathbf{X}_{hk} \geq 0 \text{ for all } \varphi_{hk} \in \mathbf{W}_{hk}. \tag{5.35}$$

We want to pass to the limit in $\mathbf{X}_{hk}$ when $h \to 0$ and $k \to 0$. So, for a fixed function φ in $\mathbf{W}$, we define the function φ_{hk} as in (2.6a)(2.6b). As in the first part of this proof, we choose φ in a dense subspace of $\mathbf{W}$ made of smooth functions, say $\varphi \in \mathcal{C}^3(\bar{\Omega} \times [0,T])$. For such a smooth φ, Lemma 6.1 gives us :

$$D_h\varphi_{hk} \to \nabla\varphi \text{ in } L^2(0,T;L^2(\Omega)), \tag{5.36}$$

$$\varphi_h \to \varphi \text{ in } L^p(0,T;L^p(\Omega)) \text{ for all } p > 1, \tag{5.37}$$

$$\gamma_0\varphi_h \to \gamma_0\varphi \text{ in } L^q(0,T;L^q(\partial\Omega)) \text{ for all } q > 1. \tag{5.38}$$

We rewrite $\mathbf{X}_{hk}$ as :

$$\mathbf{X}_{hk}(\varphi_{hk}) = I + II + III,$$

where

$$\begin{aligned} I &= \frac{1}{2}\left|u_h^{N_t}\right|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} |D_h\tilde{u}_h^{n+1}|_h^2 + \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}), \tilde{u}_h^{n+1}\right)_h \\ &+ \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}), \tilde{u}_h^{n+1}\right)_{h,\partial\Omega}, \\ II &= \frac{1}{2}\left|\varphi_h^{N_t}\right|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} |D_h\varphi_h^n|_h^2 + \Delta t \sum_{n=0}^{N_t-1} (\alpha(\varphi_h^n), \varphi_h^n)_h + \nu\Delta t \sum_{n=0}^{N_t-1} (\beta(\varphi_h^n), \varphi_h^n)_{h,\partial\Omega}, \end{aligned}$$

and

$$\begin{aligned} III &= -\left(u_h^{N_t}, \varphi_h^{N_t}\right)_h - 2\nu\Delta t \sum_{n=0}^{N_t-1} \left(D_h\tilde{u}_h^{n+1}, D_h\varphi_h^n\right)_h - \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}), \varphi_h^n\right)_h \\ &- \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\varphi_h^n), \tilde{u}_h^{n+1}\right)_h - \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}), \varphi_h^n\right)_{h,\partial\Omega} - \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\varphi_h^n), \tilde{u}_h^{n+1}\right)_{h,\partial\Omega}. \end{aligned}$$

Then, we replace φ_h by $\tilde{u}_h^{n+1} = \dfrac{u_h^{n+1} + u_h^n}{2}$ in the discrete weak formulation (3.3) and sum for $n = 0, ..., N_t - 1$; we obtain :

$$\begin{aligned} &\frac{1}{2}\left|u_h^{N_t}\right|_h^2 - \frac{1}{2}|u_h^0|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} \left|D_h\tilde{u}_h^{n+1}\right|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}), \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} \\ &+ \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}), \tilde{u}_h^{n+1}\right)_h = \nu\Delta t \sum_{n=0}^{N_t-1} \left(\frac{g^{n+1}+g^n}{2}, \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} \\ &+ \Delta t \sum_{n=0}^{N_t-1} \left(\frac{f^{n+1}+f^n}{2}, \tilde{u}_h^{n+1}\right)_h. \end{aligned} \tag{5.39}$$

Due to the above equation, we can rewrite I as :

$$I = \frac{1}{2}|u_h^0|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} \left(\frac{g^{n+1}+g^n}{2}, \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} + \Delta t \sum_{n=0}^{N_t-1} \left(\frac{f^{n+1}+f^n}{2}, \tilde{u}_h^{n+1}\right)_h.$$

Now, we study the convergence of each term in $\mathbf{X}_{hk}$ when $h \to 0$ and $k \to 0$:

- For the first term in $\mathbf{X}_{hk}$:

$$\begin{aligned} I &= \frac{1}{2}|u_h^0|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} \left(\frac{g^{n+1}+g^n}{2}, \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} + \Delta t \sum_{n=0}^{N_t-1} \left(\frac{f^{n+1}+f^n}{2}, \tilde{u}_h^{n+1}\right)_h, \\ &\to \frac{1}{2}|u_0|^2 + \nu \int_0^T \int_{\partial\Omega} gu \, d\Gamma dt + \int_0^T \int_\Omega fu \, d\Omega dt. \end{aligned}$$

For this, we have used the convergences (5.9) for the first term, (5.11) and (5.4) for the second term, (5.10) and (5.5) for the third term.

- For the second term in $\mathbf{X}_{hk}$:

$$\begin{aligned} II &= \frac{1}{2}\left|\varphi_h^{N_t}\right|_h^2 + \nu\Delta t \sum_{n=0}^{N_t-1} |D_h\varphi_h^n|_h^2 + \Delta t \sum_{n=0}^{N_t-1} (\alpha(\varphi_h^n), \varphi_h^n)_h + \nu\Delta t \sum_{n=0}^{N_t-1} (\beta(\varphi_h^n), \varphi_h^n)_{h,\partial\Omega} \\ &\to \frac{1}{2}\left|\varphi\big|_{t=T}\right|^2 + \nu\int_0^T\int_\Omega \left|\nabla\varphi\right|^2 d\Omega dt + \int_0^T\int_\Omega \alpha(\varphi)\varphi\ d\Omega dt + \nu\int_0^T\int_{\partial\Omega} \beta(\varphi)\varphi\ d\Gamma dt. \end{aligned}$$

This convergence follows from (5.36)-(5.38) and we note that α is continuous from $L^p(0,T;L^p(\Omega))$ into $L^{p'}(0,T;L^{p'}(\Omega))$ and β is continuous from $L^q(0,T;L^q(\partial\Omega))$ into $L^{q'}(0,T;L^{q'}(\partial\Omega))$.

- For the third term in $\mathbf{X}_{hk}$:

$$\begin{aligned} III &= -\left(u_h^{N_t}, \varphi_h^{N_t}\right)_h - 2\nu\Delta t \sum_{n=0}^{N_t-1} \left(D_h\tilde{u}_h^{n+1}, D_h\varphi_h^n\right)_h - \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\tilde{u}_h^{n+1}), \varphi_h^n\right)_h \\ &- \Delta t \sum_{n=0}^{N_t-1} \left(\alpha(\varphi_h^n), \tilde{u}_h^{n+1}\right)_h - \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\tilde{u}_h^{n+1}), \varphi_h^n\right)_{h,\partial\Omega} - \nu\Delta t \sum_{n=0}^{N_t-1} \left(\beta(\varphi_h^n), \tilde{u}_h^{n+1}\right)_{h,\partial\Omega} \\ &\to -\int_\Omega u\big|_{t=T}\varphi\big|_{t=T}\ d\Omega - 2\nu\int_0^T\int_\Omega \nabla u.\nabla\varphi\ d\Omega dt - \int_0^T\int_\Omega \chi\varphi\ d\Omega dt \\ &\quad -\int_0^T\int_\Omega \alpha(\varphi)u\ d\Omega dt - \nu\int_0^T\int_{\partial\Omega} \theta\varphi\ d\Gamma dt - \nu\int_0^T\int_{\partial\Omega} \beta(\varphi)u\ d\Gamma dt \end{aligned}$$

For this, we have used the convergences (5.2)-(5.8), (5.36)-(5.38), and the same arguments than for the second term II in $\mathbf{X}_{hk}$.

Finally, we have shown that :

$$\mathbf{X}_{hk} \to \mathbf{X}, \text{ when } h \to 0 \text{ and } k \to 0, \tag{5.40}$$

where

$$\begin{aligned} \mathbf{X}(\varphi) &= \frac{1}{2}|u_0|^2 + \nu\int_0^T\int_{\partial\Omega} gu\ d\Gamma dt + \int_0^T\int_\Omega fu\ d\Omega dt + \frac{1}{2}\left|\varphi\big|_{t=T}\right|^2 \\ &+ \nu\int_0^T\int_{\partial\Omega} \beta(\varphi)\varphi\ d\Gamma dt + \int_0^T\int_\Omega \alpha(\varphi)\varphi\ d\Omega dt + \nu\int_0^T\int_\Omega \left|\nabla\varphi\right|^2 d\Omega dt \\ &- \int_\Omega u\big|_{t=T}\varphi\big|_{t=T}\ d\Omega - 2\nu\int_0^T\int_\Omega \nabla u.\nabla\varphi\ d\Omega dt - \int_0^T\int_\Omega \chi\varphi\ d\Omega dt \\ &- \int_0^T\int_\Omega \alpha(\varphi)u\ d\Omega dt - \nu\int_0^T\int_{\partial\Omega} \theta\varphi\ d\Gamma dt - \nu\int_0^T\int_{\partial\Omega} \beta(\varphi)u\ d\Gamma dt. \end{aligned} \tag{5.41}$$

Note that, using a continuity argument, the equation (5.41) is satisfied for all $\varphi \in \mathbf{W}$.

Of course, $\mathbf{X}(\varphi) \geq 0$ is true for all $\varphi \in \mathbf{W}$, since $\mathbf{X}_{hk}(\varphi_{hk}) \geq 0$ for all $\varphi_{hk} \in \mathbf{W}_{hk}$.
To achieve the proof of the convergence, let $\varphi = u + \lambda w$ with $w \in \mathbf{W}$, $\lambda > 0$, then we have :

$$\begin{aligned} 0 \leq \mathbf{X}(u+\lambda w) &= \frac{1}{2}\lambda^2 \Big| w\big|_{t=T}\Big|^2_{L^2(\Omega)} + \nu\lambda^2 \int_0^T \int_\Omega |\nabla w|^2 d\Omega dt \\ &- \lambda \int_0^T \int_\Omega \big(\chi - \alpha(u+\lambda w)\big) w d\Omega dt - \lambda \int_0^T \int_{\partial\Omega} \big(\theta - \beta(u+\lambda w)\big) w d\Gamma dt. \end{aligned}$$

After dividing by $\lambda > 0$ and letting $\lambda \to 0$, we obtain :

$$\int_0^T \int_\Omega \big(\chi - \alpha(u)\big) w d\Omega dt + \int_0^T \int_{\partial\Omega} \big(\theta - \beta(u)\big) w d\Gamma dt \;\geq\; 0, \;\; \forall w \in \mathbf{W}. \tag{5.42}$$

Replacing w by $-w$ we obtain the opposite inequality, and hence :

$$\int_0^T \int_\Omega \big(\chi - \alpha(u)\big) w d\Omega dt + \int_0^T \int_{\partial\Omega} \big(\theta - \beta(u)\big) w d\Gamma dt \;=\; 0, \;\; \forall w \in \mathbf{W}. \tag{5.43}$$

Finally, considering the functions $w \in \mathbf{W}$ such that $\gamma_0 w = 0$, we obtain that $\chi = \alpha(u)$. Considering then the functions $w \in \mathbf{W}$ such that $\gamma_0 w$ is arbitrary, we find $\theta = \beta(u)$.

It means that u verify (1.1) and (1.2). Hence the convergence results (5.1) are proved, with weak convergences only. To prove the strong convergences, we observe that $\mathbf{X}_{hk}(u_{hk})$ converges to $\mathbf{X}(u) = 0$; hence $D_h u_{hk} \to \nabla u$ in $L^2(0,T;L^2(\Omega))$ strongly, and then, by the Poincare inequality, $u_{hk} \to u$ in $L^2(0,T;L^2(\Omega))$ strongly.

6 Technical lemmas

In this last section, we prove the technical lemmas used in Section 5 where we have shown the convergence.

Lemma 6.1. *For a function φ in $\boldsymbol{W}$ which is supposed enough regular, say $\varphi \in \mathcal{C}^3(\bar{\Omega} \times [0,T])$, we define its approximate function φ_{hk} in $\boldsymbol{W}_{hk}$ as in (2.6a), (2.6b). Moreover, the discrete operators used are defined in (2.2)-(2.5).*
Then, for such a function φ and for all $r > 1$, we have the following convergences :

$$\begin{aligned} &\varphi_{hk} \to \varphi \text{ in } L^r(0,T;L^r(\Omega)), \\ &\gamma_0\varphi_{hk} \to \gamma_0\varphi \text{ in } L^r(0,T;L^r(\partial\Omega)), \\ &D_h\varphi_{hk} \to \nabla\varphi \text{ in } L^r(0,T;L^r(\Omega)^2), \\ &\triangle_h\varphi_{hk} \to \triangle\varphi \text{ in } L^r(0,T;L^r(\Omega)), \\ &\frac{\partial \varphi_h}{\partial n_h} \to \frac{\partial \varphi}{\partial n} \text{ in } L^r(0,T;L^r(\partial\Omega)). \end{aligned} \tag{6.1}$$

Proof :

In this proof, we denote by C all the constants which are independent of h, k and by C_r the constants which are independent of h, k but dependent on r. We also write $t_n = n\Delta t$.

First, we have to show that the $L^r(0,T;L^r(\Omega))$-norm of $\varphi - \varphi_{hk}$ converges to 0 when $h \to 0$ and $k \to 0$. We set

$$I = \int_0^T \int_\Omega |\varphi - \varphi_{hk}|^r \, d\Omega dt,$$

and we bound I as follows :

$$\begin{aligned}
I &= \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} |\varphi - \varphi_{hk}|^r \, d\Omega dt, \\
&= \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} |\varphi - \varphi_{ij}^n|^r \, d\Omega dt, \\
&= \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \varphi(x,y,t) - \frac{1}{\Delta x \, \Delta y} \int_{K_{ij}} \varphi(x',y',t_n) dx'dy' \right|^r dxdydt, \\
&= \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \frac{1}{\Delta x \, \Delta y} \int_{K_{ij}} (\varphi(x,y,t) - \varphi(x',y',t_n)) dx'dy' \right|^r dxdydt, \\
&\le \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \frac{1}{\Delta x \, \Delta y} \int_{K_{ij}} |\varphi(x,y,t) - \varphi(x',y',t_n)| \, dx'dy' \right|^r dxdydt.
\end{aligned}$$

Since φ is in $C^1(\bar{\Omega} \times [0,T])$, we have :

$$I \le \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \frac{1}{\Delta x \, \Delta y} \int_{K_{ij}} \left| C\sqrt{\Delta x^{\,2} + \Delta y^{\,2} + \Delta t^{\,2}} \right| dx'dy' \right|^r dxdydt.$$

Hence :

$$\int_0^T \int_\Omega |\varphi - \varphi_{hk}|^r \, d\Omega dt \le |C|^r |\Omega| T (\Delta x^{\,2} + \Delta y^{\,2} + \Delta t^{\,2})^{\frac{r}{2}}.$$

This implies that $\varphi_{hk} \to \varphi$ in $L^r(0,T;L^r(\Omega))$ for all $r > 1$.

Then, for the second convergence, we show that the $L^r(0,T;L^r(\partial\Omega))$-norm of $\gamma_0\varphi - \gamma_0\varphi_{hk}$

converges to 0 when $h \to 0$ and $k \to 0$. We have :

$$
\begin{aligned}
\int_0^T \int_{\partial\Omega} |\gamma_0\varphi - \gamma_0\varphi_{hk}|^r \, d\Gamma dt &= \sum_{n=0}^{N_t-1}\sum_{i=1}^{M} \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} |\varphi(x,0,t) - \varphi_{i1}^n|^r \, dxdt \\
&+ \sum_{n=0}^{N_t-1}\sum_{i=1}^{M} \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} |\varphi(x,L_2,t) - \varphi_{iN+1}^n|^r \, dxdt \\
&+ \sum_{n=0}^{N_t-1}\sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} |\varphi(0,y,t) - \varphi_{1j}^n|^r \, dydt \\
&+ \sum_{n=0}^{N_t-1}\sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} |\varphi(L_1,j,t) - \varphi_{M+1j}^n|^r \, dxdt,
\end{aligned}
\tag{6.2}
$$

where φ_{M+1j} and φ_{iN+1} are defined by a second order extrapolation :

$$
\begin{aligned}
\varphi_{M+1j}^n &= \frac{2}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \varphi(L_1,y,t_n)\, dy - \varphi_{Mj}^n, \\
\varphi_{iN+1}^n &= \frac{2}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \varphi(x,L_2,t_n)\, dx - \varphi_{iN}^n.
\end{aligned}
$$

Now, let us see how we can majorize the two first terms of the right hand side of (6.2) ; the same arguments can be used for the two other terms :

$$
\begin{aligned}
&\sum_{n=0}^{N_t-1}\sum_{i=1}^{M} \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} |\varphi(x,0,t) - \varphi_{i1}^n|^r \, dxdt, \\
&= \sum_{n=0}^{N_t-1}\sum_{i=1}^{M} \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left|\varphi(x,0,t) - \frac{1}{\Delta x\,\Delta y}\int_{K_{i1}} \varphi(x',y',t_n)\, dx'dy'\right|^r dxdt, \\
&= \sum_{n=0}^{N_t-1}\sum_{i=1}^{M} \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left|\frac{1}{\Delta x\,\Delta y}\int_{K_{i1}} \varphi(x,0,t) - \varphi(x',y',t_n)\, dx'dy'\right|^r dxdt, \\
&\le \sum_{n=0}^{N_t-1}\sum_{i=1}^{M} \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left|\frac{1}{\Delta x\,\Delta y}\int_{K_{i1}} |\varphi(x,0,t) - \varphi(x',y',t_n)|\, dx'dy'\right|^r dxdt, \\
&\le TL_1|C|^r(\Delta x^{\,2} + \Delta y^{\,2} + \Delta t^{\,2})^{\frac{r}{2}},
\end{aligned}
$$

$$\begin{aligned}
&\sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\int_{t_n}^{t_{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left|\varphi(x,L_2,t)-\varphi_{iN+1}^n\right|^r dxdt,\\
=\ &\sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\int_{t_n}^{t_{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left|\varphi(x,L_2,t)-\frac{2}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\varphi(x',L_2,t_n)\,dx'+\varphi_{iN}^n\right|^r dxdt,\\
=\ &\sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\int_{t_n}^{t_{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left|2\Big(\varphi(x,L_2,t)-\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\varphi(x',L_2,t_n)\,dx'\Big)-\Big(\varphi(x,L_2,t)-\varphi_{iN}^n\Big)\right|^r dxdt,\\
\leq\ &\sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\int_{t_n}^{t_{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\Bigg|\frac{2}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\Big|\varphi(x,L_2,t)-\varphi(x',L_2,t_n)\Big|\,dx'\\
+\ &\frac{1}{\Delta x\,\Delta y}\int_{K_{iN}}\Big|\varphi(x,L_2,t)-\varphi(x',y',t_n)\Big|\,dx'dy'\Bigg|^r dxdt,\\
\leq\ &TL_1|3C|^r(\Delta x^{\,2}+\Delta y^{\,2}+\Delta t^{\,2})^{\frac{r}{2}}.
\end{aligned}$$

This yields :

$$\int_0^T\int_{\partial\Omega}|\gamma_0\varphi-\gamma_0\varphi_{hk}|^r\,d\Gamma dt\ \leq\ (3^r+1)|C|^r|\partial\Omega|T(\Delta x^{\,2}+\Delta y^{\,2}+\Delta t^{\,2})^{\frac{r}{2}},$$

and this implies that $\gamma_0\varphi_{hk}\to\gamma_0\varphi$ in $L^r(0,T;L^r(\partial\Omega))$ for r fixed, $r>1$, arbitrary.

About the third convergence in (6.1), we show that the $L^r(0,T;L^r(\Omega)^2)$-norm of $\nabla\varphi-D_h\varphi_{hk}$ converges to 0 when $h\to 0$ and $k\to 0$:

$$\begin{aligned}
\int_0^T\int_\Omega|\nabla\varphi-D_h\varphi_{hk}|^r\,dxdydt\ &=\ \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\sum_{j=1}^{N}\int_{t_n}^{t_{n+1}}\int_{K_{ij}}|\nabla\varphi-D_h\varphi_{hk}|^r\,dxdydt\\
&=\ \sum_{n=0}^{N_t-1}\sum_{j=1}^{N}\sum_{i=1}^{M}\int_{t_n}^{t_{n+1}}\int_{K_{ij}}\Bigg[\left|\partial_x\varphi-\frac{\varphi_{i+1j}^n-\varphi_{ij}^n}{\Delta x}\right|^r\\
&+\ \left|\partial_y\varphi-\frac{\varphi_{ij+1}^n-\varphi_{ij}^n}{\Delta y}\right|^r\Bigg]dxdydt
\end{aligned}$$

Considering for example the first term of the above equation, using the Taylor formula and the

$\mathcal{C}^2$-regularity of φ, we have :

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \partial_x \varphi - \frac{\varphi_{i+1j}^n - \varphi_{ij}^n}{\Delta x} \right|^r dxdydt \\
\leq\ & C_r \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left[\left| \partial_x \varphi - \partial_x \varphi_{ij}^n \right|^r + |C \Delta x\,|^r \right] dxdydt \\
\leq\ & C_r \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \partial_x \varphi(x,y,t) - \frac{1}{\Delta x\ \Delta y} \int_{K_{ij}} \partial_x \varphi(x',y',t_n) dx'dy' \right|^r dxdydt \\
+\ & C_r \Delta x^{\ r+1} \Delta y\ \Delta t \\
\leq\ & C_r \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \frac{1}{\Delta x\ \Delta y} \int_{K_{ij}} \left| \partial_x \varphi(x,y,t) - \partial_x \varphi(x',y',t_n) \right| dx'dy' \right|^r dxdydt \\
+\ & C_r \Delta x^{\ r+1} \Delta y\ \Delta t \\
\leq\ & C_r \Delta x\ \Delta y\ \Delta t\ \left(\Delta x^{\ 2} + \Delta y^{\ 2} + \Delta t^{\ 2} \right)^{\frac{r}{2}} + C_r \Delta x^{\ r+1} \Delta y\ \Delta t\ .
\end{aligned}$$

We can write similar inequalities for each term of the $L^r(0,T;L^r(\Omega)^2)$-norm of $\nabla\varphi - D_h \varphi_{hk}$; it is then easy to conclude that the norm converges to 0 when $h \to 0$ and $k \to 0$.

For the fourth convergence in (6.1), we show that the $L^r(0,T;L^r(\Omega))$-norm of $\Delta\varphi - \Delta_h \varphi_{hk}$ converges to 0 when $h \to 0$ and $k \to 0$:

$$\begin{aligned}
\int_0^T \int_\Omega |\Delta\varphi - \Delta_h \varphi_{hk}|^r dxdydt \ =\ & \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} |\Delta\varphi - \Delta_h \varphi_{hk}|^r dxdydt \\
=\ & \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{t_n}^{t_{n+1}} \int_{K_{ij}} \left| \Delta\varphi - \frac{1}{\Delta x\ \Delta y} \left[\Delta y\ \frac{\varphi_{i+1j}^n - \varphi_{ij}^n}{\Delta x} \right.\right. \\
+\ & \left.\left. \Delta y\ \frac{\varphi_{i-1j}^n - \varphi_{ij}^n}{\Delta x} + \Delta x\ \frac{\varphi_{ij+1}^n - \varphi_{ij}^n}{\Delta y} + \Delta x\ \frac{\varphi_{ij-1}^n - \varphi_{ij}^n}{\Delta y} \right] \right|^r dxdydt \\
\leq\ & C_r \sum_{n=0}^{N_t-1} \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{K_{ij}} |\Delta\varphi - \Delta\varphi_{ij}^n|^r dxdydt \\
+\ & C_r |\Omega| T (\Delta x^{\ 2r} + \Delta y^{\ 2r}),
\end{aligned}$$

due to the Taylor formula.
Now, according to the $\mathcal{C}^3$ regularity of φ, we can write that :

$$\int_0^T \int_\Omega |\Delta\varphi - \Delta_h \varphi_{hk}|^r dxdydt \ \leq\ C_r |\Omega| T \left[\left(\Delta x^{\ 2} + \Delta y^{\ 2} + \Delta t^{\ 2} \right)^{\frac{r}{2}} + \Delta x^{\ 2r} + \Delta y^{\ 2r} \right].$$

So, the $L^r(0,T;L^r(\Omega))$-norm of $\Delta\varphi - \Delta_h \varphi_{hk}$ converges to 0 when $h \to 0$ and $k \to 0$.

For the last convergence in (6.1), we show that the $L^r(0,T;L^r(\partial\Omega))$-norm of $\dfrac{\partial\varphi}{\partial n}-\dfrac{\partial\varphi_{hk}}{\partial n_h}$ converges to 0 when $h\to 0$ and $k\to 0$:

$$\begin{aligned}
\int_0^T\int_{\partial\Omega}\left|\frac{\partial\varphi}{\partial n}-\frac{\partial\varphi_{hk}}{\partial n_h}\right|^r d\Gamma dt &= \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\int_{t_n}^{t_{n+1}}\left[\int_{x_{i-\frac12}}^{x_{i+\frac12}}\left|-\partial_y\varphi(x,0,t)-\frac{\varphi_{i0}^n-\varphi_{i1}^n}{\Delta y}\right|^r dx\right.\\
&\quad\left.+\int_{x_{i-\frac12}}^{x_{i+\frac12}}\left|\partial_y\varphi(x,L_2,t)-\frac{\varphi_{iN+1}^n-\varphi_{iN}^n}{\Delta y}\right|^r dx\right]dt\\
&\quad+\sum_{n=0}^{N_t-1}\sum_{j=1}^{N}\int_{t_n}^{t_{n+1}}\left[\int_{y_{j-\frac12}}^{y_{j+\frac12}}\left|-\partial_x\varphi(0,y,t)-\frac{\varphi_{0j}^n-\varphi_{1j}^n}{\Delta x}\right|^r dy\right.\\
&\quad\left.+\int_{y_{j-\frac12}}^{y_{j+\frac12}}\left|\partial_x\varphi(L_1,y,t)-\frac{\varphi_{M+1j}^n-\varphi_{Mj}^n}{\Delta x}\right|^r dx\right]dt.
\end{aligned}$$

If we consider one of the above terms and use the Taylor formula, we obtain :

$$\begin{aligned}
&\sum_{n=0}^{N_t-1}\sum_{j=1}^{N}\int_{t_n}^{t_{n+1}}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\left|-\partial_x\varphi(0,y,t)-\frac{\varphi_{0j}^n-\varphi_{1j}^n}{\Delta x}\right|^r dydt\\
\le\ &\sum_{n=0}^{N_t-1}\sum_{j=1}^{N}\int_{t_n}^{t_{n+1}}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\left|-\partial_x\varphi(0,y,t)+\frac{1}{\Delta y}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\partial_x\varphi(0,y',t_n)\,dy'\right|^r dydt+C_rTL_2\Delta x^{\,2r},\\
\le\ &C_rTL_2\left(\Delta x^{\,2}+\Delta y^{\,2}+\Delta t^{\,2}\right)^{\frac r2}+C_rTL_2\Delta x^{\,2r}.
\end{aligned}$$

We proceed similarly for all the terms and we deduce that the $L^r(0,T;L^r(\partial\Omega))$-norm of $\partial\varphi/\partial n-\partial\varphi_{hk}/\partial n_h$ converges to 0 when $h\to 0$ and $k\to 0$.

Lemma 6.2. *For $q>1$ fixed and for u_{hk} in $\boldsymbol{W}_{hk}$ such that :*

$$u_{hk}\rightharpoonup u \text{ in } L^2(0,T;L^2(\Omega)),$$
$$D_hu_{hk}\rightharpoonup \nabla u \text{ in } L^2(0,T;L^2(\Omega)),$$
$$\gamma_0u_{hk} \text{ is bounded in } L^q(0,T;L^q(\partial\Omega)),$$

we have :

$$\gamma_0u_{hk}\rightharpoonup\gamma_0u \text{ in } L^q(0,T;L^q(\partial\Omega)).$$

Proof :

For a function φ in $\mathbf{W}$ supposed regular enough, say $\varphi \in \mathcal{C}^3(\bar{\Omega}\times[0,T])$, we define its approximate function φ_{hk} in $\mathbf{W}_{hk}$ as in (2.6a), (2.6b). We have :

$$\begin{aligned}
\int_0^T \int_\Omega u_{hk}\, \Delta_h \varphi_{hk}\, dxdydt &= \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\sum_{j=1}^{N}\int_{t_n}^{t_{n+1}}\int_{K_{ij}} u_{hk}\Delta_h\, \varphi_{hk}\, dxdydt,\\
&= \Delta t \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\sum_{j=1}^{N}\Bigg[\Delta x\left(u_{ij}^n \frac{\varphi_{ij+1}^n - \varphi_{ij}^n}{\Delta y} + u_{ij}^n \frac{\varphi_{ij-1}^n - \varphi_{ij}^n}{\Delta y}\right)\\
&+ \Delta y\left(u_{ij}^n \frac{\varphi_{i+1j}^n - \varphi_{ij}^n}{\Delta x} + u_{ij}^n \frac{\varphi_{i-1j}^n - \varphi_{ij}^n}{\Delta x}\right)\Bigg],\\
&= \Delta t \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\left[\sum_{j=1}^{N}\Delta x\, u_{ij}^n \frac{\varphi_{ij+1}^n - \varphi_{ij}^n}{\Delta y} + \sum_{j=0}^{N-1}\Delta x\, u_{ij+1}^n \frac{\varphi_{ij}^n - \varphi_{ij+1}^n}{\Delta y}\right]\\
&+ \Delta t \sum_{n=0}^{N_t-1}\sum_{j=1}^{N}\left[\sum_{i=1}^{M}\Delta y\, u_{ij}^n \frac{\varphi_{i+1j}^n - \varphi_{ij}^n}{\Delta x} + \sum_{i=0}^{M-1}\Delta y\, u_{i+1j}^n \frac{\varphi_{ij}^n - \varphi_{i+1j}^n}{\Delta x}\right],\\
&= -\Delta t \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\,\Delta y\left[\frac{u_{ij+1}^n - u_{ij}^n}{\Delta y}\frac{\varphi_{ij+1}^n - \varphi_{ij}^n}{\Delta y}\right]\\
&- \Delta t \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\,\Delta y\left[\frac{u_{i+1j}^n - u_{ij}^n}{\Delta x}\frac{\varphi_{i+1j}^n - \varphi_{ij}^n}{\Delta x}\right]\\
&+ \Delta t \sum_{n=0}^{N_t-1}\sum_{i=1}^{M}\Delta x\left[u_{iN+1}^n \frac{\varphi_{iN+1}^n - \varphi_{iN}^n}{\Delta y} + u_{i1}^n \frac{\varphi_{i0}^n - \varphi_{i1}^n}{\Delta y}\right]\\
&+ \Delta t \sum_{n=0}^{N_t-1}\sum_{j=1}^{N}\Delta y\left[u_{M+1j}^n \frac{\varphi_{M+1j}^n - \varphi_{Mj}^n}{\Delta x} + u_{1j}^n \frac{\varphi_{0j}^n - \varphi_{1j}^n}{\Delta x}\right],\\
&= -\int_0^T\int_\Omega D_h u_{hk} D_h \varphi_{hk}\, dxdydt + \int_0^T\int_{\partial\Omega}\gamma_0 u_{hk}\frac{\partial \varphi_{hk}}{\partial n_h}\, d\Gamma dt.
\end{aligned}$$

Then, we pass to the limit when $h \to 0$ and $k \to 0$ in the above equation. To do so, we use the following strong convergences (see Lemma 6.1) :

$$\begin{aligned}
&D_h\varphi_{hk} \to \nabla\varphi \text{ in } L^2(0,T;L^2(\Omega)^2),\\
&\Delta_h\varphi_{hk} \to \Delta\varphi \text{ in } L^2(0,T;L^2(\Omega)),\\
&\frac{\partial\varphi_{hk}}{\partial n_h} \to \frac{\partial\varphi}{\partial n} \text{ in } L^{q'}(0,T;L^{q'}(\partial\Omega)),\ (q' > 1),
\end{aligned}$$

and we have the following weak convergences :

$$\begin{aligned}
&u_h \rightharpoonup u \text{ in } L^2(0,T;L^2(\Omega)),\\
&D_h u_h \rightharpoonup \nabla u \text{ in } L^2(0,T;L^2(\Omega)^2),\\
&\gamma_0 u_h \rightharpoonup \tau \text{ in } L^q(0,T;L^q(\partial\Omega)).
\end{aligned}$$

This yields :

$$\int_0^T \int_\Omega u \Delta\varphi \, dx dy dt \quad = \quad \int_0^T \int_\Omega \nabla u \nabla \varphi \, dx dy dt + \int_0^T \int_{\partial\Omega} \tau \frac{\partial \varphi}{\partial n} \, d\Gamma dt,$$

Thanks to the Stokes formula, we find :

$$\int_0^T \int_{\partial\Omega} (\tau - \gamma_0 u) \frac{\partial \varphi}{\partial n} \, d\Gamma dt \quad = \quad 0. \tag{6.3}$$

Finally, the functions $\dfrac{\partial \varphi}{\partial n}$ over $\partial\Omega$, for $\varphi \in \mathbf{W}$ regular, are dense in $L^q(0,T;L^q(\partial\Omega))$. Then (6.3) implies that τ is equal to $\gamma_0 u$ in $L^q(0,T;L^q(\Omega))$.

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Chapitre 2

Stability of a collocated finite volume scheme for the Navier-Stokes equations

Stability of a collocated finite volume scheme for the Navier-Stokes equations

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Abstract : The aim of this article is to describe a collocated finite volume approximation of the incompressible Navier-Stokes equation and study its stability. One of the advantages of collocated finite volume schemes over staggered schemes is that all variables share the same location, hence, the possibility to use more easily complex geometries and hierarchical space discretizations. The time discretization used in the scheme studied here is a projection method [Tem69].
First, we give the full discretization of the incompressible Navier-Stokes equations, then, we state the stability result and prove it following the methods of [MT98].

1 Introduction

In this article, we consider the Navier-Stokes equations in their velocity-pressure formulation and the continuity equation written for an incompressible viscous fluid; Ω is an open and bounded domain in $\mathbb{R}^2$. For a given volume force $\mathbf{f}$, we look for the velocity vector $\mathbf{u}$ and the pressure p such that :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \times [0, T], \tag{1.1}$$

$$div\ \mathbf{u} = 0, \tag{1.2}$$

where ν is the viscosity and, in space dimension two, $\mathbf{u} = \big(u(x,y,t), v(x,y,t)\big)$, $t \geq 0$.
On $\partial\Omega$, we use here boundary conditions of the Dirichlet type which we assume homogeneous, although the nonhomogeneous case can be treated in a very similar way :

$$\mathbf{u}\Big|_{\partial\Omega} = 0. \tag{1.3}$$

We define the function spaces $\mathbf{W} = \{\varphi \in H_0^1(\Omega)^2\}$, $\mathbf{V} = \{\varphi \in H_0^1(\Omega)^2; div\ \varphi = 0\}$ and $\mathbf{H} = \{\varphi \in L^2(\Omega)^2;\ div\ \varphi = 0,\ \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. We note $(\cdot,\ \cdot)$, $|\cdot|$ the dot-product and norm of $L^2(\Omega)^2$, and $((\cdot,\ \cdot))$, $||\cdot||$ the dot-product and norm of $H_0^1(\Omega)^2$. The weak formulation of the problem (1.1)-(1.3) is (see [Ler33] [Ler34a], [Ler34b] and [Tem01]) : for $\mathbf{f} \in L^2(0,T;\mathbf{V}')$ and $u_0 \in \mathbf{H}$ given, we look for $\mathbf{u}$ satisfying $\mathbf{u} \in L^2(0,T;\mathbf{V})$ and

$$\frac{d}{dt}(\mathbf{u},\ \varphi) + \nu\left((\mathbf{u},\ \varphi)\right) + \mathbf{b}(\mathbf{u},\mathbf{u},\varphi) = (\mathbf{f},\ \varphi)\,, \forall\varphi \in \mathbf{V}, \tag{1.4}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{1.5}$$

where, the definition of the trilinear form $\mathbf{b}$ is :

$$\mathbf{b}(\mathbf{u},\varphi,\boldsymbol{\xi}) = \sum_{i,j=1}^{2} \int_\Omega u_i(x)\frac{\partial\varphi_j(x)}{\partial x_i}\xi_j(x)dx. \tag{1.6}$$

It is well-known that in the two-dimensional case, this problem possesses a unique solution (see e.g. [Tem01]).

The aim of this paper is to introduce a finite volume scheme adapted to this problem and prove its stability. Our scheme is a collocated scheme, this means that the space unknowns are only located at the centers of the control volumes. Our motivation for collocated schemes comes from the fact that it is easier to then develop a multilevel method with them than with staggered schemes. Indeed, as the unknowns share the same location, we can make a simple hierarchical space discretization, see e.g. Chapter 4. Moreover, the extension to the three dimensional case do not add new conceptual difficulties.
Other finite volume schemes for the Navier-Stokes equations exist in the literature : see e.g. [BH96a], [BH96b], [LL94a] and [LL94b]. Here, the results are presented for a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$ discretized by a rectangular finite volumes for the sake of simplicity. In view of extending these results to more general situations, we use convenient notations and concepts of [EGH00]. It is also important to note that more general domains are also handled by finite volumes adapted to curvilinear coordinates, see e.g. in meteorology [RT03].

In Section 2, we describe the projection method used for the time discretization and the finite volume scheme used for the space discretization. Then, in Section 3, we introduce the notations, we give the weak formulation and we state the main stability result for our scheme. Section 4 contains the technical lemmas used in Section 3, and Section 5 contains the a priori estimates. In Section 6, we give the proof of stability.

2 Full discretization : finite volumes in space and the projection method in time

In this section, we give the time and space discretizations used in order to solve the Navier-Stokes equations.
The time discretization is a projection method. For the sake of simplicity in the presentation, we will describe here the original projection method of Chorin and Temam [Cho68], [Tem69] ; however in actual computations we used the improved form of the projection method of Van

Kan [VK86], see Chapter 3. The first step consists in computing the intermediate velocity vector $\mathbf{u}^{n+\frac{1}{2}}$ with :

$$\frac{\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}^n}{\Delta t} = \nu\Delta\Big(\frac{\mathbf{u}^{n+\frac{1}{2}} + \mathbf{u}^n}{2}\Big) - \frac{3}{2}\mathbf{C}^n + \frac{1}{2}\mathbf{C}^{n-1} + \frac{1}{2}(\mathbf{f}^{n+1} + \mathbf{f}^n), \tag{2.1}$$

$$\mathbf{u}^{n+\frac{1}{2}}\Big|_{\partial\Omega} = 0, \tag{2.2}$$

where :

$\Delta t = T/N_t$ is the time step, N_t being an integer,
$\mathbf{u}^n(x,y) \simeq \mathbf{u}(x,y,n\Delta t)$ if $t \in [n\Delta t, (n+1)\Delta t[$,
$\mathbf{C}^n = (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n$ is the nonlinear term.

Then, in a second step, we compute the pressure from :

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+\frac{1}{2}}}{\Delta t} = -\nabla p^{n+1}, \tag{2.3}$$

$$div\ \mathbf{u}^{n+1} = 0, \tag{2.4}$$

$$\mathbf{u}^{n+1} \cdot \mathbf{n}\Big|_{\partial\Omega} = 0, \tag{2.5}$$

where $p^n(x,y) \simeq p(x,y,n\Delta t)$ if $t \in [n\Delta t, (n+1)\Delta t[$, and $\mathbf{n}$ is the unit outward normal on $\partial\Omega$.

Equations (2.3) and (2.4) imply the projection equation :

$$\Delta p^{n+1} = \frac{1}{\Delta t} div\ \mathbf{u}^{n+\frac{1}{2}}, \tag{2.6}$$

Due to (2.2), (2.3) and (2.5), we have a Neumann boundary condition for p^{n+1} :

$$\frac{\partial p^{n+1}}{\partial \mathbf{n}} = 0. \tag{2.7}$$

Consequently, we are able to compute p^{n+1} from (2.6) and (2.7), up to an additive constant. Note that the necessary compatibility condition is satisfied :

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial p^{n+1}}{\partial \mathbf{n}}\, d(\partial\Omega) &= \int_{\Omega} \Delta p^{n+1}\, d\Omega, \\
&= \frac{1}{\Delta t}\int_{\Omega} div\ \mathbf{u}^{n+\frac{1}{2}}\, d\Omega, \\
&= \frac{1}{\Delta t}\int_{\partial\Omega} \mathbf{u}^{n+\frac{1}{2}} \cdot \mathbf{n}\, d(\partial\Omega), \\
&= 0.
\end{aligned}$$

Finally, we compute the new velocity vector $\mathbf{u}^{n+1}$ from (2.3).

Now, we describe the space discretization. In order to discretize the domain $\Omega = (0, L_1) \times (0, L_2)$, we use rectangular finite volumes. More general domains are handled by curvilinear finite volumes based on orthogonal coordinates [RT03]. We assume that all the volumes have the same dimensions $\Delta x \Delta y$ with $M\Delta x = L_1$ and $N\Delta y = L_2$ where M, N are given integers. Hence, we have MN volumes defined by :

$$\left(K_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\right)_{i=1,\dots M,\, j=1,\dots,N},$$

where

$$\begin{aligned} x_{i+\frac{1}{2}} &= i\Delta x \text{ for } i = 0, \dots, M, \\ y_{j+\frac{1}{2}} &= j\Delta y \text{ for } j = 0, \dots, N. \end{aligned}$$

As the scheme studied uses a collocated grid, all the unknowns are located at the center of the volume. So, for $i = 1, ..., M$ and $j = 1, ..., N$, the unknowns are meant to be approximations of the cell averages :

$$\mathbf{u}_{ij} \simeq \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{u}(x, y, t) dx dy \text{ for the velocity,} \tag{2.8}$$

$$p_{ij} \simeq \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} p(x, y, t) dx dy \text{ for the pressure.} \tag{2.9}$$

Moreover, we will use the normal fluxes for the velocity, for example in the discrete nonlinear term. They are not considered like independent unknowns because they are obtained by interpolation, see Remark 2.1. The horizontal fluxes are defined for $i = 0, ..., M$ by :

$$F_{u\, i+\frac{1}{2}j}(t) \simeq \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x_{i+\frac{1}{2}}, y, t) dy, \tag{2.10}$$

and the vertical fluxes are defined for $j = 0, ..., N$ by :

$$F_{v\, ij+\frac{1}{2}}(t) \simeq \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v(x, y_{j+\frac{1}{2}}, t) dx. \tag{2.11}$$

To still improve the notations, we introduce the velocities at the edges obtained by linear interpolation :

$$\begin{aligned} \mathbf{u}_{i+\frac{1}{2}j}(t) &= \frac{\mathbf{u}_{i+1j}(t) + \mathbf{u}_{ij}(t)}{2}, \\ \mathbf{u}_{ij+\frac{1}{2}}(t) &= \frac{\mathbf{u}_{ij+1}(t) + \mathbf{u}_{ij}(t)}{2}. \end{aligned} \tag{2.12}$$

The integration of equation (2.1) over a control volume K_{ij}, for $i = 1, ..., M$ and $j = 1, ..., N$, suggests to consider the following scheme (first half step) :

$$\begin{aligned}
\Delta x\Delta y\frac{\mathbf{u}_{ij}^{n+\frac12}-\mathbf{u}_{ij}^{n}}{\Delta t} &= \frac{\nu}{2}\Big[\Delta y\frac{\mathbf{u}_{i+1j}^{n+\frac12}-\mathbf{u}_{ij}^{n+\frac12}}{\Delta x}+\Delta y\frac{\mathbf{u}_{i-1j}^{n+\frac12}-\mathbf{u}_{ij}^{n+\frac12}}{\Delta x}+\Delta x\frac{\mathbf{u}_{ij+1}^{n+\frac12}-\mathbf{u}_{ij}^{n+\frac12}}{\Delta y}\\
&+ \Delta x\frac{\mathbf{u}_{ij-1}^{n+\frac12}-\mathbf{u}_{ij}^{n+\frac12}}{\Delta y}+\Delta y\frac{\mathbf{u}_{i+1j}^{n}-\mathbf{u}_{ij}^{n}}{\Delta x}+\Delta y\frac{\mathbf{u}_{i-1j}^{n}-\mathbf{u}_{ij}^{n}}{\Delta x}\\
&+ \Delta x\frac{\mathbf{u}_{ij+1}^{n}-\mathbf{u}_{ij}^{n}}{\Delta y}+\Delta x\frac{\mathbf{u}_{ij-1}^{n}-\mathbf{u}_{ij}^{n}}{\Delta y}\Big]\\
&- \frac{3}{2}\Big[\Delta y\big(F^{n}_{u\,i+\frac12 j}\boldsymbol{u}^{n}_{i+\frac12 j}-F^{n}_{u\,i-\frac12 j}\boldsymbol{u}^{n}_{i-\frac12 j}\big)+\Delta x\big(F^{n}_{v\,ij+\frac12}\boldsymbol{u}^{n}_{ij+\frac12}-F^{n}_{v\,ij-\frac12}\boldsymbol{u}^{n}_{ij-\frac12}\big)\Big]\\
&+ \frac{1}{2}\Big[\Delta y\big(F^{n-1}_{u\,i+\frac12 j}\boldsymbol{u}^{n-1}_{i+\frac12 j}-F^{n-1}_{u\,i-\frac12 j}\boldsymbol{u}^{n-1}_{i-\frac12 j}\big)+\Delta x\big(F^{n-1}_{v\,ij+\frac12}\boldsymbol{u}^{n-1}_{ij+\frac12}-F^{n-1}_{v\,ij-\frac12}\boldsymbol{u}^{n-1}_{ij-\frac12}\big)\Big]\\
&+ \frac{1}{2}\big(\mathbf{f}_{ij}^{n+1}+\mathbf{f}_{ij}^{n}\big). \qquad (2.13)
\end{aligned}$$

Because of (2.2), we propose the following Dirichlet boundary condition at step $n+\frac12$:

$$\mathbf{u}_{\frac12 j}^{n+\frac12}=\mathbf{u}_{M+\frac12 j}^{n+\frac12}=\mathbf{u}_{i\frac12}^{n+\frac12}=\mathbf{u}_{iN+\frac12}^{n+\frac12}=0, \qquad (2.14)$$

which also means, in accordance with the definition (2.12), that :

$$\mathbf{u}_{0j}^{n+\frac12}=-\mathbf{u}_{1j}^{n+\frac12},\quad \mathbf{u}_{M+1j}^{n+\frac12}=-\mathbf{u}_{Mj}^{n+\frac12},\quad \mathbf{u}_{i0}^{n+\frac12}=-\mathbf{u}_{i1}^{n+\frac12},\quad \mathbf{u}_{iN+1}^{n+\frac12}=-\mathbf{u}_{iN}^{n+\frac12}. \qquad (2.15)$$

Notice that when $i=0$ or $i=M+1$ or $j=0$ or $j=N+1$, the control volume K_{ij} is a fictitious control volume outside of the domain Ω.

Similarly, we integrate equation (2.4) over a control volume K_{ij}, for $i=1,...,M$ and $j=1,...,N$, which suggests the following discrete equation :

$$\Delta y\Big(F^{n+1}_{u\,i+\frac12 j}-F^{n+1}_{u\,i-\frac12 j}\Big)+\Delta x\Big(F^{n+1}_{v\,ij+\frac12}-F^{n+1}_{v\,ij-\frac12}\Big)=0. \qquad (2.16)$$

To proceed, we need to define the fluxes $F^{n+1}_{u\,i+\frac12 j}$ and $F^{n+1}_{v\,ij+\frac12}$. Based on (2.10) and (2.11), it is natural to define these fluxes by linear interpolation of the velocities (see however Remark 2.1), that is, at any point :

$$F^{n+1}_{u\,i+\frac12 j}=u^{n+1}_{i+\frac12 j},\quad F^{n+1}_{v\,ij+\frac12}=v^{n+1}_{ij+\frac12}. \qquad (2.17)$$

Hence, with (2.5), we choose at the boundary points :

$$F^{n+1}_{u\,\frac12 j}=F^{n+1}_{u\,M+\frac12 j}=F^{n+1}_{v\,i\frac12}=F^{n+1}_{v\,iN+\frac12}=0. \qquad (2.18)$$

Consequently, the incompressibility condition (2.16) becomes, for $i=1,...,M$ and $j=1,...,N$:

$$\frac{\Delta y}{2}\Big(u^{n+1}_{i+1j}-u^{n+1}_{i-1j}\Big)+\frac{\Delta x}{2}\Big(v^{n+1}_{ij+1}-v^{n+1}_{ij-1}\Big)=0, \qquad (2.19)$$

with :

$$u_{0j}^{n+1} = -u_{1j}^{n+1}, \quad u_{M+1j}^{n+1} = -u_{Mj}^{n+1}, \quad v_{i0}^{n+1} = -v_{i1}^{n+1}, \quad v_{iN+1}^{n+1} = -v_{iN}^{n+1}. \tag{2.20}$$

Furthermore, we integrate equation (2.3) over a control volume K_{ij}, for $i = 1, ..., M$ and $j = 1, ..., N$, which leads us to propose the following discrete equations for the second half step :

$$\Delta x \Delta y \frac{\boldsymbol{u}_{ij}^{n+1} - \boldsymbol{u}_{ij}^{n+\frac{1}{2}}}{\Delta t} = - \begin{pmatrix} \Delta y \left[p_{i+\frac{1}{2}j}^{n+1} - p_{i-\frac{1}{2}j}^{n+1}\right] \\ \Delta x \left[p_{ij+\frac{1}{2}}^{n+1} - p_{ij-\frac{1}{2}}^{n+1}\right] \end{pmatrix}. \tag{2.21}$$

To proceed with (2.21), we must interpolate the pressures $p_{i+\frac{1}{2}j}^{n+1}$, $p_{ij+\frac{1}{2}}^{n+1}$ at the edges. By analogy with the fluxes, we define these pressures by linear interpolation :

$$p_{i+\frac{1}{2}j}^{n+1} = \frac{p_{i+1j}^{n+1} + p_{ij}^{n+1}}{2} \text{ for i=0,...,M,} \tag{2.22}$$

$$p_{ij+\frac{1}{2}}^{n+1} = \frac{p_{ij+1}^{n+1} + p_{ij}^{n+1}}{2} \text{ for j=0,...,N,} \tag{2.23}$$

where the pressure on the fictitious control volumes is defined by : $p_{M+1j}^{n+1} = p_{Mj}^{n+1}$, $p_{0j}^{n+1} = p_{1j}^{n+1}$, $p_{iN+1}^{n+1} = p_{iN}^{n+1}$, $p_{i0}^{n+1} = p_{i1}^{n+1}$. These definitions are consistent with the pressure Neumann boundary condition (2.7) and allow us to rewritte the equation (2.21) like, for $i = 1, ..., M$ and $j = 1, ..., N$:

$$\Delta x \Delta y \frac{\boldsymbol{u}_{ij}^{n+1} - \boldsymbol{u}_{ij}^{n+\frac{1}{2}}}{\Delta t} = - \begin{pmatrix} \frac{\Delta y}{2} \left[p_{i+1j}^{n+1} - p_{i-1j}^{n+1}\right] \\ \frac{\Delta x}{2} \left[p_{ij+1}^{n+1} - p_{ij-1}^{n+1}\right] \end{pmatrix}, \tag{2.24}$$

where $p_{M+1j}^{n+1} = p_{Mj}^{n+1}$, $p_{0j}^{n+1} = p_{1j}^{n+1}$, $p_{iN+1}^{n+1} = p_{iN}^{n+1}$, $p_{i0}^{n+1} = p_{i1}^{n+1}$.

Now, to obtain the pressure equation, we take the discrete divergence (defined as for (2.19)) of equation (2.24). First, inside the domain i.e. for $i = 2, ..., M-1$ and $j = 2, ..., N-1$, we have :

$$\begin{aligned} &\frac{1}{2\Delta x}\left[\frac{\Delta x \Delta y}{\Delta t}\left(u_{i+1j}^{n+1} - u_{i+1j}^{n+\frac{1}{2}}\right) + \frac{\Delta y}{2}\left(p_{i+2j}^{n+1} - p_{ij}^{n+1}\right) - \frac{\Delta x \Delta y}{\Delta t}\left(u_{i-1j}^{n+1} - u_{i-1j}^{n+\frac{1}{2}}\right)\right. \\ &- \left.\frac{\Delta y}{2}\left(p_{ij}^{n+1} - p_{i-2j}^{n+1}\right)\right] + \frac{1}{2\Delta y}\left[\frac{\Delta x \Delta y}{\Delta t}\left(v_{ij+1}^{n+1} - v_{ij+1}^{n+\frac{1}{2}}\right) + \frac{\Delta x}{2}\left(p_{ij+2}^{n+1} - p_{ij}^{n+1}\right)\right. \\ &- \left.\frac{\Delta x \Delta y}{\Delta t}\left(v_{ij-1}^{n+1} - v_{ij-1}^{n+\frac{1}{2}}\right) - \frac{\Delta x}{2}\left(p_{ij}^{n+1} - p_{ij-2}^{n+1}\right)\right] = 0. \end{aligned}$$

Hence, using the incompressibility condition (2.19) :

$$\Delta y\frac{p_{i+2j}^{n+1}-p_{ij}^{n+1}}{2\Delta x}+\Delta y\frac{p_{i-2j}^{n+1}-p_{ij}^{n+1}}{2\Delta x}+\Delta x\frac{p_{ij+2}^{n+1}-p_{ij}^{n+1}}{2\Delta y}+\Delta x\frac{p_{ij-2}^{n+1}-p_{ij}^{n+1}}{2\Delta y}$$
$$=\frac{4}{\Delta t}\left[\Delta y\left(u_{i+1j}^{n+\frac{1}{2}}-u_{i-1j}^{n+\frac{1}{2}}\right)+\Delta x\left(v_{ij+1}^{n+\frac{1}{2}}-v_{ij-1}^{n+\frac{1}{2}}\right)\right]. \tag{2.25}$$

Then, near the boundary, for example when $i=1$, (2.19) becomes in accordance with the boundary conditions (2.20) :

$$\frac{\Delta y}{2}\left(u_{2j}^{n+1}+u_{1j}^{n+1}\right)+\frac{\Delta x}{2}\left(v_{1j+1}^{n+1}-v_{1j-1}^{n+1}\right)=0, \tag{2.26}$$

So, when we apply this discrete divergence to the equation (2.24), we find, for $i=1$ and $j=2,...,N-1$:

$$\Delta y\frac{p_{3j}^{n+1}-p_{1j}^{n+1}}{2\Delta x}+\Delta y\frac{p_{2j}^{n+1}-p_{1j}^{n+1}}{2\Delta x}+\Delta x\frac{p_{1j+2}^{n+1}-p_{1j}^{n+1}}{2\Delta y}+\Delta x\frac{p_{1j-2}^{n+1}-p_{1j}^{n+1}}{2\Delta y}$$
$$=\frac{4}{\Delta t}\left[\Delta y\left(u_{2j}^{n+\frac{1}{2}}+u_{1j}^{n+\frac{1}{2}}\right)+\Delta x\left(v_{1j+1}^{n+\frac{1}{2}}-v_{1j-1}^{n+\frac{1}{2}}\right)\right]. \tag{2.27}$$

The pressure equation can be finally written like, for $i=1,...,M$ and $j=1,...,M$:

$$\Delta y\frac{p_{i+2j}^{n+1}-p_{ij}^{n+1}}{2\Delta x}+\Delta y\frac{p_{i-2j}^{n+1}-p_{ij}^{n+1}}{2\Delta x}+\Delta x\frac{p_{ij+2}^{n+1}-p_{ij}^{n+1}}{2\Delta y}+\Delta x\frac{p_{ij-2}^{n+1}-p_{ij}^{n+1}}{2\Delta y}$$
$$=\frac{4}{\Delta t}\left[\Delta y\left(u_{i+1j}^{n+\frac{1}{2}}-u_{i-1j}^{n+\frac{1}{2}}\right)+\Delta x\left(v_{ij+1}^{n+\frac{1}{2}}-v_{ij-1}^{n+\frac{1}{2}}\right)\right], \tag{2.28}$$

with the following definitions for the pressure on the fictitious control volumes :

$$p_{M+1j}^{n+1}=p_{Mj}^{n+1},p_{0j}^{n+1}=p_{1j}^{n+1},p_{iN+1}^{n+1}=p_{iN}^{n+1},p_{i0}^{n+1}=p_{i1}^{n+1},$$
$$p_{M+2j}^{n+1}=p_{M-1j}^{n+1},p_{-1j}^{n+1}=p_{2j}^{n+1},p_{iN+2}^{n+1}=p_{iN-1}^{n+1},p_{i\,-1}^{n+1}=p_{i2}^{n+1}. \tag{2.29}$$

Remark 2.1 (Decoupling). Considering the pressure equation (2.27), we see that, when the velocity fluxes and the pressure are computed using linear interpolations, the pressure obtained may not be physical due to the odd/even decoupling of the modes of checkerboard type. For this reason, it is recommended in practice not to use linear interpolations for the intermediate velocity fluxes $F_u^{n+1/2}$, $F_v^{n+1/2}$ in a numerical simulation with a large Reynolds number in order to improve the coupling between the velocities and the pressure. A specific interpolation not given here (see [RC83], Chapter 3) has been used for the intermediate fluxes $F_u^{n+1/2}$, $F_v^{n+1/2}$, and then, the velocity fluxes F_u^{n+1}, F_v^{n+1}, are derived from the following equations :

$$\begin{pmatrix}\Delta x\Delta y\dfrac{F_{u\,i+\frac{1}{2}j}^{n+1}-F_{u\,i+\frac{1}{2}j}^{n+\frac{1}{2}}}{\Delta t}\\[2ex] \Delta x\Delta y\dfrac{F_{v\,ij+\frac{1}{2}}^{n+1}-F_{u\,ij+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta t}\end{pmatrix}=-\begin{pmatrix}\Delta y\left[p_{i+1j}^{n+1}-p_{ij}^{n+1}\right]\\[2ex] \Delta x\left[p_{ij+1}^{n+1}-p_{ij}^{n+1}\right]\end{pmatrix}. \tag{2.30}$$

For more details about the implementation of such a collocated scheme and for numerical results, see e.g. Chapter 3.

3 The main result : stability of the scheme

3.1 Notations

The mesh considered here is clearly an admissible finite mesh in the sense of [EGH00]. For a function $\varphi \in L^2(\Omega)^2$, we can define its approximate function φ_h which is constant over each control volume K_{ij} of the mesh :

$$\varphi_h(x,y) = \varphi_{ij} \quad \text{if } (x,y) \in K_{ij},$$

where $(i,j) \in \left\{(i,j),\ (\frac{1}{2},j),\ (M+\frac{1}{2},j),\ (i,\frac{1}{2}),\ (i,N+\frac{1}{2})\ ;\ i=1,...,M \text{ and } j=1,...,N\right\}$. In the following, we write h for $(\Delta x, \Delta y)$.

We define an approximation $\mathbf{W}_h$ of $\mathbf{W} = H_0^1(\Omega)^2$:

$$\begin{aligned} \mathbf{W}_h &= \left\{\varphi_h\ ;\ \varphi_{\frac{1}{2}j} = \varphi_{M+\frac{1}{2}j} = \varphi_{i\frac{1}{2}} = \varphi_{iN+\frac{1}{2}} = 0,\ i=1,...,M,\ j=1,...,N\right\}, \qquad (3.1)\\ &= \left\{\varphi_h\ ;\ \varphi_{0j} = -\varphi_{1j},\ \varphi_{M+1j} = -\varphi_{Mj},\ \varphi_{i0} = -\varphi_{i1},\ \varphi_{iN+1} = -\varphi_{iN},\ i=1,...,M,\ j=1,...,N\right\}. \end{aligned}$$

and an approximation $\mathbf{V}_h$ of $\mathbf{V} = \left\{\varphi \in H_0^1(\Omega)^2; div\ \varphi = 0\right\}$:

$$\mathbf{V}_h = \left\{\varphi_h \in \mathbf{W}_h\ ;\ \frac{\Delta y}{2}\left(\varphi_{i+1j} - \varphi_{i-1j}\right) + \frac{\Delta x}{2}\left(\varphi_{ij+1} - \varphi_{ij-1}\right),\ i=1,...,M,\ j=1,...,N\right\}. \qquad (3.2)$$

Then, for φ_h and ψ_h in $\mathbf{W}_h$, we introduce the discrete dot-products and norms of the function spaces used :

- Discrete dot-product and norm corresponding to the dot-product $(\cdot,\ \cdot)$ and norm $|\cdot|$ of the space $L^2(\Omega)$ are written :

$$(\varphi_h,\ \psi_h)_h = \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x \Delta y\ \varphi_{ij} \cdot \psi_{ij}, \qquad (3.3)$$

$$|\varphi_h|_h^2 = (\varphi_h,\ \varphi_h)_h. \qquad (3.4)$$

- Discrete dot-product and norm corresponding to the dot-product $((\cdot,\ \cdot))$ and norm $||\cdot||$ of the space $H_0^1(\Omega)$ are written :

$$
\begin{aligned}
((\varphi_h,\, \psi_h))_h \;=\;& \sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x \Delta y \frac{\varphi_{i+1j}-\varphi_{ij}}{\Delta x} \cdot \frac{\psi_{i+1j}-\psi_{ij}}{\Delta x} \\
+\;& \sum_{i=1}^{M}\sum_{j=1}^{N-1} \Delta x \Delta y \frac{\varphi_{ij+1}-\varphi_{ij}}{\Delta y} \cdot \frac{\psi_{ij+1}-\psi_{ij}}{\Delta y} \\
+\;& \frac{1}{2}\sum_{j=1}^{N} \Delta x \Delta y \Big[\frac{\varphi_{M+1j}-\varphi_{Mj}}{\Delta x} \cdot \frac{\psi_{M+1j}-\psi_{Mj}}{\Delta x} + \frac{\varphi_{1j}-\varphi_{0j}}{\Delta x} \cdot \frac{\psi_{1j}-\psi_{0j}}{\Delta x}\Big] \\
+\;& \frac{1}{2}\sum_{i=1}^{M} \Delta x \Delta y \Big[\frac{\varphi_{iN+1}-\varphi_{iN}}{\Delta y} \cdot \frac{\psi_{iN+1}-\psi_{iN}}{\Delta y} + \frac{\varphi_{i1}-\varphi_{i0}}{\Delta y} \cdot \frac{\psi_{i1}-\psi_{i0}}{\Delta y}\Big],
\end{aligned}
\tag{3.5}
$$

$$
||\varphi_h||_h^2 \;=\; ((\varphi_h,\, \varphi_h))_h\,. \tag{3.6}
$$

Finally, the discrete operator $\mathbf{G}_h$ which will be use for the pressure gradient is defined for $i = 1, ..., M$ and $j = 1, ..., N$ by :

$$
\mathbf{G}_h p_h(x,y) = \begin{pmatrix} \dfrac{p_{i+1j}-p_{i-1j}}{2\Delta x} \\ \dfrac{p_{ij+1}-p_{ij-1}}{2\Delta y} \end{pmatrix} \text{ if } (x,y) \in K_{ij}\ , \tag{3.7}
$$

with $p_{M+1j} = p_{Mj}$, $p_{0j} = p_{1j}$, $p_{iN+1} = p_{iN}$ and $p_{i0} = p_{i1}$.

3.2 Weak formulation

First, we must introduce the discrete form corresponding to the nonlinear term $\mathbf{b}(\mathbf{u},\, \varphi,\, \boldsymbol{\xi})$:

$$
\begin{aligned}
\mathbf{b}_h(\mathbf{u}_h,\, \varphi_h,\, \xi_h) \;=\;& \sum_{i=1}^{M}\sum_{j=1}^{N} \Big[\Delta y u_{i+\frac{1}{2}j}\varphi_{i+\frac{1}{2}j}\cdot \boldsymbol{\xi}_{ij} - \Delta y u_{i-\frac{1}{2}j}\varphi_{i-\frac{1}{2}j}\cdot \boldsymbol{\xi}_{ij} \\
+\;& \Delta x v_{ij+\frac{1}{2}}\varphi_{ij+\frac{1}{2}}\cdot \boldsymbol{\xi}_{ij} - \Delta x v_{ij-\frac{1}{2}}\varphi_{ij-\frac{1}{2}}\cdot \boldsymbol{\xi}_{ij}\Big].
\end{aligned}
\tag{3.8}
$$

Then, we write the weak formulation :

Lemma 3.1 (Weak formulation). *With the previous notations, equations (2.13) and (2.21) can be reinterpreted in the following way, for all φ_h in $\mathbf{W}_h$:*

$$
\begin{aligned}
\boldsymbol{u}_h^{n+\frac{1}{2}} \in \mathbf{W}_h,\ \frac{1}{\Delta t}\left(\boldsymbol{u}_h^{n+\frac{1}{2}} - \boldsymbol{u}_h^n,\, \varphi_h\right)_h \;=\;& -\nu\left(\left(\frac{\boldsymbol{u}_h^{n+\frac{1}{2}}+\boldsymbol{u}_h^n}{2},\, \varphi_h\right)\right)_h - \frac{3}{2}\boldsymbol{b}_h(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \varphi_h) \\
+\;& \frac{1}{2}\boldsymbol{b}_h(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^{n-1}, \varphi_h) + \left(\frac{\boldsymbol{f}^{n+1}+\boldsymbol{f}^n}{2},\, \varphi_h\right)_h,
\end{aligned}
\tag{3.9}
$$

and

$$\boldsymbol{u}_h^{n+1} \in \mathbf{V}_h,\ \frac{1}{\Delta t}\left(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n+\frac{1}{2}},\ \varphi_h\right)_h \quad = \quad -\left(\boldsymbol{G}_h p_h^{n+1},\ \varphi_h\right)_h. \qquad (3.10)$$

Proof :

In order to find the weak formulation (3.9), we multiply by φ_{ij} our finite volume scheme (2.13) and sum for $i = 1, ..., M$ and $j = 1, ..., N$. Regarding each term, we have :

– For the time derivative :

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\boldsymbol{u}_{ij}^{n+\frac{1}{2}} - \boldsymbol{u}_{ij}^{n}}{\Delta t} \cdot \varphi_{ij} = \frac{1}{\Delta t}\left(\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^{n},\ \varphi_h\right)_h.$$

– For the diffusive term :

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\left[\Delta y \frac{\boldsymbol{u}_{i+1j} - \boldsymbol{u}_{ij}}{\Delta x} + \Delta y \frac{\boldsymbol{u}_{i-1j} - \boldsymbol{u}_{ij}}{\Delta x} + \Delta x \frac{\boldsymbol{u}_{ij+1} - \boldsymbol{u}_{ij}}{\Delta y} + \Delta x \frac{\boldsymbol{u}_{ij-1} - \boldsymbol{u}_{ij}}{\Delta y}\right] \cdot \varphi_{ij},$$

$$= \quad -\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x \Delta y \frac{\boldsymbol{u}_{i+1j} - \boldsymbol{u}_{ij}}{\Delta x} \cdot \frac{\varphi_{i+1j} - \varphi_{ij}}{\Delta x} - \sum_{i=1}^{M}\sum_{j=1}^{N-1} \Delta x \Delta y \frac{\boldsymbol{u}_{ij+1} - \boldsymbol{u}_{ij}}{\Delta y} \cdot \frac{\varphi_{ij+1} - \varphi_{ij}}{\Delta y}$$

$$+ \quad \sum_{j=1}^{N} \Delta y \left[\frac{\mathbf{u}_{M+1j} - \mathbf{u}_{Mj}}{\Delta x} \cdot \varphi_{Mj} + \frac{\mathbf{u}_{0j} - \mathbf{u}_{1j}}{\Delta x} \cdot \varphi_{1j}\right]$$

$$+ \quad \sum_{i=1}^{M} \Delta x \left[\frac{\mathbf{u}_{iN+1} - \mathbf{u}_{iN}}{\Delta y} \cdot \varphi_{iN} + \frac{\mathbf{u}_{i0} - \mathbf{u}_{i1}}{\Delta y} \cdot \varphi_{i1}\right],$$

$$= \quad -\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x \Delta y \frac{\boldsymbol{u}_{i+1j} - \boldsymbol{u}_{ij}}{\Delta x} \cdot \frac{\varphi_{i+1j} - \varphi_{ij}}{\Delta x} - \sum_{i=1}^{M}\sum_{j=1}^{N-1} \Delta x \Delta y \frac{\boldsymbol{u}_{ij+1} - \boldsymbol{u}_{ij}}{\Delta y} \cdot \frac{\varphi_{ij+1} - \varphi_{ij}}{\Delta y}$$

$$- \quad \sum_{j=1}^{N} \Delta x \Delta y \left[\frac{\mathbf{u}_{M+1j} - \mathbf{u}_{Mj}}{\Delta x} \cdot \frac{\varphi_{M+1j} - \varphi_{Mj}}{2\Delta x} + \frac{\mathbf{u}_{0j} - \mathbf{u}_{1j}}{\Delta x} \cdot \frac{\varphi_{0j} - \varphi_{1j}}{2\Delta x}\right]$$

$$- \quad \sum_{i=1}^{M} \Delta x \Delta y \left[\frac{\mathbf{u}_{iN+1} - \mathbf{u}_{iN}}{\Delta y} \cdot \frac{\varphi_{iN+1} - \varphi_{iN}}{2\Delta y} + \frac{\mathbf{u}_{i0} - \mathbf{u}_{i1}}{\Delta y} \cdot \frac{\varphi_{i0} - \varphi_{i1}}{2\Delta y}\right]$$

$=$ (due to the boundary conditions (2.14) written for φ_h),

$$= \quad -((\mathbf{u}_h,\ \varphi_h))_h.$$

– For the nonlinear term :

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\left[\Delta y u_{i+\frac{1}{2}j} \mathbf{u}_{i+\frac{1}{2}j} \cdot \varphi_{ij} - \Delta y u_{i-\frac{1}{2}j} \mathbf{u}_{i-\frac{1}{2}j} \cdot \varphi_{ij} + \Delta x v_{ij+\frac{1}{2}} \mathbf{u}_{ij+\frac{1}{2}} \cdot \varphi_{ij}\right.$$

$$\left. - \quad \Delta x v_{ij-\frac{1}{2}} \mathbf{u}_{ij-\frac{1}{2}} \cdot \varphi_{ij}\right] = \mathbf{b}_h(\mathbf{u}_h,\ \mathbf{u}_h,\ \varphi_h).$$

– For the right-hand side :

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\frac{\boldsymbol{f}_{ij}^{n+1}+\boldsymbol{f}_{ij}^{n}}{2}\cdot\varphi_{ij}=\frac{1}{2}\left(\mathbf{f}^{n+1}+\mathbf{f}^{n},\,\varphi_h\right)_h.$$

Finally, we obtain the following weak formulation (3.9) :

$$\begin{aligned}\frac{1}{\Delta t}\left(\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n,\,\varphi_h\right)_h &= \nu\left(\left(\frac{\mathbf{u}_h^{n+\frac{1}{2}}+\mathbf{u}_h^n}{2},\,\varphi_h\right)\right)_h-\frac{3}{2}\mathbf{b}_h(\mathbf{u}_h^n,\mathbf{u}_h^n,\varphi_h)+\frac{1}{2}\mathbf{b}_h(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},\varphi_h)\\ &+\left(\frac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2},\,\varphi_h\right)_h.\end{aligned}$$

To find the weak formulation (3.10), we multiply by $\varphi_{ij}=(\varphi_{ij},\psi_{ij})$ our finite volume scheme (2.21) and sum for $i=1,...,M$ and $j=1,...,N$. Regarding each term, we have :

– For the left-hand side :

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\frac{\boldsymbol{u}_{ij}^{n+1}-\boldsymbol{u}_{ij}^{n+\frac{1}{2}}}{\Delta t}\cdot\varphi_{ij} = \frac{1}{\Delta t}\left(\mathbf{u}_h^{n+1}-\mathbf{u}_h^{n+\frac{1}{2}},\,\varphi_h\right)_h.$$

– For the right-hand side :

$$\begin{aligned}-\sum_{i=1}^{M}\sum_{j=1}^{N}\begin{pmatrix}\frac{\Delta y}{2}\left[p_{i+1j}^{n+1}-p_{i-1j}^{n+1}\right]\\ \frac{\Delta x}{2}\left[p_{ij+1}^{n+1}-p_{ij-1}^{n+1}\right]\end{pmatrix}\cdot\varphi_{ij} &= -\sum_{i=1}^{M}\sum_{j=1}^{N}\Big(\frac{\Delta y}{2}\left[p_{i+1j}^{n+1}-p_{i-1j}^{n+1}\right]\varphi_{ij}\\ &+\frac{\Delta x}{2}\left[p_{ij+1}^{n+1}-p_{ij-1}^{n+1}\right]\psi_{ij}^{n+1}\Big),\\ &= -\left(\mathbf{G}_h p_h^{n+1},\,\varphi_h\right)_h.\end{aligned}$$

So, we have obtained the second weak formulation (3.10).

3.3 Stability result

Our aim in this section is to perform the stability analysis of this scheme. We know [EGH00] that there exist four constants K_1, K_2, K_3 and $\tilde{K}_3$, which are independent of h and k, such that :

$$|\mathbf{u}_h^0|_h \leq K_1, \tag{3.11}$$

$$\Delta t\sum_{n=0}^{N_t-1}|\mathbf{f}_h^n|_h \leq K_2, \tag{3.12}$$

$$|\mathbf{u}_h^1|_h^2+\nu\Delta t\,||\mathbf{u}_h^1||_h^2 \leq K_3, \tag{3.13}$$

$$\left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2+\nu\Delta t\left|\left|\mathbf{u}_h^{1+\frac{1}{2}}\right|\right|_h^2 \leq \tilde{K}_3. \tag{3.14}$$

Theorem 3.1 (Stability). *We assume that (3.11)-(3.14) hold. Then there exist K_4 and K_5 independent of h and k such that, if :*

$$\Delta t \leq \frac{2c_1^2}{\nu}, \tag{3.15}$$

where c_1 is the constant of the Poincaré inequality (Lemma 4.1), and,

$$\Delta t S_h^2 \leq min(\frac{1}{64\nu}, \frac{\nu}{5120K_4}), \tag{3.16}$$

where S_h is the constant of the inverse inequality (Lemma 4.2), then

$$|\boldsymbol{u}_h^n|_h^2 \leq K_4 \quad \text{for } n = 0, ..., N_t, \tag{3.17}$$

$$\sum_{n=0}^{N_t-1} |\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n|_h^2 \leq K_5, \tag{3.18}$$

$$\nu \Delta t \sum_{n=0}^{N_t-1} ||\boldsymbol{u}_h^{n+1}||_h^2 \leq K_5. \tag{3.19}$$

Proof :

As the results of Sections 4 and 5 are needed to show this stability result, we have postponed the proof of Theorem 3.1 to Section 6.

4 Technical lemmas

This section contains the proofs of lemmas used to show the stability theorem. We start by the Poincaré inequality and an inverse inequality. Then, we prove an usual lemma for the Navier-Stokes equations : the discrete nonlinear term properties. At last, we provide an estimate for the discrete nonlinear term and we prove a specific lemma required for the proof of the discrete nonlinear term estimation.

4.1 The Poincaré inequality and an inverse inequality

Lemma 4.1 (Poincaré inequality). *For $\boldsymbol{u}_h \in \boldsymbol{W}_h$, there exists a constant $c_1 = diam(\Omega)$, such that :*

$$|\boldsymbol{u}_h|_h \leq c_1 \, ||\boldsymbol{u}_h||_h \, .$$

Proof :

As these discrete norms are usual in a finite volume context, the reader could find the proof of this Poincaré inequality in [EGH00]. Here, due to our discretization, the constant c_1 is equal to $max(N\Delta x, M\Delta y) = max(L_1, L_2)$.

Lemma 4.2 (Inverse inequality). *For $u_h \in \mathbf{W}_h$, there exists a constant S_h, depending on h, such that :*

$$||u_h||_h \leq S_h \, |u_h|_h \, .$$

Proof :

We recall the definition of the discrete norm corresponding to the continuous norm $H_0^1(\Omega)^2$:

$$\begin{aligned} ||\mathbf{u}_h||_h^2 &= \sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x \Delta y \left(\frac{\mathbf{u}_{i+1j}-\mathbf{u}_{ij}}{\Delta x}\right)^2 + \sum_{i=1}^{M}\sum_{j=1}^{N-1} \Delta x \Delta y \left(\frac{\mathbf{u}_{ij+1}-\mathbf{u}_{ij}}{\Delta y}\right)^2 \\ &+ \frac{1}{2}\sum_{j=1}^{N} \Delta x \Delta y \left[\left(\frac{\mathbf{u}_{M+1j}-\mathbf{u}_{Mj}}{\Delta x}\right)^2 + \left(\frac{\mathbf{u}_{1j}-\mathbf{u}_{0j}}{\Delta x}\right)^2\right] \\ &+ \frac{1}{2}\sum_{i=1}^{M} \Delta x \Delta y \left[\left(\frac{\mathbf{u}_{iN+1}-\mathbf{u}_{iN}}{\Delta y}\right)^2 + \left(\frac{\mathbf{u}_{i1}-\mathbf{u}_{i0}}{\Delta y}\right)^2\right]. \end{aligned}$$

Then we use the boundary conditions (2.14) and the inequality $(a-b)^2 \leq 2a^2 + 2b^2$, we find :

$$\begin{aligned} ||\mathbf{u}_h||_h^2 &\leq \frac{2}{\Delta x^2}\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x \Delta y (\boldsymbol{u}_{i+1j}^2 + \boldsymbol{u}_{ij}^2) + \frac{2}{\Delta y^2}\sum_{i=1}^{M}\sum_{j=1}^{N-1} \Delta x \Delta y (\boldsymbol{u}_{ij+1}^2 + \boldsymbol{u}_{ij}^2) \\ &+ \frac{2}{\Delta x^2}\sum_{j=1}^{N} \Delta x \Delta y (\mathbf{u}_{Mj}^2 + \mathbf{u}_{1j}^2) + \frac{2}{\Delta y^2}\sum_{i=1}^{M} \Delta x \Delta y (\mathbf{u}_{iN}^2 + \mathbf{u}_{i1}^2), \\ &\leq 4\Big(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\Big) \, |\mathbf{u}_h|_h^2 \, . \end{aligned}$$

So, the inequality is proved and $S_h = \sqrt{4\Big(\dfrac{1}{\Delta x^2} + \dfrac{1}{\Delta y^2}\Big)}$.

4.2 Standard lemma

Lemma 4.3 (Properties of the discrete nonlinear term). *For all approximate functions $u_h \in \mathbf{V}_h$ and φ_h, $\xi_h \in \mathbf{W}_h$, the discrete nonlinear term satisfies the properties :*

$$b_h(u_h, \varphi_h, \varphi_h) = 0, \tag{4.1}$$

$$b_h(u_h, \varphi_h, \xi_h) = -b_h(u_h, \xi_h, \varphi_h). \tag{4.2}$$

Proof :

We recall the definition of the discrete nonlinear term :

$$\begin{aligned} b_h(\mathbf{u}_h, \varphi_h, \xi_h) &= \sum_{i=1}^{M}\sum_{j=1}^{N} \Big[\Delta y u_{i+\frac{1}{2}j} \varphi_{i+\frac{1}{2}j} \cdot \xi_{ij} - \Delta y u_{i-\frac{1}{2}j} \varphi_{i-\frac{1}{2}j} \cdot \xi_{ij} \\ &+ \Delta x v_{ij+\frac{1}{2}} \varphi_{ij+\frac{1}{2}} \cdot \xi_{ij} - \Delta x v_{ij-\frac{1}{2}} \varphi_{ij-\frac{1}{2}} \cdot \xi_{ij}\Big]. \end{aligned}$$

If we set $\xi_h = \varphi_h$ then :

$$\begin{aligned} b_h(\mathbf{u}_h, \varphi_h, \varphi_h) &= \sum_{i=1}^{M}\sum_{j=1}^{N}\Big[\Delta y u_{i+\frac12 j}\varphi_{i+\frac12 j}\cdot\varphi_{ij} - \Delta y u_{i-\frac12 j}\varphi_{i-\frac12 j}\cdot\varphi_{ij} \\ &+ \Delta x v_{ij+\frac12}\varphi_{ij+\frac12}\cdot\varphi_{ij} - \Delta x v_{ij-\frac12}\varphi_{ij-\frac12}\cdot\varphi_{ij}\Big]. \end{aligned}$$

The fluxes being linearly interpolated, it follows :

$$\begin{aligned} b_h(\mathbf{u}_h, \varphi_h, \varphi_h) &= \sum_{j=1}^{N}\Big[\sum_{i=1}^{M-1}\Delta y u_{i+\frac12 j}\frac{\varphi_{i+1j}+\varphi_{ij}}{2}\cdot\varphi_{ij} - \sum_{i=2}^{M}\Delta y u_{i-\frac12 j}\frac{\varphi_{i-1j}+\varphi_{ij}}{2}\cdot\varphi_{ij}\Big] \\ &+ \sum_{i=1}^{M}\Big[\sum_{j=1}^{N-1}\Delta x v_{ij+\frac12}\frac{\varphi_{ij+1}+\varphi_{ij}}{2}\cdot\varphi_{ij} - \sum_{j=2}^{N}\Delta x v_{ij-\frac12}\frac{\varphi_{ij-1}+\varphi_{ij}}{2}\cdot\varphi_{ij}\Big], \end{aligned}$$

which can be rewritten as :

$$\begin{aligned} b_h(\mathbf{u}_h, \varphi_h, \varphi_h) &= \sum_{j=1}^{N}\Big[\sum_{i=1}^{M}\frac{\Delta y}{2}(u_{i+\frac12 j} - u_{i-\frac12 j})\varphi_{ij}^2 + \sum_{i=1}^{M-1}\frac{\Delta y}{2}u_{i+\frac12 j}\varphi_{i+1j}\cdot\varphi_{ij} \\ &- \sum_{i=2}^{M}\frac{\Delta y}{2}u_{i-\frac12 j}\varphi_{i-1j}\cdot\varphi_{ij}\Big] + \sum_{i=1}^{M}\Big[\sum_{j=1}^{N}\frac{\Delta x}{2}(v_{ij+\frac12} - v_{ij-\frac12})\varphi_{ij}^2 \\ &+ \sum_{j=1}^{N-1}\frac{\Delta x}{2}v_{ij+\frac12}\varphi_{ij+1}\cdot\varphi_{ij} - \sum_{j=2}^{N}\frac{\Delta x}{2}v_{ij-\frac12}\varphi_{ij-1}\cdot\varphi_{ij}\Big]. \end{aligned}$$

In each square brackets, the two last terms cancel out and we obtain :

$$\begin{aligned} b_h(\mathbf{u}_h, \varphi_h, \varphi_h) &= \frac12\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\Big(\frac{u_{i+\frac12 j} - u_{i-\frac12 j}}{\Delta x} + \frac{v_{ij+\frac12} - v_{ij-\frac12}}{\Delta y}\Big)\varphi_{ij}^2, \\ &= \frac14\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\Big(\frac{u_{i+1j} - u_{i-1j}}{\Delta x} + \frac{v_{ij+1} - v_{ij-1}}{\Delta y}\Big)\varphi_{ij}^2. \end{aligned}$$

Finally, according to the incompressibility condition (2.19), the right-hand side is equal to zero, so :

$$b_h(\mathbf{u}_h, \varphi_h, \varphi_h) = 0.$$

Moreover,considering that $b_h(\mathbf{u}_h,\ \varphi_h + \xi_h\ , \varphi_h + \xi_h)$, we see that (4.1) classically implies (4.2). The lemma is proved.

4.3 Specific lemmas for the Navier-Stokes equations

Lemma 4.4 (Estimates on the discrete nonlinear term). *For all functions $\boldsymbol{u}_h = (u_h, v_h)$ in $\mathbf{V}_h$, $\varphi_h = (\varphi_h, \psi_h)$ and $\boldsymbol{\xi}_h = (\xi_h, \eta_h)$ in $\boldsymbol{W}_h$,*

$$\begin{aligned} \Big|b_h(\boldsymbol{u}_h, \varphi_h, \boldsymbol{\xi}_h)\Big| &\le 2\sqrt{2}\,|\boldsymbol{u}_h|_h^{\frac12}\,||\boldsymbol{u}_h||_h^{\frac12}\,|\varphi_h|_h^{\frac12}\,||\varphi_h||_h^{\frac12}\,||\boldsymbol{\xi}_h||_h\,, \\ &\le 2\sqrt{2}\,|\boldsymbol{u}_h|_h^{\frac12}\,||\boldsymbol{u}_h||_h^{\frac12}\,|\boldsymbol{\xi}_h|_h^{\frac12}\,||\boldsymbol{\xi}_h||_h^{\frac12}\,||\varphi_h||_h\,. \end{aligned}$$

Proof :

Remark 4.1 (Notation). In this proof, we will use the same notation for the discrete norms of $L^2(\Omega)$ and of $L^2(\Omega)^2$. So, we will write for $\mathbf{u}_h = (u_h, v_h)$, $|u_h|_h$ for the discrete $L^2(\Omega)$-norm ofu_h and $|\mathbf{u}_h|_h$ for the discrete $L^2(\Omega)^2$-norm of $\mathbf{u}_h$.

We want to bound properly the discrete nonlinear term :

$$\begin{aligned} |b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| &= \Bigg| \sum_{i=1}^{M} \sum_{j=1}^{N} \Big(\Delta y u_{i+\frac{1}{2}j} \varphi_{i+\frac{1}{2}j} \cdot \boldsymbol{\xi}_{ij} - \Delta y u_{i-\frac{1}{2}j} \varphi_{i-\frac{1}{2}j} \cdot \boldsymbol{\xi}_{ij} \\ &+ \Delta x v_{ij+\frac{1}{2}} \varphi_{ij+\frac{1}{2}} \cdot \boldsymbol{\xi}_{ij} - \Delta x v_{ij-\frac{1}{2}} \varphi_{ij-\frac{1}{2}} \cdot \boldsymbol{\xi}_{ij} \Big) \Bigg|. \end{aligned}$$

We simplify it using the Dirichlet boundary conditions $\mathbf{u}_{\frac{1}{2}j} = \mathbf{u}_{N+\frac{1}{2}j} = \mathbf{u}_{i\frac{1}{2}} = \mathbf{u}_{iM+\frac{1}{2}} = 0$:

$$\begin{aligned} |b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| &= \Bigg| \sum_{j=1}^{N} \left(\sum_{i=1}^{M-1} \Delta y u_{i+\frac{1}{2}j} \varphi_{i+\frac{1}{2}j} \cdot \boldsymbol{\xi}_{ij} - \sum_{i=2}^{M} \Delta y u_{i-\frac{1}{2}j} \varphi_{i-\frac{1}{2}j} \cdot \boldsymbol{\xi}_{ij} \right) \\ &+ \sum_{i=1}^{M} \left(\sum_{j=1}^{N-1} \Delta x v_{ij+\frac{1}{2}} \varphi_{ij+\frac{1}{2}} \cdot \boldsymbol{\xi}_{ij} - \sum_{j=2}^{N} \Delta x v_{ij-\frac{1}{2}} \varphi_{ij-\frac{1}{2}} \cdot \boldsymbol{\xi}_{ij} \right) \Bigg|, \end{aligned}$$

and we arrange this expression and write :

$$\begin{aligned} |b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| &= \Bigg| \underbrace{\sum_{j=1}^{N} \sum_{i=1}^{M-1} \Delta x \Delta y u_{i+\frac{1}{2}j} \varphi_{i+\frac{1}{2}j} \cdot \frac{\boldsymbol{\xi}_{ij} - \boldsymbol{\xi}_{i+1j}}{\Delta x}}_{I} \\ &+ \underbrace{\sum_{i=1}^{M} \sum_{j=1}^{N-1} \Delta x \Delta y v_{ij+\frac{1}{2}} \varphi_{ij+\frac{1}{2}} \cdot \frac{\boldsymbol{\xi}_{ij} - \boldsymbol{\xi}_{ij+1}}{\Delta y}}_{II} \Bigg|. \end{aligned}$$

Then, the method used to bound these two terms I and II being very similar, we explain how to bound the first term I and we will briefly do the same for the second term II. For I, we have :

$$|I| = \left| \sum_{j=1}^{N} \sum_{i=1}^{M-1} \Delta x \Delta y u_{i+\frac{1}{2}j} \Big(\varphi_{i+\frac{1}{2}j} \frac{\xi_{ij} - \xi_{i+1j}}{\Delta x} + \psi_{i+\frac{1}{2}j} \frac{\eta_{ij} - \eta_{i+1j}}{\Delta x} \Big) \right|,$$

which can be bounded as :

$$\begin{aligned} |I| &\leq \left| \sum_{j=1}^{N} \sum_{i=1}^{M-1} \Big(\sqrt{\Delta x \Delta y} u_{i+\frac{1}{2}j} \varphi_{i+\frac{1}{2}j} \Big) \Big(\sqrt{\Delta x \Delta y} \frac{\xi_{ij} - \xi_{i+1j}}{\Delta x} \Big) \right| \\ &+ \left| \sum_{j=1}^{N} \sum_{i=1}^{M-1} \Big(\sqrt{\Delta x \Delta y} u_{i+\frac{1}{2}j} \psi_{i+\frac{1}{2}j} \Big) \Big(\sqrt{\Delta x \Delta y} \frac{\eta_{ij} - \eta_{i+1j}}{\Delta x} \Big) \right|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain :

$$\begin{aligned}|I| &\leq \Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x\Delta y u^2_{i+\frac{1}{2}j}\varphi^2_{i+\frac{1}{2}j}\Big]^{\frac{1}{2}}\Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\xi_{i+1j}-\xi_{ij}}{\Delta x}\Big)^2\Big]^{\frac{1}{2}}\\ &+ \Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x\Delta y u^2_{i+\frac{1}{2}j}\psi^2_{i+\frac{1}{2}j}\Big]^{\frac{1}{2}}\Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\eta_{i+1j}-\eta_{ij}}{\Delta x}\Big)^2\Big]^{\frac{1}{2}};\end{aligned}$$

using again the Cauchy-Schwarz inequality, we find :

$$\begin{aligned}|I| &\leq \Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x\Delta y u^4_{i+\frac{1}{2}j}\Big]^{\frac{1}{4}}\Bigg(\Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\varphi^4_{i+\frac{1}{2}j}\Big]^{\frac{1}{4}}\Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\xi_{i+1j}-\xi_{ij}}{\Delta x}\Big)^2\Big]^{\frac{1}{2}}\\ &+ \Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1} \Delta x\Delta y\psi^4_{i+\frac{1}{2}j}\Big]^{\frac{1}{4}}\Big[\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\eta_{i+1j}-\eta_{ij}}{\Delta x}\Big)^2\Big]^{\frac{1}{2}}\Bigg).\end{aligned}$$

Moreover, due to the inequality $(a+b)^4 \leq 8a^4 + 8b^4$ and the definition of the fluxes by linear interpolation, we obtain the following upper bound :

$$\begin{aligned}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y u^4_{i+\frac{1}{2}j}\Big)^{\frac{1}{4}} &= \Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{u_{i+1j}+u_{ij}}{2}\Big)^4\Big)^{\frac{1}{4}},\\ &\leq \Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{u^4_{i+1j}+u^4_{ij}}{2}\Big)\Big)^{\frac{1}{4}},\\ &= \Big(\sum_{j=1}^{N}\sum_{i=2}^{M}\Delta x\Delta y\frac{1}{2}u^4_{ij} + \sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\frac{1}{2}u^4_{ij}\Big)^{\frac{1}{4}},\\ &\leq \Big(\sum_{j=1}^{N}\sum_{i=1}^{M}\Delta x\Delta y u^4_{ij}\Big)^{\frac{1}{4}},\\ &= |u^2_h|_h^{\frac{1}{2}}.\end{aligned}$$

So, from this inequality, we can majorize the first term as follows :

$$\begin{aligned}|I| &\leq |u^2_h|_h^{\frac{1}{2}}\Bigg[|\varphi^2_h|_h^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\xi_{i+1j}-\xi_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\\ &+ |\psi^2_h|_h^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\eta_{i+1j}-\eta_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\Bigg].\end{aligned}$$

Proceeding similarly for the second term II, we find :

$$\begin{aligned}|II| &\leq |v_h^2|_h^{\frac{1}{2}}\Bigg[|\varphi_h^2|_h^{\frac{1}{2}}\Big(\sum_{i=1}^{M}\sum_{j=1}^{N-1}\Delta x\Delta y\Big(\frac{\xi_{ij+1}-\xi_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}} \\ &+ |\psi_h^2|_h^{\frac{1}{2}}\Big(\sum_{i=1}^{M}\sum_{j=1}^{N-1}\Delta x\Delta y\Big(\frac{\eta_{ij+1}-\eta_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\Bigg].\end{aligned}$$

Thus, the nonlinear term b_h can be bounded as follows :

$$\begin{aligned}|b_h(\mathbf{u}_h,\varphi_h,\boldsymbol{\xi}_h)| &\leq |u_h^2|_h^{\frac{1}{2}}\Bigg[|\varphi_h^2|_h^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\xi_{i+1j}-\xi_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}} \\ &\quad+|\psi_h^2|_h^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M-1}\Delta x\Delta y\Big(\frac{\eta_{i+1j}-\eta_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\Bigg] \\ &\quad+|v_h^2|_h^{\frac{1}{2}}\Bigg[|\varphi_h^2|_h^{\frac{1}{2}}\Big(\sum_{i=1}^{M}\sum_{j=1}^{N-1}\Delta x\Delta y\Big(\frac{\xi_{ij+1}-\xi_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}} \\ &\quad+|\psi_h^2|_h^{\frac{1}{2}}\Big(\sum_{i=1}^{M}\sum_{j=1}^{N-1}\Delta x\Delta y\Big(\frac{\eta_{ij+1}-\eta_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\Bigg], \\ &\leq |u_h^2|_h^{\frac{1}{2}}\Bigg[|\varphi_h^2|_h^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M}\Delta x\Delta y\Big(\frac{\xi_{i+1j}-\xi_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}} \\ &\quad+|\psi_h^2|_h^{\frac{1}{2}}\Big(\sum_{j=1}^{N}\sum_{i=1}^{M}\Delta x\Delta y\Big(\frac{\eta_{i+1j}-\eta_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\Bigg] \\ &\quad+|v_h^2|_h^{\frac{1}{2}}\Bigg[|\varphi_h^2|_h^{\frac{1}{2}}\Big(\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\Big(\frac{\xi_{ij+1}-\xi_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}} \\ &\quad+|\psi_h^2|_h^{\frac{1}{2}}\Big(\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\Big(\frac{\eta_{ij+1}-\eta_{ij}}{\Delta x}\Big)^2\Big)^{\frac{1}{2}}\Bigg].\end{aligned}$$

It is convenient to introduce the following notation :

$$D_{x,h}\varphi_h(x,y) = \frac{\varphi_{i+1j}-\varphi_{ij}}{\Delta x} \text{ if } (x,y)\in K_{ij}, \tag{4.3}$$

$$D_{y,h}\varphi_h(x,y) = \frac{\varphi_{ij+1}-\varphi_{ij}}{\Delta y} \text{ if } (x,y)\in K_{ij}. \tag{4.4}$$

Considering the above notations, the discrete nonlinear term satisfies :

$$\begin{aligned}|b_h(\mathbf{u}_h,\varphi_h,\boldsymbol{\xi}_h)| &\leq |u_h^2|_h^{\frac{1}{2}}\Big[|\varphi_h^2|_h^{\frac{1}{2}}|D_{x,h}\xi_h|_h + |\psi_h^2|_h^{\frac{1}{2}}|D_{x,h}\eta_h|_h\Big] \\ &+ |v_h^2|_h^{\frac{1}{2}}\Big[|\varphi_h^2|_h^{\frac{1}{2}}|D_{y,h}\xi_h|_h + |\psi_h^2|_h^{\frac{1}{2}}|D_{y,h}\eta_h|_h\Big].\end{aligned}$$

Due to the Cauchy-Schwarz inequality :

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \leq \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}\sqrt{b_1^2 + b_2^2 + b_3^2 + b_4^2},$$

we have :

$$\begin{aligned} |b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| &\leq \left(|u_h^2|_h^{\frac{1}{2}} |\varphi_h^2|_h^{\frac{1}{2}}\right) |D_{x,h}\xi_h|_h + \left(|u_h^2|_h^{\frac{1}{2}} |\psi_h^2|_h^{\frac{1}{2}}\right) |D_{x,h}\eta_h|_h \\ &+ \left(|v_h^2|_h^{\frac{1}{2}} |\varphi_h^2|_h^{\frac{1}{2}}\right) |D_{y,h}\xi_h|_h + \left(|v_h^2|_h^{\frac{1}{2}} |\psi_h^2|_h^{\frac{1}{2}}\right) |D_{y,h}\eta_h|_h \, , \\ &\leq \left[|u_h^2|_h + |v_h^2|_h\right]^{\frac{1}{2}} \left[|\varphi_h^2|_h + |\psi_h^2|_h\right]^{\frac{1}{2}} \\ &\cdot \left[|D_{x,h}\xi_h|_h^2 + |D_{x,h}\eta_h|_h^2 + |D_{y,h}\xi_h|_h^2 + |D_{y,h}\eta_h|_h^2\right]^{\frac{1}{2}} . \end{aligned}$$

Thanks to Lemma 4.5 :

$$\begin{aligned} |b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| &\leq 2\left[|u_h|_h \, ||u_h||_h + |v_h|_h \, ||v_h||_h\right]^{\frac{1}{2}} \left[|\varphi_h|_h \, ||\varphi_h||_h + |\psi_h|_h \, ||\psi_h||_h\right]^{\frac{1}{2}} \\ &\cdot \left[|D_{x,h}\xi_h|_h^2 + |D_{x,h}\eta_h|_h^2 + |D_{y,h}\xi_h|_h^2 + |D_{y,h}\eta_h|_h^2\right]^{\frac{1}{2}} , \end{aligned}$$

and due to the Cauchy-Schwarz inequality $a_1b_1 + a_2b_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$:

$$\begin{aligned} |b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| &\leq 2\left[|u_h|_h^2 + |v_h|_h^2\right]^{\frac{1}{4}} \left[||u_h||_h^2 + ||v_h||_h^2\right]^{\frac{1}{4}} \left[|\varphi_h|_h^2 + |\psi_h|_h^2\right]^{\frac{1}{4}} \left[||\varphi_h||_h^2 + ||\psi_h||_h^2\right]^{\frac{1}{4}} \\ &\cdot \left[|D_{x,h}\xi_h|_h^2 + |D_{x,h}\eta_h|_h^2 + |D_{y,h}\xi_h|_h^2 + |D_{y,h}\eta_h|_h^2\right]^{\frac{1}{2}} . \end{aligned}$$

Therefore, we note that :

$$\left[|D_{x,h}\xi_h|_h^2 + |D_{x,h}\eta_h|_h^2 + |D_{y,h}\xi_h|_h^2 + |D_{y,h}\eta_h|_h^2\right] \leq 2\,||\boldsymbol{\xi}_h||_h^2 .$$

Finally :

$$|b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)| \leq 2\sqrt{2}\,|\mathbf{u}_h|_h^{\frac{1}{2}} \, ||\mathbf{u}_h||_h^{\frac{1}{2}} \, |\varphi_h|_h^{\frac{1}{2}} \, ||\varphi_h||_h^{\frac{1}{2}} \, ||\boldsymbol{\xi}_h||_h .$$

Also, Lemma 4.3 implies :

$$\begin{aligned} \Big|b_h(\mathbf{u}_h, \varphi_h, \boldsymbol{\xi}_h)\Big| &= \Big| - b_h(\mathbf{u}_h, \boldsymbol{\xi}_h, \varphi_h)\Big|, \\ &= \Big|b_h(\mathbf{u}_h, \boldsymbol{\xi}_h, \varphi_h)\Big|, \\ &\leq 2\sqrt{2}\,|\mathbf{u}_h|_h^{\frac{1}{2}} \, ||\mathbf{u}_h||_h^{\frac{1}{2}} \, |\boldsymbol{\xi}_h|_h^{\frac{1}{2}} \, ||\boldsymbol{\xi}_h||_h^{\frac{1}{2}} \, ||\varphi_h||_h . \end{aligned}$$

Lemma 4.5.

$$|u_h^2|_h \leq 2\,|u_h|_h \, ||u_h||_h .$$

Proof :

First, we note that :

$$\begin{aligned} u_{ij}^2 &= u_{1j}^2 + \sum_{k=1}^{i-1} \big(u_{k+1j}^2 - u_{kj}^2\big), \\ &= u_{1j}^2 + \sum_{k=1}^{i-1} \big(u_{k+1j} - u_{kj}\big)\big(u_{k+1j} + u_{kj}\big), \\ &\leq u_{1j}^2 + \sum_{k=1}^{i-1} \big|u_{k+1j}\big|\big|u_{k+1j} - u_{kj}\big| + \big|u_{kj}\big|\big|u_{k+1j} - u_{kj}\big|, \\ &\leq u_{1j}^2 + \sum_{k=2}^{i} \big|u_{kj}\big|\big|u_{kj} - u_{k-1j}\big| + \sum_{k=1}^{i-1} \big|u_{kj}\big|\big|u_{k+1j} - u_{kj}\big|. \end{aligned}$$

Then, using the boundary conditions (2.14), we find :

$$u_{1j}^2 = \frac{1}{2} u_{1j}\big(u_{1j} - u_{0j}\big), \tag{4.5}$$

this yields :

$$u_{ij}^2 \leq \sum_{k=1}^{M} \Big[\big|u_{kj}\big|\big|u_{kj} - u_{k-1j}\big| + \big|u_{kj}\big|\big|u_{k+1j} - u_{kj}\big|\Big].$$

Hence :

$$\Delta y u_{ij}^2 \leq \Delta y \sum_{k=1}^{M} \Big[\big|u_{kj}\big|\big|u_{kj} - u_{k-1j}\big| + \big|u_{kj}\big|\big|u_{k+1j} - u_{kj}\big|\Big].$$

Proceeding similarly in the other direction, we get :

$$\Delta x u_{ij}^2 \leq \Delta x \sum_{l=1}^{N} \Big[\big|u_{il}\big|\big|u_{il} - u_{il-1}\big| + \big|u_{il}\big|\big|u_{il+1} - u_{il}\big|\Big].$$

So :

$$\begin{aligned} \Delta x \Delta y u_{ij}^4 \leq & \left(\Delta y \sum_{k=1}^{M} \Big[\big|u_{kj}\big|\big|u_{kj} - u_{k-1j}\big| + \big|u_{kj}\big|\big|u_{k+1j} - u_{kj}\big|\Big]\right) \\ & \cdot \left(\Delta x \sum_{l=1}^{N} \Big[\big|u_{il}\big|\big|u_{il} - u_{il-1}\big| + \big|u_{il}\big|\big|u_{il+1} - u_{il}\big|\Big]\right). \end{aligned}$$

Now, summing for $i = 1, ..., M$ and $j = 1, ..., N$, we obtain :

$$\begin{aligned} \sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x \Delta y u_{ij}^4 \leq & \left(\sum_{j=1}^{N}\sum_{i=1}^{M} \Delta y \Big[\big|u_{ij}\big|\big|u_{ij} - u_{i-1j}\big| + \big|u_{ij}\big|\big|u_{i+1j} - u_{ij}\big|\Big]\right) \\ & \cdot \left(\sum_{i=1}^{M}\sum_{j=1}^{N} \Delta x \Big[\big|u_{ij}\big|\big|u_{ij} - u_{ij-1}\big| + \big|u_{ij}\big|\big|u_{ij+1} - u_{ij}\big|\Big]\right). \end{aligned}$$

Using the Cauchy-Schwarz inequality for each term, like for example :

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y|u_{ij}|\frac{|u_{ij}-u_{i-1j}|}{\Delta x} \leq \left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y|u_{ij}|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\left(\frac{u_{ij}-u_{i-1j}}{\Delta x}\right)^2\right]^{\frac{1}{2}},$$

we find :

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y u_{ij}^4 \leq & \left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y|u_{ij}|^2\right]\\ & \cdot\left[\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\left(\frac{u_{ij}-u_{i-1j}}{\Delta x}\right)^2\right]^{\frac{1}{2}}+\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\left(\frac{u_{i+1j}-u_{ij}}{\Delta x}\right)^2\right]^{\frac{1}{2}}\right]\\ & \cdot\left[\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\left(\frac{u_{ij}-u_{ij-1}}{\Delta y}\right)^2\right]^{\frac{1}{2}}+\left[\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\left(\frac{u_{ij+1}-u_{ij}}{\Delta y}\right)^2\right]^{\frac{1}{2}}\right].\end{aligned}$$

Then, we majorize each term as follows :

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y\left(\frac{u_{ij}-u_{i-1j}}{\Delta x}\right)^2 &= \sum_{j=1}^{N}\sum_{i=0}^{M-1}\Delta x\Delta y\left(\frac{u_{i+1j}-u_{ij}}{\Delta x}\right)^2,\\ &\leq \sum_{j=1}^{N}\sum_{i=0}^{M}\Delta x\Delta y\left(\frac{u_{i+1j}-u_{ij}}{\Delta x}\right)^2\end{aligned}$$

hence the inequality :

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y u_{ij}^4 \leq\ & 4\,|u_h|_h^2\left[\sum_{j=1}^{N}\sum_{i=0}^{M}\Delta x\Delta y\left(\frac{u_{i+1j}-u_{ij}}{\Delta x}\right)^2\right]^{\frac{1}{2}}\\ & \cdot\left[\sum_{i=1}^{M}\sum_{j=0}^{N}\Delta x\Delta y\left(\frac{u_{ij+1}-u_{ij}}{\Delta y}\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Due to the well-known inequality $ab \leq (a^2+b^2)/2$, we have :

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{N}\Delta x\Delta y u_{ij}^4 &\leq 2\,|u_h|_h^2\left[\sum_{j=1}^{N}\sum_{i=0}^{M}\Delta x\Delta y\left(\frac{u_{i+1j}-u_{ij}}{\Delta x}\right)^2+\sum_{i=1}^{M}\sum_{j=0}^{N}\Delta x\Delta y\left(\frac{u_{ij+1}-u_{ij}}{\Delta y}\right)^2\right],\\ &\leq 4\,|u_h|_h^2\,||u_h||_h^2.\end{aligned}$$

The above inequality implies that :

$$|u_h^2|_h \leq 2\,|u_h|_h\,||u_h||_h.$$

The proof is achieved.

5 A priori estimates

In this section, we prove the a priori estimates required for the proof of the stability theorem. We start by a lemma which contains the properties deduced from the scheme equation (2.21). Then, we will use the scheme equation (2.13) and this lemma in order to obtain two a priori estimates : one for the intermediate velocities $\mathbf{u}_h^{n+\frac{1}{2}} \in \mathbf{W}_h$ and one for the new velocities $\mathbf{u}_h^{n+1} \in \mathbf{V}_h$.

Lemma 5.1. *The second step (2.21) of our finite volume scheme implies :*

$$\left|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n+\frac{1}{2}}\right|_h^2 = \left|\boldsymbol{u}_h^{n+\frac{1}{2}}\right|_h^2 - \left|\boldsymbol{u}_h^{n+1}\right|_h^2, \tag{5.1}$$

$$\left|\boldsymbol{u}_h^{n+1}\right|_h \leq \left|\boldsymbol{u}_h^{n+\frac{1}{2}}\right|_h, \tag{5.2}$$

$$\left|\boldsymbol{u}_h^{n+1}\right|_h^2 - \left(\boldsymbol{u}_h^{n+\frac{1}{2}},\, \boldsymbol{u}_h^{n+1}\right)_h = 0. \tag{5.3}$$

Proof :

We note that (5.1) is obtained from (5.3) :

$$\begin{aligned}
(5.3) \quad &\Longrightarrow \quad \left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}},\, \mathbf{u}_h^{n+1}\right)_h = 0,\\
&\Longrightarrow \quad \left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}},\, 2\mathbf{u}_h^{n+1}\right)_h = 0,\\
&\Longrightarrow \quad \left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}},\, \mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}} + \mathbf{u}_h^{n+\frac{1}{2}}\right)_h + \left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}},\, \mathbf{u}_h^{n+1}\right)_h = 0,\\
&\Longrightarrow \quad \left|\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 - \left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 + \left|\mathbf{u}_h^{n+1}\right|_h^2 = 0.
\end{aligned}$$

Moreover, we also note that (5.2) is obtained from (5.3) :

$$\begin{aligned}
(5.3) \quad &\Longrightarrow \quad \left|\mathbf{u}_h^{n+1}\right|_h^2 = \left(\mathbf{u}_h^{n+\frac{1}{2}},\, \mathbf{u}_h^{n+1}\right)_h,\\
&\Longrightarrow \quad \left|\mathbf{u}_h^{n+1}\right|_h^2 \leq \left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h \left|\mathbf{u}_h^{n+1}\right|_h \text{ due to the Cauchy-Schwarz inequality,}\\
&\Longrightarrow \quad \left|\mathbf{u}_h^{n+1}\right|_h \leq \left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h.
\end{aligned}$$

Finally, we just need to show (5.3). To do so, we replace φ_h by $\mathbf{u}_h^{n+1}$ in (3.10) :

$$\frac{1}{\Delta t}\left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+\frac{1}{2}},\, \mathbf{u}_h^{n+1}\right)_h \quad = \quad -\left(\mathrm{G}_h p_h^{n+1},\, \mathbf{u}_h^{n+1}\right)_h,$$

and we reorganize the right-hand side as follows :

$$
\begin{aligned}
\left(\mathbf{G}_h p_h^{n+1},\ \mathbf{u}_h^{n+1}\right)_h &= \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\Delta y}{2}\left(p_{i+1j}^{n+1} - p_{i-1j}^{n+1}\right) u_{ij}^{n+1} + \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\Delta x}{2}\left(p_{ij+1}^{n+1} - p_{ij-1}^{n+1}\right) v_{ij}^{n+1}, \\
&= \sum_{j=1}^{N} \frac{\Delta y}{2}\left(\sum_{i=2}^{M+1} p_{ij}^{n+1} u_{i-1j}^{n+1} - \sum_{i=0}^{M-1} p_{ij}^{n+1} u_{i+1j}^{n+1}\right) \\
&+ \sum_{j=1}^{N} \frac{\Delta x}{2}\left(\sum_{j=2}^{N+1} p_{ij}^{n+1} v_{ij-1}^{n+1} - \sum_{j=0}^{N-1} p_{ij}^{n+1} v_{ij+1}^{n+1}\right), \\
&= \sum_{j=1}^{N}\sum_{i=1}^{M} p_{ij}^{n+1}\left(\frac{\Delta y}{2}\left[u_{i-1j}^{n+1} - u_{i+1j}^{n+1}\right] + \frac{\Delta x}{2}\left[v_{ij-1}^{n+1} - v_{ij+1}^{n+1}\right]\right) \\
&+ \sum_{j=1}^{N} \frac{\Delta y}{2}\left(-p_{1j}^{n+1} u_{0j}^{n+1} + p_{M+1j}^{n+1} u_{Mj}^{n+1} - p_{0j}^{n+1} u_{1j}^{n+1} + p_{Mj}^{n+1} u_{M+1j}^{n+1}\right) \\
&+ \sum_{i=1}^{M} \frac{\Delta x}{2}\left(-p_{i1}^{n+1} v_{i0}^{n+1} + p_{iN+1}^{n+1} v_{iN}^{n+1} - p_{i0}^{n+1} v_{i1}^{n+1} + p_{iN}^{n+1} v_{iN+1}^{n+1}\right).
\end{aligned}
$$

Now, using the incompressibility condition (2.19), we find :

$$
\begin{aligned}
\left(\mathbf{G}_h p_h^{n+1},\ \mathbf{u}_h^{n+1}\right)_h &= \sum_{j=1}^{N} \frac{\Delta y}{2}\left(-p_{1j}^{n+1} u_{0j}^{n+1} + p_{M+1j}^{n+1} u_{Mj}^{n+1} - p_{0j}^{n+1} u_{1j}^{n+1} + p_{Mj}^{n+1} u_{M+1j}^{n+1}\right) \\
&+ \sum_{i=1}^{M} \frac{\Delta x}{2}\left(-p_{i1}^{n+1} v_{i0}^{n+1} + p_{iN+1}^{n+1} v_{iN}^{n+1} - p_{i0}^{n+1} v_{i1}^{n+1} + p_{iN}^{n+1} v_{iN+1}^{n+1}\right),
\end{aligned}
$$

and using the boundary conditions (2.20) and (2.29), we have :

$$
\begin{aligned}
\left(\mathbf{G}_h p_h^{n+1},\ \mathbf{u}_h^{n+1}\right)_h &= \sum_{j=1}^{N} \frac{\Delta y}{2}\left(-p_{1j}^{n+1} u_{0j}^{n+1} + p_{Mj}^{n+1} u_{Mj}^{n+1} - p_{1j}^{n+1} u_{1j}^{n+1} + p_{Mj}^{n+1} u_{M+1j}^{n+1}\right) \\
&+ \sum_{i=1}^{M} \frac{\Delta x}{2}\left(-p_{i1}^{n+1} v_{i0}^{n+1} + p_{iN}^{n+1} v_{iN}^{n+1} - p_{i1}^{n+1} v_{i1}^{n+1} + p_{iN}^{n+1} v_{iN+1}^{n+1}\right), \\
&= \sum_{j=1}^{N} \frac{\Delta y}{2}\left(-2p_{1j}^{n+1} F_{u\,\frac{1}{2}j}^{n+1} + 2p_{Mj}^{n+1} F_{u\,M+\frac{1}{2}j}^{n+1}\right) \\
&+ \sum_{i=1}^{M} \frac{\Delta x}{2}\left(-2p_{i1}^{n+1} F_{v\,i\frac{1}{2}}^{n+1} + 2p_{iN}^{n+1} F_{v\,iN+\frac{1}{2}}^{n+1}\right), \\
&= 0.
\end{aligned}
$$

Finally, $\left(\mathbf{G}_h p_h^{n+1},\ \mathbf{u}_h^{n+1}\right)_h = 0$ and (5.3) is proved.

Now, we show the a priori estimates used in the stability section.

Lemma 5.2 (A priori estimates). *We have two a priori estimates, one for the intermediate velocities :*

$$\frac{3}{4}\left|\frac{\boldsymbol{u}_h^{n+\frac{1}{2}}-\boldsymbol{u}_h^n}{2}\right|_h^2+\left|\boldsymbol{u}_h^{n+\frac{1}{2}}\right|_h^2+2\nu\Delta t\left|\left|\boldsymbol{u}_h^{n+\frac{1}{2}}\right|\right|_h^2\left(1-\left(4\nu\Delta t S_h^2+\frac{145}{\nu}\Delta t S_h^2[|\boldsymbol{u}_h^n|_h^2+|\boldsymbol{u}_h^{n-1}|_h^2]\right)\right)$$
$$\leq \frac{c_1^2\Delta t}{2\nu}\left(|\boldsymbol{f}_h^{n+1}|_h^2+|\boldsymbol{f}_h^n|_h^2\right)+\left|\boldsymbol{u}_h^{n-\frac{1}{2}}\right|_h^2+\frac{1}{4}\left|\frac{\boldsymbol{u}_h^{n-\frac{1}{2}}-\boldsymbol{u}_h^{n-1}}{2}\right|_h^2, \tag{5.4}$$

and one for the new velocities :

$$\frac{3}{4}\left|\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{2}\right|_h^2+|\boldsymbol{u}_h^{n+1}|_h^2+\frac{\nu\Delta t}{2}||\boldsymbol{u}_h^{n+1}||_h^2\left(1-\left(16\nu\Delta t S_h^2+\frac{640}{\nu}\Delta t S_h^2[|\boldsymbol{u}_h^n|_h^2+|\boldsymbol{u}_h^{n-1}|_h^2]\right)\right)$$
$$\leq \frac{2c_1^2\Delta t}{\nu}\left(|\boldsymbol{f}_h^{n+1}|_h^2+|\boldsymbol{f}_h^n|_h^2\right)+|\boldsymbol{u}_h^n|_h^2+\frac{1}{4}\left|\frac{\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}}{2}\right|_h^2+\frac{\nu\Delta t}{2}\left|\left|\boldsymbol{u}_h^{n+\frac{1}{2}}\right|\right|_h^2. \tag{5.5}$$

Proof :

We recall the discrete weak formulation (Lemma 3.1) :

$$\frac{1}{\Delta t}\left(\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n,\,\varphi_h\right)_h = -\nu\left(\left(\frac{\mathbf{u}_h^{n+\frac{1}{2}}+\mathbf{u}_h^n}{2},\,\varphi_h\right)\right)_h-\frac{3}{2}\mathbf{b}_h(\mathbf{u}_h^n,\mathbf{u}_h^n,\varphi_h)+\frac{1}{2}\mathbf{b}_h(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},\varphi_h)$$
$$+\left(\frac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2},\,\varphi_h\right)_h.$$

To find the first a priori estimate, for the intermediate velocity, we replace in the above formulation φ_h by $2\Delta t\mathbf{u}_h^{n+\frac{1}{2}}$:

$$\left(\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n,\,2\mathbf{u}_h^{n+\frac{1}{2}}\right)_h = -\nu\left(\left(\frac{\mathbf{u}_h^{n+\frac{1}{2}}+\mathbf{u}_h^n}{2},\,2\Delta t\mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h-3\Delta t\mathbf{b}_h(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{u}_h^{n+\frac{1}{2}})$$
$$+\Delta t\mathbf{b}_h(\mathbf{u}_h^{n-1},\mathbf{u}_h^{n-1},\mathbf{u}_h^{n+\frac{1}{2}})+\left(\frac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2},\,2\Delta t\mathbf{u}_h^{n+\frac{1}{2}}\right)_h.$$

Moreover, we can modify each term as follows :

$$\left(\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n,\,2\mathbf{u}_h^{n+\frac{1}{2}}\right)_h = \left(\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n,\,\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n+\mathbf{u}_h^n\right)_h+\left(\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n,\,\mathbf{u}_h^{n+\frac{1}{2}}\right)_h,$$
$$= \left|\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n\right|_h^2-|\mathbf{u}_h^n|_h^2+\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2,$$

$$\left(\left(\mathbf{u}_h^{n+\frac{1}{2}} + \mathbf{u}_h^n,\ \Delta t \mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h = 2\Delta t \left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 - \Delta t \left(\left(\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n,\ \mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h,$$

$$\begin{aligned}
-3\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+\frac{1}{2}}) + \Delta t \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+\frac{1}{2}}) &= 4\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}, \mathbf{u}_h^{n+\frac{1}{2}}) \\
&- \Delta t [\mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+\frac{1}{2}}) \\
&- \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+\frac{1}{2}})],
\end{aligned}$$

according to $\mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^{n+\frac{1}{2}}, \mathbf{u}_h^{n+\frac{1}{2}}) = 0$ (Lemma 4.3).
So, we have the equality :

$$\begin{aligned}
&\left|\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n\right|_h^2 - |\mathbf{u}_h^n|_h^2 + \left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 + 2\nu\Delta t \left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 = \underbrace{\nu \Delta t \left(\left(\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n,\ \mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h}_{I} \\
&+ \underbrace{4\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}, \mathbf{u}_h^{n+\frac{1}{2}})}_{II} \underbrace{-\Delta t [\mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+\frac{1}{2}}) - \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+\frac{1}{2}})]}_{III} \quad (5.6) \\
&+ \underbrace{\Delta t \left(\mathbf{f}^{n+1} + \mathbf{f}^n,\ \mathbf{u}_h^{n+\frac{1}{2}}\right)_h}_{IV}.
\end{aligned}$$

(5.7)

To find the a priori estimate, we need to bound the terms which are in the right handside :
– For the first term :

$$\begin{aligned}
I &= 2\nu\Delta t \left(\left(\frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2},\ \mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h, \\
&\leq 2\nu\Delta t \left\|\frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}\right\|_h \left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h \quad \text{with Cauchy-Schwarz,} \\
&\leq 2\nu\Delta t S_h \left|\frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}\right|_h \left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h \quad \text{with the inverse inequality (Lemma 4.2),} \\
&\leq \frac{1}{8}\left|\frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}\right|_h^2 + 8\nu^2\Delta t^2 S_h^2 \left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 \quad \text{with Young.}
\end{aligned}$$

– For the second term :

$$
\begin{aligned}
II &= 4\Delta t\, \mathbf{b}_h\Big(\mathbf{u}_h^n, \frac{\mathbf{u}_h^{n+\frac12}-\mathbf{u}_h^n}{2}, \mathbf{u}_h^{n+\frac12}\Big),\\
&\le 8\sqrt{2}\Delta t\, |\mathbf{u}_h^n|_h^{\frac12}\, \|\mathbf{u}_h^n\|_h^{\frac12} \left|\frac{\mathbf{u}_h^{n+\frac12}-\mathbf{u}_h^n}{2}\right|_h^{\frac12} \left\|\frac{\mathbf{u}_h^{n+\frac12}-\mathbf{u}_h^n}{2}\right\|_h^{\frac12} \left\|\mathbf{u}_h^{n+\frac12}\right\|_h \text{ with Lemma 4.4,}\\
&\le 8\sqrt{2}\Delta t S_h\, |\mathbf{u}_h^n|_h \left|\frac{\mathbf{u}_h^{n+\frac12}-\mathbf{u}_h^n}{2}\right|_h \left\|\mathbf{u}_h^{n+\frac12}\right\|_h \text{ with the inverse inequality (Lemma 4.2),}\\
&\le \frac18 \left|\frac{\mathbf{u}_h^{n+\frac12}-\mathbf{u}_h^n}{2}\right|_h^2 + 256\Delta t^2 S_h^2 \left\|\mathbf{u}_h^{n+\frac12}\right\|_h^2 |\mathbf{u}_h^n|_h^2 \text{ with Young,}\\
&\le \frac18 \left|\frac{\mathbf{u}_h^{n+\frac12}-\mathbf{u}_h^n}{2}\right|_h^2 + 256\Delta t^2 S_h^2 \left\|\mathbf{u}_h^{n+\frac12}\right\|_h^2 \big(|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2\big).
\end{aligned}
$$

– For the third term :

$$
\begin{aligned}
III &= -\Delta t\Big[\mathbf{b}_h\big(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+\frac12}\big) - \mathbf{b}_h\big(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+\frac12}\big)\Big],\\
&= -2\Delta t\Big[\mathbf{b}_h\Big(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}, \mathbf{u}_h^n, \mathbf{u}_h^{n+\frac12}\Big) - \mathbf{b}_h\Big(\mathbf{u}_h^{n-1}, \frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}, \mathbf{u}_h^{n+\frac12}\Big)\Big],\\
&\le 4\sqrt{2}\Delta t \left|\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}\right|_h^{\frac12} \left\|\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}\right\|_h^{\frac12} \left\|\mathbf{u}_h^{n+\frac12}\right\|_h \Big[|\mathbf{u}_h^n|_h^{\frac12}\, \|\mathbf{u}_h^n\|_h^{\frac12} + |\mathbf{u}_h^{n-1}|_h^{\frac12}\, \|\mathbf{u}_h^{n-1}\|_h^{\frac12}\Big]\\
&\quad \text{with Lemma 4.4,}\\
&\le 2\sqrt{2}\Delta t S_h\, |\mathbf{u}_h^n-\mathbf{u}_h^{n-1}|_h \left\|\mathbf{u}_h^{n+\frac12}\right\|_h \big[|\mathbf{u}_h^n|_h + |\mathbf{u}_h^{n-1}|_h\big]\\
&\quad \text{with the inverse inequality (Lemma 4.2),}\\
&\le 2\sqrt{2}\Delta t S_h \Big[\left|\mathbf{u}_h^n - \mathbf{u}_h^{n-\frac12}\right|_h + \left|\mathbf{u}_h^{n-\frac12} - \mathbf{u}_h^{n-1}\right|_h\Big] \left\|\mathbf{u}_h^{n+\frac12}\right\|_h \big[|\mathbf{u}_h^n|_h + |\mathbf{u}_h^{n-1}|_h\big],\\
&\le \left|\mathbf{u}_h^n - \mathbf{u}_h^{n-\frac12}\right|_h^2 + \frac1{16}\left|\mathbf{u}_h^{n-\frac12} - \mathbf{u}_h^{n-1}\right|_h^2 + 34\Delta t^2 S_h^2 \left\|\mathbf{u}_h^{n+\frac12}\right\|_h^2 \big[|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2\big]\\
&\quad \text{with Young,}\\
&\le \left|\mathbf{u}_h^{n-\frac12}\right|_h^2 - |\mathbf{u}_h^n|_h^2 + \frac1{16}\left|\mathbf{u}_h^{n-\frac12} - \mathbf{u}_h^{n-1}\right|_h^2 + 34\Delta t^2 S_h^2 \left\|\mathbf{u}_h^{n+\frac12}\right\|_h^2 \big[|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2\big]\\
&\quad \text{with (5.1).}
\end{aligned}
$$

– For the fourth term :

$$\begin{aligned} IV &= \Delta t \left(\mathbf{f}^{n+1} + \mathbf{f}^{n},\ \mathbf{u}_h^{n+\frac{1}{2}} \right)_h, \\ &\leq \Delta t \left(|\mathbf{f}^{n+1}|_h + |\mathbf{f}^{n}|_h \right) \left| \mathbf{u}_h^{n+\frac{1}{2}} \right|_h \text{ with Cauchy-Schwarz,} \\ &\leq c_1 \Delta t \left(|\mathbf{f}^{n+1}|_h + |\mathbf{f}^{n}|_h \right) \left\| \mathbf{u}_h^{n+\frac{1}{2}} \right\|_h \text{ with Poincaré (Lemma 4.1),} \\ &\leq \frac{\nu \Delta t}{2} \left\| \mathbf{u}_h^{n+\frac{1}{2}} \right\|_h^2 + \frac{c_1^2 \Delta t}{2\nu} \left(|\mathbf{f}^{n+1}|_h^2 + |\mathbf{f}^{n}|_h^2 \right) \text{ with Young.} \end{aligned}$$

So, due to these previous inequalities we obtain the first a priori estimate :

$$\begin{aligned} &\frac{3}{4} \left| \frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2} \right|_h^2 + \left| \mathbf{u}_h^{n+\frac{1}{2}} \right|_h^2 + 2\nu\Delta t \left\| \mathbf{u}_h^{n+\frac{1}{2}} \right\|_h^2 \left(1 - (4\nu\Delta t S_h^2 + \frac{145\Delta t}{\nu} S_h^2 [|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2] \right) \\ &\leq \frac{c_1^2 \Delta t}{2\nu} \left(|\mathbf{f}_h^{n+1}|_h^2 + |\mathbf{f}_h^n|_h^2 \right) + \left| \mathbf{u}_h^{n-\frac{1}{2}} \right|_h^2 + \frac{1}{4} \left| \frac{\mathbf{u}_h^{n-\frac{1}{2}} - \mathbf{u}_h^{n-1}}{2} \right|_h^2 . \end{aligned}$$

To get the second a priori estimate about the new velocities $\mathbf{u}_h^{n+1}$, we apply the same method except that we replace in the discrete weak formulation (Lemma 3.1) φ_h by $2\Delta t \mathbf{u}_h^{n+1}$ and not by $2\Delta t \mathbf{u}_h^{n+\frac{1}{2}}$:

$$\begin{aligned} \left(\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n,\ 2\mathbf{u}_h^{n+1} \right)_h &= -\nu \left(\left(\frac{\mathbf{u}_h^{n+\frac{1}{2}} + \mathbf{u}_h^n}{2},\ 2\Delta t \mathbf{u}_h^{n+1} \right) \right)_h - 3\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1}) \\ &+ \Delta t \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}) + \left(\frac{\mathbf{f}^{n+1} + \mathbf{f}^n}{2},\ 2\Delta t \mathbf{u}_h^{n+1} \right)_h . \end{aligned}$$

Moreover, if we follow the same way that for the previous estimate, we rewrite each term :

$$\begin{aligned} \left(\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n,\ 2\mathbf{u}_h^{n+1} \right)_h &= 2 \left(\mathbf{u}_h^{n+\frac{1}{2}},\ \mathbf{u}_h^{n+1} \right)_h - \left(\mathbf{u}_h^n,\ 2\mathbf{u}_h^{n+1} \right)_h, \\ &= 2 \left(\mathbf{u}_h^{n+1},\ \mathbf{u}_h^{n+1} \right)_h - \left(\mathbf{u}_h^n,\ 2\mathbf{u}_h^{n+1} \right)_h \text{ with (5.3),} \\ &= \left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n,\ 2\mathbf{u}_h^{n+1} \right)_h, \\ &= |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 - |\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n+1}|_h^2, \end{aligned}$$

$$\begin{aligned} \left(\left(\frac{\mathbf{u}_h^{n+\frac{1}{2}} + \mathbf{u}_h^n}{2},\ 2\mathbf{u}_h^{n+1} \right) \right)_h &= \left(\left(\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^{n+1} + \mathbf{u}_h^{n+1} + \mathbf{u}_h^n,\ \mathbf{u}_h^{n+1} \right) \right)_h, \\ &= \|\mathbf{u}_h^{n+1}\|_h^2 - ((\mathbf{u}_h^{n+1} - \mathbf{u}_h^n,\ \mathbf{u}_h^{n+1}))_h + \left(\left(\mathbf{u}_h^{n+\frac{1}{2}},\ \mathbf{u}_h^{n+1} \right) \right)_h, \end{aligned}$$

$$\begin{aligned} & -3\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1}) + \Delta t \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}) \\ = \ & 4\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}, \mathbf{u}_h^{n+1}) - \Delta t[\mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1}) - \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1})], \end{aligned}$$

according to $\mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) = 0$ (Lemma 4.3).
Hence, the equality obtained from the weak formulation becomes :

$$\begin{aligned} & |\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|_h^2 - |\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n+1}|_h^2 + \nu\Delta t \, ||\mathbf{u}_h^{n+1}||_h^2 = \underbrace{2\nu\Delta t \left(\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}, \, \mathbf{u}_h^{n+1}\right)\right)_h}_{I} \\ & \underbrace{-\nu\Delta t \left(\left(\mathbf{u}_h^{n+1}, \, \mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h}_{II} \underbrace{+4\Delta t \mathbf{b}_h(\mathbf{u}_h^n, \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}, \mathbf{u}_h^{n+1})}_{III} \\ & \underbrace{-\Delta t[\mathbf{b}_h(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1}) - \mathbf{b}_h(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1})]}_{IV} + \underbrace{\Delta t \, (\mathbf{f}^{n+1} + \mathbf{f}^n, \, \mathbf{u}_h^{n+1})_h}_{V}, \end{aligned} \tag{5.8}$$

and we have to bound the terms I-V which are in the right handside.

– For the first term :

$$\begin{aligned} I &= 2\nu\Delta t \left(\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}, \, \mathbf{u}_h^{n+1}\right)\right)_h, \\ &\leq 2\nu\Delta t \left|\left|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}\right|\right|_h ||\mathbf{u}_h^{n+1}||_h \text{ with Cauchy-Schwarz,} \\ &\leq 2\nu\Delta t S_h \left|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}\right|_h ||\mathbf{u}_h^{n+1}||_h \text{ with the inverse inequality (Lemma 4.2),} \\ &\leq \frac{1}{8}\left|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{2}\right|_h^2 + 8\nu^2\Delta t^2 S_h^2 \, ||\mathbf{u}_h^{n+1}||_h^2 \text{ with Young.} \end{aligned}$$

– For the second term

$$\begin{aligned} II &= -\nu\Delta t \left(\left(\mathbf{u}_h^{n+1}, \, \mathbf{u}_h^{n+\frac{1}{2}}\right)\right)_h, \\ &\leq \nu\Delta t \, ||\mathbf{u}_h^{n+1}||_h \left|\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|\right|_h \text{ with Cauchy-Schwarz,} \\ &\leq \frac{\nu\Delta t}{2} ||\mathbf{u}_h^{n+1}||_h^2 + \frac{\nu\Delta t}{2}\left|\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|\right|_h^2 \text{ with Young.} \end{aligned}$$

– For the third term :

$$\begin{aligned}
III &= 4\Delta t\, \mathbf{b}_h\big(\mathbf{u}_h^n, \frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}, \mathbf{u}_h^{n+1}\big),\\
&\le 8\sqrt{2}\Delta t\, |\mathbf{u}_h^n|_h^{\frac12}\, ||\mathbf{u}_h^n||_h^{\frac12} \left|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}\right|_h^{\frac12} \left|\left|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}\right|\right|_h^{\frac12} ||\mathbf{u}_h^{n+1}||_h \text{ with Lemma 4.4},\\
&\le 8\sqrt{2}\Delta t S_h\, |\mathbf{u}_h^n|_h \left|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}\right|_h ||\mathbf{u}_h^{n+1}||_h \text{ with the inverse inequality (Lemma 4.2)},\\
&\le \frac18 \left|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}\right|_h^2 + 256\Delta t^2 S_h^2\, ||\mathbf{u}_h^{n+1}||_h^2\, |\mathbf{u}_h^n|_h^2 \text{ with Young},\\
&\le \frac18 \left|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}\right|_h^2 + 256\Delta t^2 S_h^2\, ||\mathbf{u}_h^{n+1}||_h^2 \left(|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2\right).
\end{aligned}$$

– For the fourth term :

$$\begin{aligned}
IV &= -\Delta t\Big[\mathbf{b}_h\big(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1}\big) - \mathbf{b}_h\big(\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}\big)\Big],\\
&= -2\Delta t\Big[\mathbf{b}_h\big(\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}, \mathbf{u}_h^n, \mathbf{u}_h^{n+1}\big) + \mathbf{b}_h\big(\mathbf{u}_h^{n-1}, \frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}, \mathbf{u}_h^{n+1}\big)\Big],\\
&\le 4\sqrt{2}\Delta t \left|\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}\right|_h^{\frac12} \left|\left|\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}\right|\right|_h^{\frac12} ||\mathbf{u}_h^{n+1}||_h \left[|\mathbf{u}_h^n|_h^{\frac12}\, ||\mathbf{u}_h^n||_h^{\frac12} + |\mathbf{u}_h^{n-1}|_h^{\frac12}\, ||\mathbf{u}_h^{n-1}||_h^{\frac12}\right]\\
&\quad \text{with Lemma 4.4},\\
&\le 2\sqrt{2}\Delta t S_h\, |\mathbf{u}_h^n - \mathbf{u}_h^{n-1}|_h\, ||\mathbf{u}_h^{n+1}||_h \left[|\mathbf{u}_h^n|_h + |\mathbf{u}_h^{n-1}|_h\right]\\
&\quad \text{with the inverse inequality (Lemma 4.2)},\\
&\le \frac{1}{16} |\mathbf{u}_h^n - \mathbf{u}_h^{n-1}|_h^2 + 64\Delta t^2 S_h^2\, ||\mathbf{u}_h^{n+1}||_h^2 \left[|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2\right] \text{ with Young}.
\end{aligned}$$

– For the fifth term :

$$\begin{aligned}
V &= \Delta t\left(\mathbf{f}^{n+1} + \mathbf{f}^n,\, \mathbf{u}_h^{n+1}\right)_h,\\
&\le \Delta t\left(|\mathbf{f}^{n+1}|_h + |\mathbf{f}^n|_h\right) |\mathbf{u}_h^{n+1}|_h \text{ with Cauchy-Schwarz},\\
&\le c_1\Delta t\left(|\mathbf{f}^{n+1}|_h + |\mathbf{f}^n|_h\right) ||\mathbf{u}_h^{n+1}||_h \text{ with Poincaré (Lemma 4.1)},\\
&\le \frac{\nu\Delta t}{8} ||\mathbf{u}_h^{n+1}||_h^2 + \frac{2c_1^2\Delta t}{\nu}\left(|\mathbf{f}^{n+1}|_h^2 + |\mathbf{f}^n|_h^2\right) \text{ with Young}.
\end{aligned}$$

So, due to these previous inequalities we obtain the second a priori estimate :

$$\begin{aligned}
&\frac34 \left|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{2}\right|_h^2 + |\mathbf{u}_h^{n+1}|_h^2 + \frac{\nu\Delta t}{2} ||\mathbf{u}_h^{n+1}||_h^2 \left(1 - (16\nu\Delta t S_h^2 + \frac{640\Delta t}{\nu} S_h^2[|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2]\right)\\
\le\ &\frac{2c_1^2\Delta t}{\nu}\left(|\mathbf{f}_h^{n+1}|_h^2 + |\mathbf{f}_h^n|_h^2\right) + |\mathbf{u}_h^n|_h^2 + \frac14 \left|\frac{\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}\right|_h^2 + \frac{\nu\Delta t}{2} \left|\left|\mathbf{u}_h^{n+\frac12}\right|\right|_h^2.
\end{aligned}$$

6 Stability proof

In this last section, we present the proof of the Stability Theorem 3.1 which is stated in Section 3.

First, we show the inequality (3.17) step by step. We know that the inequality is true for $n = 0$ and $n = 1$. We suppose that $|\mathbf{u}_h^p|_h^2 \leq K_4$ for all $p = 0, ..., n$ and we show that $|\mathbf{u}_h^{n+1}|_h^2 \leq K_4$. The sketch of the proof consists in using the a priori estimate (5.4) in order to show that $\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 \leq K_4$, and then, due to (5.2), we have $|\mathbf{u}_h^{n+1}|_h^2 \leq K_4$. The hypothesis (3.16) and the recurrence hypothesis imply that

$$\begin{aligned} 1 - \left(4\nu\Delta t S_h^2 + \frac{145\Delta t}{\nu} S_h^2[|\mathbf{u}_h^n|_h^2 + |\mathbf{u}_h^{n-1}|_h^2]\right) &\geq 1 - \left(4\nu\Delta t S_h^2 + \frac{290\Delta t}{\nu} S_h^2 K_4\right) \\ &\geq \frac{1}{2}. \end{aligned}$$

Indeed :

$$\begin{aligned} (3.16) \quad &\Longrightarrow \quad \Delta t S_h^2 \leq \min\left(\frac{1}{64\nu}, \frac{\nu}{5120 K_4}\right), \\ &\Longrightarrow \quad 4\nu\Delta t S_h^2 \leq \frac{1}{4} \text{ and } \frac{290 K_4}{\nu}\Delta t S_h^2 \leq \frac{1}{4}, \\ &\Longrightarrow \quad \Delta t S_h^2\left(4\nu + \frac{290 K_4}{\nu}\right) \leq \frac{1}{2}, \\ &\Longrightarrow \quad 1 - \Delta t S_h^2\left(4\nu + \frac{290 K_4}{\nu}\right) \geq \frac{1}{2}. \end{aligned}$$

So, the a priori estimate (5.4) can be rewrite as :

$$\begin{aligned} \frac{3}{4}\left|\frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}\right|_h^2 + \left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 + \nu\Delta t \left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 &\leq \frac{c_1^2\Delta t}{2\nu}\left(|\mathbf{f}_h^{n+1}|_h^2 + |\mathbf{f}_h^n|_h^2\right) + \left|\mathbf{u}_h^{n-\frac{1}{2}}\right|_h^2 \\ &+ \frac{1}{4}\left|\frac{\mathbf{u}_h^{n-\frac{1}{2}} - \mathbf{u}_h^{n-1}}{2}\right|_h^2, \end{aligned} \tag{6.1}$$

and then, due to the Poincaré inequality (Lemma 4.1) :

$$\begin{aligned} \frac{3}{4}\left|\frac{\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n}{2}\right|_h^2 + \left(1 + \frac{\nu\Delta t}{c_1^2}\right)\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 &\leq \frac{c_1^2\Delta t}{2\nu}\left(|\mathbf{f}_h^{n+1}|_h^2 + |\mathbf{f}_h^n|_h^2\right) + \left|\mathbf{u}_h^{n-\frac{1}{2}}\right|_h^2 \\ &+ \frac{1}{4}\left|\frac{\mathbf{u}_h^{n-\frac{1}{2}} - \mathbf{u}_h^{n-1}}{2}\right|_h^2. \end{aligned} \tag{6.2}$$

Using the hypothesis (3.15), the above inequality can be rewritten as follows :

$$\left(1+\frac{\nu\Delta t}{c_1^2}\right)\xi^{n+\frac{1}{2}} \quad \leq \quad \xi^{n-\frac{1}{2}}+r^{n+\frac{1}{2}}, \tag{6.3}$$

where :

$$\begin{aligned}
\xi^{n+\frac{1}{2}} &= \left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2+\frac{1}{4}\left|\frac{\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n}{2}\right|_h^2, \\
r^{n+\frac{1}{2}} &= \frac{c_1^2\Delta t}{2\nu}\left(\left|\mathbf{f}_h^{n+1}\right|_h^2+|\mathbf{f}_h^n|_h^2\right).
\end{aligned}$$

Now, we can use the following lemma which is proved in [MT98] :

Lemma 6.1. *Consider two sequences of number ξ^k, $r^k \geq 0$ such that*

$$(1-\alpha)\xi^k \leq (1+\beta)\xi^{k-1}+r^k \ \forall k \geq 1, \ \alpha<1 \textit{ and } \beta>-1,$$

then :

$$\begin{aligned}
\xi^k &\leq \left(\frac{1+\beta}{1-\alpha}\right)^k \xi^0+\frac{(1+\beta)^{k-1}}{(1-\alpha)^k}\sum_{j=1}^k r^j, \\
\xi^k &\leq \left(\frac{1+\beta}{1-\alpha}\right)^k\left(\xi^0+\frac{r}{\beta+\alpha}\right) \quad \textit{if } r^j \leq r \ \forall j.
\end{aligned}$$

Here, we take $\beta=0$ and $\alpha=-\dfrac{\nu\Delta t}{c_1^2}$, so :

$$\xi^{n+\frac{1}{2}} \leq \left(1+\frac{\nu\Delta t}{c_1^2}\right)^{-(n-1)}\xi^{1+\frac{1}{2}}+\left(1+\frac{\nu\Delta t}{c_1^2}\right)^{-(n-1)}\sum_{j=2}^n r^{j+\frac{1}{2}},$$

that means :

$$\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2+\frac{1}{4}\left|\frac{\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n}{2}\right|_h^2 \leq \left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2+\frac{1}{4}\left|\frac{\mathbf{u}_h^{1+\frac{1}{2}}-\mathbf{u}_h^1}{2}\right|_h^2+\frac{c_1^2\Delta t}{\nu}\sum_{j=2}^{n+1}|\mathbf{f}^n|_h^2. \tag{6.4}$$

Then, due to the hypotheses (3.11)-(3.14) :

$$\begin{aligned}
\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2+\frac{1}{4}\left|\frac{\mathbf{u}_h^{n+\frac{1}{2}}-\mathbf{u}_h^n}{2}\right|_h^2 &\leq \left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2+\frac{1}{8}\left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2+\frac{1}{8}\left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2+\frac{c_1^2}{\nu}K_2, \\
&\leq \underbrace{\frac{9}{8}\tilde{K}_3+\frac{1}{8}K_3+\frac{c_1^2}{\nu}K_2}_{K_4}.
\end{aligned}$$

Finally, using (5.2), we have shown that $\left|\mathbf{u}_h^{n+1}\right|_h^2 \leq K_4$ for all $n = 0, ..., N_t$.

We have proved that (3.17) is satisfied, and now, we want to prove (3.18) and (3.19). In order to do it, in a first time, we sum for $n = 2, ..., N_t$ the inequality (6.1) obtained due to the a priori estimate (5.4) and due to the stability theorem hypotheses :

$$\frac{3}{16}\sum_{n=2}^{N_t}\left|\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n\right|_h^2 + \sum_{n=2}^{N_t}\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|_h^2 + \nu\Delta t\sum_{n=2}^{N_t}\left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 \leq \frac{c_1^2\Delta t}{2\nu}\sum_{n=2}^{N_t}\left(\left|\mathbf{f}_h^{n+1}\right|_h^2 + |\mathbf{f}_h^n|_h^2\right)$$
$$+ \sum_{n=2}^{N_t}\left|\mathbf{u}_h^{n-\frac{1}{2}}\right|_h^2 + \frac{1}{16}\sum_{n=2}^{N_t}\left|\mathbf{u}_h^{n-\frac{1}{2}} - \mathbf{u}_h^{n-1}\right|_h^2.$$

We simplify the above inequality and use the stability theorem hypotheses (3.11)-(3.14), we write :

$$\begin{aligned}
&\frac{1}{8}\sum_{n=2}^{N_t-1}\left|\mathbf{u}_h^{n+\frac{1}{2}} - \mathbf{u}_h^n\right|_h^2 + \frac{3}{16}\left|\mathbf{u}_h^{N_t+\frac{1}{2}} - \mathbf{u}_h^{N_t}\right|_h^2 + \left|\mathbf{u}_h^{N_t+\frac{1}{2}}\right|_h^2 + \nu\Delta t\sum_{n=2}^{N_t}\left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 \\
\leq\;& \frac{c_1^2\Delta t}{2\nu}\sum_{n=2}^{N_t}\left(\left|\mathbf{f}_h^{n+1}\right|_h^2 + |\mathbf{f}_h^n|_h^2\right) + \left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2 + \frac{1}{16}\left|\mathbf{u}_h^{1+\frac{1}{2}} - \mathbf{u}_h^1\right|_h^2, \\
\leq\;& \frac{c_1^2\Delta t}{\nu}K_2 + \left|\mathbf{u}_h^{1+\frac{1}{2}}\right|_h^2 + \frac{1}{16}\left|\mathbf{u}_h^{1+\frac{1}{2}} - \mathbf{u}_h^1\right|_h^2, \\
\leq\;& K_4.
\end{aligned}$$

At last, we obtain the following inequality :

$$\nu\Delta t\sum_{n=1}^{N_t}\left\|\mathbf{u}_h^{n+\frac{1}{2}}\right\|_h^2 \leq K_4 + \tilde{K}_3. \tag{6.5}$$

Now, we are able to prove (3.18) and (3.19). The hypothesis (3.16) and the stability theorem inequality (3.17) imply that

$$\begin{aligned}
1 - \left(16\nu\Delta t S_h^2 + \frac{640\Delta t}{\nu}S_h^2\left[|\mathbf{u}_h^n|_h^2 + \left|\mathbf{u}_h^{n-1}\right|_h^2\right]\right) &\geq 1 - \left(16\nu\Delta t S_h^2 + \frac{1280\Delta t}{\nu}S_h^2 K_4\right), \\
&\geq \frac{1}{2}.
\end{aligned}$$

Indeed :

$$\begin{aligned}
(3.16) \implies\;& \Delta t S_h^2 \leq \min\left(\frac{1}{64\nu}, \frac{\nu}{5120K_4}\right), \\
\implies\;& 16\nu\Delta t S_h^2 \leq \frac{1}{4} \text{ and } \frac{1280K_4}{\nu}\Delta t S_h^2 \leq \frac{1}{4}, \\
\implies\;& \Delta t S_h^2\left(16\nu + \frac{1280K_4}{\nu}\right) \leq \frac{1}{2}, \\
\implies\;& 1 - \Delta t S_h^2\left(16\nu + \frac{1280K_4}{\nu}\right) \geq \frac{1}{2}.
\end{aligned}$$

So, the a priori estimate (5.5) can be rewrite as :

$$\frac{3}{16}\left|\mathbf{u}_h^{n+1}-\mathbf{u}_h^n\right|_h^2+\left|\mathbf{u}_h^{n+1}\right|_h^2+\frac{\nu\Delta t}{4}\left|\left|\mathbf{u}_h^{n+1}\right|\right|_h^2\leq\frac{2c_1^2\Delta t}{\nu}\left(\left|\mathbf{f}_h^{n+1}\right|_h^2+\left|\mathbf{f}_h^n\right|_h^2\right)+\left|\mathbf{u}_h^n\right|_h^2$$
$$+\ \frac{1}{16}\left|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\right|_h^2+\frac{\nu\Delta t}{2}\left|\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|\right|_h^2. \tag{6.6}$$

We sum the above inequality for $n=1,...,N_t$ and use the hypotheses (3.11)-(3.14), (6.5) :

$$\frac{1}{8}\sum_{n=1}^{N_t}\left|\mathbf{u}_h^{n+1}-\mathbf{u}_h^n\right|_h^2+\frac{3}{16}\left|\mathbf{u}_h^{N_t+1}-\mathbf{u}_h^{N_t}\right|_h^2+\left|\mathbf{u}_h^{N_t+1}\right|_h^2+\frac{\nu\Delta t}{4}\sum_{n=1}^{N_t}\left|\left|\mathbf{u}_h^{n+1}\right|\right|_h^2,$$
$$\leq\ \frac{2c_1^2\Delta t}{\nu}\left(\left|\mathbf{f}_h^{n+1}\right|_h^2+\left|\mathbf{f}_h^n\right|_h^2\right)+\left|\mathbf{u}_h^1\right|_h^2+\frac{1}{16}\left|\mathbf{u}_h^1-\mathbf{u}_h^0\right|_h^2+\frac{\nu\Delta t}{2}\sum_{n=1}^{N_t}\left|\left|\mathbf{u}_h^{n+\frac{1}{2}}\right|\right|_h^2,$$
$$\leq\ \frac{4c_1^2}{\nu}K_2+\frac{9}{8}K_3+\frac{1}{8}K_1^2+K_4+\tilde{K}_3.$$

We can conclude that there exists a constant K_5 such that :

$$\sum_{n=0}^{N_t-1}\left|\mathbf{u}_h^{n+1}-\mathbf{u}_h^n\right|_h^2\ \leq\ K_5,$$
$$\nu\Delta t\sum_{n=0}^{N_t}\left|\left|\mathbf{u}_h^{n+1}\right|\right|_h^2\ \leq\ K_5.$$

So, the stability theorem inequalities (3.18) and (3.19) are shown and the proof is completed.

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Chapitre 3

Collocated finite volume schemes for fluid flows

Collocated finite volume schemes for fluid flows

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Abstract : Our aim in this article is to build a collocated finite volume scheme for the incompressible Navier-Stokes equations. The main advantage of collocated schemes over staggered schemes is that all variables share the same location, hence, the possibility to use complex geometries and hierarchical space discretizations more easily. We plan, in the future, to use this collocated scheme and the multilevel method of Chapter 4 initially applied to the two dimensional Burgers problem, in order to solve the incompressible Navier-Stokes equations.
An important issue when the velocities and the pressure are computed at the same place, i.e. at the centers of the control volumes, is that these unknowns must be properly coupled. Consequently, the choice of the time discretization and the method used to interpolate the fluxes at the edges of the control volumes are essentials. Traditionally, the time discretization used is often a projection method, see Temam [Tem69] and Chorin [Cho68], so, in a first part, we describe a collocated scheme using such a projection method. Then, in a second part, we describe a collocated scheme using the new splitting method of Guermond and Shen [GS03a]. Finally, after the description of these collocated schemes, we compare the numerical results obtained in the case of a driven cavity.

1 Introduction

We consider the Navier-Stokes equations in their velocity-pressure formulation and the continuity equation written for an incompressible viscous fluid; Ω is an open bounded domain in $\mathbb{R}^2$, for our simulations we use a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$.
For a given force $\mathbf{f} = (f_u, f_v)$, and we look for the velocity vector $\mathbf{u}$ and the pressure p such that :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times [0, T], \tag{1.1}$$

$$div\ \mathbf{u} = 0, \tag{1.2}$$

where ν is the kinematic viscosity and, in space dimension two, $\mathbf{u} = \big(u(x,y,t), v(x,y,t)\big)$, $t \geq 0$. On the boundary $\partial\Omega$ of Ω, we impose a Dirichlet no-slip boundary condition :

$$\mathbf{u}\Big|_{\partial\Omega} = \mathbf{g}, \tag{1.3}$$

where $\mathbf{g} = (g_u, g_v)$ is a given function defined on $\partial\Omega$.

Usually, a staggered variable arrangement is prefered to a collocated variable arrangement. Indeed, the collocated arrangement has long been considered impraticable as collocated schemes are known to generate a decoupling between the velocities and the pressure. The first successful collocated finite volume schemes were introduced in 1981 by Hsu [Hsu81], Prakash [Pra81] and Rhie [Rhi81]. One reason to still study collocated schemes is that staggered schemes become very complicated for complex geometries [ZSK94]. Moreover, multigrid techniques, which result in a significant reduction of computing time on fine grids, are also much easier applied to the collocated arrangement. See [PKS88] for a more detailed comparison between staggered and collocated schemes. For theoretical aspects of the finite volume methods, see [EGH00].

The purpose of the present paper is to describe two collocated schemes which have different time discretizations. For each scheme, we will study if there exists a decoupling problem between the velocities and the pressure. Then, we will comment the numerical results obtained in the case of a driven cavity.

In the following, the domain Ω is discretized by rectangular finite volumes of same dimensions $\Delta x \Delta y$ with $M\Delta x = L_1$ and $N\Delta y = L_2$ (M, N are given integers). Hence, we have MN volumes which are defined (see Figure 1.1) by :

$$\Big(K_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\Big)_{i=1,\ldots M, j=1,\ldots,N},$$

where

$$x_{i+\frac{1}{2}} = i\Delta x \text{ for } i = 0, \ldots, M,$$
$$y_{j+\frac{1}{2}} = j\Delta y \text{ for } j = 0, \ldots, N.$$

The edges of the control volumes are defined by :

$$\Gamma_{i+\frac{1}{2}j} = \Big\{(x,y)\ ;\ x = x_{i+\frac{1}{2}}, y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\Big\} \text{ for } i = 0, \ldots, M,$$
$$\Gamma_{ij+\frac{1}{2}} = \Big\{(x,y)\ ;\ x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], y = y_{j+\frac{1}{2}},\Big\} \text{ for } j = 0, \ldots, N.$$

When a collocated method is used, all the unknowns, i.e. the velocities and the pressure, are defined at the center of the cells. Hence, for $i = 1, ..., M$ and $j = 1, ..., N$, the unknowns are meant to be approximations of the cell averages :

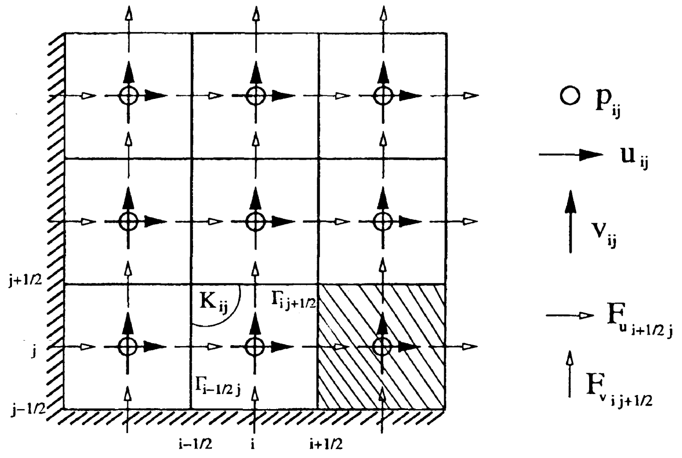

FIG. 1.1 – Collocated mesh.

$$\mathbf{u}_{ij}(t) \simeq \frac{1}{\Delta x \Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathbf{u}(x,y,t)\,dxdy \quad \text{for the velocity,} \tag{1.4}$$

$$p_{ij}(t) \simeq \frac{1}{\Delta x \Delta y}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} p(x,y,t)\,dxdy \quad \text{for the pressure.} \tag{1.5}$$

Furthermore, as they occur in the schemes, in the nonlinear term for example, we also define the velocity fluxes :

$$F_{u\,i+\frac{1}{2}j}(t) \simeq \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x_{i+\frac{1}{2}},y,t)\,dy \quad \text{for the horizontal fluxes, } i=0,...,M, \tag{1.6}$$

$$F_{v\,ij+\frac{1}{2}}(t) \simeq \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v(x,y_{j+\frac{1}{2}},t)\,dx \quad \text{for the vertical fluxes, } j=0,...,N. \tag{1.7}$$

However, the velocity fluxes are not considered as independant unknowns : they will be computed either directly or by interpolation from the above cell centered unknowns (1.4), (1.5).

Finally, concerning the time discretization of the problem (1.1)-(1.3), let $T > 0$ be fixed, and let the time step be $\Delta t = T/N_t$ where N_t is an integer. For $k = 0,...,N_t$, we define $\mathbf{u}^k$ as the approximate value of $\mathbf{u}$ at the time $t_k = k\Delta t$. In the following we will use a time finite difference schemes to approximate the time derivative; we will not use a space-time finite volume discretization but a space finite volume discretization with a time finite difference scheme. A consequence of this choice is that implicit time discretizations for the diffusive terms can be considered.

In the next section, we describe the collocated scheme using the projection method of Van Kan [VK86]. In the third section, we present the scheme using the new splitting method of Guermond and Shen [GS03a]. Then, in the last section we present the numerical results obtained with these schemes and give some comments.

2 A collocated finite volume scheme with a projection method for the time discretization

2.1 Time discretization

The concept of projection methods was introduced by [Cho68] and [Tem69] in the late 60s. Then, several projection methods have been elaborated, see [KM85], [MT98], [She96] and [GS03b] producing in general schemes of order two in time. In this section, we consider for simplicity the projection method of Van Kan [VK86] ; we will study later on the improvements resulting from more recent projection methods.
We start from a Crank-Nicholson scheme for the linear part and an Adams-Bashforth scheme for the nonlinear part, that is :

$$\left\{\begin{array}{l} \dfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} - \nu\Delta\Big(\dfrac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}\Big) + \mathbf{B}\left(\mathbf{u}^n,\mathbf{u}^{n-1}\right) + \nabla\Big(\dfrac{\mathbf{p}^{n+1}+\mathbf{p}^n}{2}\Big) = \dfrac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2}, \\ \mathbf{u}^{n+1}\Big|_{\partial\Omega} = \mathbf{g}, \\ div\ \mathbf{u}^{n+1} = 0, \end{array}\right. \tag{2.1}$$

where $\mathbf{B}\left(\mathbf{u}^n,\mathbf{u}^{n-1}\right) = \dfrac{3}{2}(\mathbf{u}^n.\nabla)\mathbf{u}^n - \dfrac{1}{2}(\mathbf{u}^{n-1}.\nabla)\mathbf{u}^{n-1}$.
Combining this with the projection scheme we obtain :

$$\left\{\begin{array}{l} \dfrac{\mathbf{u}^{n+\frac{1}{2}}-\mathbf{u}^n}{\Delta t} - \nu\Delta\Big(\dfrac{\mathbf{u}^{n+\frac{1}{2}}+\mathbf{u}^n}{2}\Big) + B\left(\mathbf{u}^n,\mathbf{u}^{n-1}\right) + \nabla p^n = \dfrac{\mathbf{f}^{n+1}+\mathbf{f}^n}{2}, \\ \mathbf{u}^{n+\frac{1}{2}}\Big|_{\partial\Omega} = \mathbf{u}^{n+1}\Big|_{\partial\Omega} = \mathbf{g}, \end{array}\right. \tag{2.2}$$

$$\left\{\begin{array}{l} \dfrac{\mathbf{u}^{n+1}-\mathbf{u}^{n+\frac{1}{2}}}{\Delta t} = -\dfrac{1}{2}\nabla\delta p^{n+1} \text{ where } \delta p^{n+1} = p^{n+1}-p^n, \\ div\ \mathbf{u}^{n+1} = 0, \end{array}\right. \tag{2.3}$$

where $\mathbf{u}^{n+\frac{1}{2}}$ is called the intermediate velocity.
Moreover, from (2.2) and (2.3), we deduce the following system used to compute the pressure :

$$\left\{\begin{array}{l} \Delta\delta p^{n+1} = \dfrac{2}{\Delta t} div\ \mathbf{u}^{n+\frac{1}{2}}, \\ \dfrac{\partial\delta p^{n+1}}{\partial\mathbf{n}} = 0. \end{array}\right. \tag{2.4}$$

The algorithm starts by computing the intermediate velocities with (2.2). Then, in a second step, the projection step, we compute the pressure with (2.4). The new velocities are derived from (2.3). Consequently, the computations of the velocity and the pressure are decoupled. As the nonlinear terms are made explicit, at each time step, we only have to solve a set of Helmholtz-type equations for the velocity and a scalar Poisson equation with Neumann boundary condition for the pressure. Another advantage of this method is that the incompressibility condition is satisfied with the computer accuracy.

2.2 Finite volume discretization

Now, we describe the collocated finite volume implementation of the previous time discretization. As usual in a finite volume context, we integrate the equations on each control volume K_{ij} for $i = 1, ..., M$, $j = 1, ..., N$, and we find :

– For the time derivative term in (2.2) :

$$\int_{K_{ij}} \frac{\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}^n}{\Delta t} \, dxdy \simeq \Delta x \Delta y \frac{\mathbf{u}_{ij}^{n+\frac{1}{2}} - \mathbf{u}_{ij}^n}{\Delta t}.$$

– For the Laplacian operator used with the velocities in (2.2) and with the pressure variation in (2.4) :

$$\begin{aligned}
\int_{K_{ij}} \Delta \mathbf{u} \, dxdy &= \int_{\partial K_{ij}} \frac{\partial \mathbf{u}}{\partial n} d\Gamma, \\
&\simeq \Delta y \frac{\mathbf{u}_{i+1j} - \mathbf{u}_{ij}}{\Delta x} + \Delta y \frac{\mathbf{u}_{i-1j} - \mathbf{u}_{ij}}{\Delta x} + \Delta x \frac{\mathbf{u}_{ij+1} - \mathbf{u}_{ij}}{\Delta y} + \Delta x \frac{\mathbf{u}_{ij-1} - \mathbf{u}_{ij}}{\Delta y}, \\
\int_{K_{ij}} \Delta \, \delta p \, dxdy &\simeq \Delta y \frac{\delta p_{i+1j} - \delta p_{ij}}{\Delta x} + \Delta y \frac{\delta p_{i-1j} - \delta p_{ij}}{\Delta x} + \Delta x \frac{\delta p_{ij+1} - \delta p_{ij}}{\Delta y} \\
&+ \Delta x \frac{\delta p_{ij-1} - \delta p_{ij}}{\Delta y}.
\end{aligned}$$

– For the pressure gradient used in (2.2) and (2.3) :

$$\begin{aligned}
\int_{K_{ij}} \nabla p \, dxdy &= \int_{\partial K_{ij}} \begin{pmatrix} p n_x \\ p n_y \end{pmatrix} d\Gamma, \\
&\simeq \begin{pmatrix} \frac{\Delta y}{2} \left(p_{i+1j} - p_{i-1j} \right) \\ \frac{\Delta x}{2} \left(p_{ij+1} - p_{ij-1} \right) \end{pmatrix},
\end{aligned}$$

using a linear interpolation for the pressure at the edges.

– For the nonlinear term in (2.2) :

$$\begin{aligned}
\int_{K_{ij}} (\mathbf{u}.\nabla)\mathbf{u}\, dxdy &= \int_{K_{ij}} \begin{pmatrix} \partial_x(u^2) + \partial_y(vu) \\ \partial_x(uv) + \partial_y(v^2) \end{pmatrix} dxdy, \\
&= \int_{\partial K_{ij}} \begin{pmatrix} (u^2)n_x + (vu)n_y \\ (uv)n_x + (v^2)n_y \end{pmatrix} d\Gamma, \\
&\simeq \left(\begin{array}{l} \Delta y F_{u\, i+\frac{1}{2}j} \dfrac{u_{i+1j} + u_{ij}}{2} - \Delta y F_{u\, i-\frac{1}{2}j} \dfrac{u_{ij} + u_{i-1j}}{2} \\ \Delta y F_{u\, i+\frac{1}{2}j} \dfrac{v_{i+1j} + v_{ij}}{2} - \Delta y F_{u\, i-\frac{1}{2}j} \dfrac{v_{ij} + v_{i-1j}}{2} \\ +\Delta x F_{v\, ij+\frac{1}{2}} \dfrac{u_{ij+1} + u_{ij}}{2} - \Delta x F_{v\, ij-\frac{1}{2}} \dfrac{u_{ij} + u_{ij-1}}{2} \\ +\Delta x F_{v\, ij+\frac{1}{2}} \dfrac{v_{ij+1} + v_{ij}}{2} - \Delta x F_{v\, ij-\frac{1}{2}} \dfrac{v_{ij} + v_{ij-1}}{2} \end{array} \right),
\end{aligned}$$

using again a linear interpolation for the velocities at the edges.

– For the divergence term in (2.4) :

$$\begin{aligned}
\int_{K_{ij}} div\, \mathbf{u}\, dxdy &= \int_{\partial K_{ij}} \mathbf{u} \cdot \mathbf{n}\, d\Gamma, \\
&\simeq \left[\Delta y \Big(F_{u\, i+\frac{1}{2}j} - F_{u\, i-\frac{1}{2}j} \Big) + \Delta x \Big(F_{v\, ij+\frac{1}{2}} - F_{v\, ij-\frac{1}{2}} \Big) \right].
\end{aligned}$$

In the above equations, the boundary conditions are not taken into account. The Dirichlet boundary condition for the intermediate velocity $\mathbf{u}^{n+\frac{1}{2}}$ in (2.2) means that for $i = 1, \ldots, M$ and $j = 1, \ldots, N$, we have :

$$\begin{aligned}
\Delta y \frac{\mathbf{u}_{M+1j}^{n+\frac{1}{2}} - \mathbf{u}_{Mj}^{n+\frac{1}{2}}}{\Delta x} &= \Delta y \frac{\mathbf{g}_{M+\frac{1}{2}j} - \mathbf{u}_{Mj}^{n+\frac{1}{2}}}{\frac{\Delta x}{2}}, & \Delta y \frac{\mathbf{u}_{0j}^{n+\frac{1}{2}} - \mathbf{u}_{1j}^{n+\frac{1}{2}}}{\Delta x} &= \Delta y \frac{\mathbf{g}_{\frac{1}{2}j} - \mathbf{u}_{1j}^{n+\frac{1}{2}}}{\frac{\Delta x}{2}}, \\
\Delta x \frac{\mathbf{u}_{iN+1}^{n+\frac{1}{2}} - \mathbf{u}_{iN}^{n+\frac{1}{2}}}{\Delta y} &= \Delta x \frac{\mathbf{g}_{iN+\frac{1}{2}} - \mathbf{u}_{iN}^{n+\frac{1}{2}}}{\frac{\Delta y}{2}}, & \Delta x \frac{\mathbf{u}_{i0}^{n+\frac{1}{2}} - \mathbf{u}_{i1}^{n+\frac{1}{2}}}{\Delta y} &= \Delta x \frac{\mathbf{g}_{i\frac{1}{2}} - \mathbf{u}_{i1}^{n+\frac{1}{2}}}{\frac{\Delta y}{2}}, \qquad (2.5)
\end{aligned}$$

where $\mathbf{g}_{M+\frac{1}{2}j}$, $\mathbf{g}_{\frac{1}{2}j}$, $\mathbf{g}_{iN+\frac{1}{2}j}$ and $\mathbf{g}_{i\frac{1}{2}j}$ are given by :

$$\begin{aligned}
\mathbf{g}_{M+\frac{1}{2}j} &= \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} g(L_1, y)\, dy, \quad \mathbf{g}_{\frac{1}{2}j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} g(0, y)\, dy, \\
\mathbf{g}_{iN+\frac{1}{2}j} &= \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(x, L_2)\, dx, \quad \mathbf{g}_{i\frac{1}{2}j} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(x, 0)\, dx.
\end{aligned}$$

It is well known that when we use finite volumes with Dirichlet boundary conditions, we consider the edges which determine the frontier of the domain as control volumes.

Moreover, the Neumann boundary condition for the pressure variation δp^{n+1} in (2.4) means that :

$$\begin{aligned}
\Delta y \frac{\delta p^{n+1}_{M+1j} - \delta p^{n+1}_{Mj}}{\Delta x} &= 0, \quad \Delta y \frac{\delta p^{n+1}_{0j} - \delta p^{n+1}_{1j}}{\Delta x} = 0, \\
\Delta x \frac{\delta p^{n+1}_{iN+1} - \delta p^{n+1}_{iN}}{\Delta y} &= 0, \quad \Delta x \frac{\delta p^{n+1}_{i0} - \delta p^{n+1}_{i1}}{\Delta y} = 0.
\end{aligned} \tag{2.6}$$

When Neumann boundary conditions are required in a finite volume context, we introduce virtual control volumes outside the domain.

Remark 2.1. In (2.2), the terms p^n_{0j}, p^n_{M+1j}, p^n_{i0} and p^n_{iN+1} occur in the gradient of the pressure, but they are not given by the boundary conditions. Indeed, the Neumann boundary condition is for δp and not for p. Consequently, we use a second order compact scheme [Lel92] to compute them :

$$\begin{aligned}
p^n_{0j} &= \frac{5}{2}p^n_{1j} - 2p^n_{2j} + \frac{1}{2}p^n_{3j}, \quad p^n_{M+1j} = \frac{5}{2}p^n_{Mj} - 2p^n_{M-1j} + \frac{1}{2}p^n_{M-2j}, \\
p^n_{i0} &= \frac{5}{2}p^n_{i1} - 2p^n_{i2} + \frac{1}{2}p^n_{i3}, \quad p^n_{iN+1} = \frac{5}{2}p^n_{iN} - 2p^n_{iN-1} + \frac{1}{2}p^n_{iN-2}.
\end{aligned} \tag{2.7}$$

Finally, the three steps of the method are the following :

(*i*) We compute the intermediate velocities $\mathbf{u}^{n+\frac{1}{2}} = \left(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}\right)$ from :

$$\begin{aligned}
&\Delta x \Delta y \frac{u^{n+\frac{1}{2}}_{ij} - u^n_{ij}}{\Delta t} - \frac{\nu}{2}\left[\Delta y \frac{u^{n+\frac{1}{2}}_{i+1j} - u^{n+\frac{1}{2}}_{ij}}{\Delta x} + \Delta y \frac{u^{n+\frac{1}{2}}_{i-1j} - u^{n+\frac{1}{2}}_{ij}}{\Delta x} + \Delta x \frac{u^{n+\frac{1}{2}}_{ij+1} - u^{n+\frac{1}{2}}_{ij}}{\Delta y}\right. \\
&+ \left.\Delta x \frac{u^{n+\frac{1}{2}}_{ij-1} - u^{n+\frac{1}{2}}_{ij}}{\Delta y} + \Delta y \frac{u^n_{i+1j} - u^n_{ij}}{\Delta x} + \Delta y \frac{u^n_{i-1j} - u^n_{ij}}{\Delta x} + \Delta x \frac{u^n_{ij+1} - u^n_{ij}}{\Delta y} + \Delta x \frac{u^n_{ij-1} - u^n_{ij}}{\Delta y}\right] \\
&+ \frac{3}{2}\left[\Delta y F^n_{u\, i+\frac{1}{2}j} \frac{u^n_{i+1j} + u^n_{ij}}{2} - \Delta y F^n_{u\, i-\frac{1}{2}j} \frac{u^n_{ij} + u^n_{i-1j}}{2} + \Delta x F^n_{v\, ij+\frac{1}{2}} \frac{u^n_{ij+1} + u^n_{ij}}{2}\right. \\
&- \left.\Delta x F^n_{v\, ij-\frac{1}{2}} \frac{u^n_{ij} + u^n_{ij-1}}{2}\right] - \frac{1}{2}\left[\Delta y F^{n-1}_{u\, i+\frac{1}{2}j} \frac{u^{n-1}_{i+1j} + u^{n-1}_{ij}}{2} - \Delta y F^{n-1}_{u\, i-\frac{1}{2}j} \frac{u^{n-1}_{ij} + u^{n-1}_{i-1j}}{2}\right. \\
&+ \left.\Delta x F^{n-1}_{v\, ij+\frac{1}{2}} \frac{u^{n-1}_{ij+1} + u^{n-1}_{ij}}{2} - \Delta x F^{n-1}_{v\, ij-\frac{1}{2}} \frac{u^{n-1}_{ij} + u^{n-1}_{ij-1}}{2}\right] + \frac{\Delta y}{2}\left(p^n_{i+1j} - p^n_{i-1j}\right) \\
&= \Delta x \Delta y \frac{f^{n+1}_{u\, ij} + f^n_{u\, ij}}{2},
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\Delta x\Delta y\frac{v_{ij}^{n+\frac{1}{2}}-v_{ij}^{n}}{\Delta t}-\frac{\nu}{2}\Bigg[\Delta y\frac{v_{i+1j}^{n+\frac{1}{2}}-v_{ij}^{n+\frac{1}{2}}}{\Delta x}+\Delta y\frac{v_{i-1j}^{n+\frac{1}{2}}-v_{ij}^{n+\frac{1}{2}}}{\Delta x}+\Delta x\frac{v_{ij+1}^{n+\frac{1}{2}}-v_{ij}^{n+\frac{1}{2}}}{\Delta y}\\
&+\ \Delta x\frac{v_{ij-1}^{n+\frac{1}{2}}-v_{ij}^{n+\frac{1}{2}}}{\Delta y}+\Delta y\frac{v_{i+1j}^{n}-v_{ij}^{n}}{\Delta x}+\Delta y\frac{v_{i-1j}^{n}-v_{ij}^{n}}{\Delta x}+\Delta x\frac{v_{ij+1}^{n}-v_{ij}^{n}}{\Delta y}+\Delta x\frac{v_{ij-1}^{n}-v_{ij}^{n}}{\Delta y}\Bigg]\\
&+\ \frac{3}{2}\Bigg[\Delta yF_{u\,i+\frac{1}{2}j}^{n}\frac{v_{i+1j}^{n}+v_{ij}^{n}}{2}-\Delta yF_{u\,i-\frac{1}{2}j}^{n}\frac{v_{ij}^{n}+v_{i-1j}^{n}}{2}+\Delta xF_{v\,ij+\frac{1}{2}}^{n}\frac{v_{ij+1}^{n}+v_{ij}^{n}}{2}\\
&-\ \Delta xF_{v\,ij-\frac{1}{2}}^{n}\frac{v_{ij}^{n}+v_{ij-1}^{n}}{2}\Bigg]-\frac{1}{2}\Bigg[\Delta yF_{u\,i+\frac{1}{2}j}^{n-1}\frac{v_{i+1j}^{n-1}+v_{ij}^{n-1}}{2}-\Delta yF_{u\,i-\frac{1}{2}j}^{n-1}\frac{v_{ij}^{n-1}+v_{i-1j}^{n-1}}{2}\\
&+\ \Delta xF_{v\,ij+\frac{1}{2}}^{n-1}\frac{v_{ij+1}^{n-1}+v_{ij}^{n-1}}{2}-\Delta xF_{v\,ij-\frac{1}{2}}^{n-1}\frac{v_{ij}^{n-1}+v_{ij-1}^{n-1}}{2}\Bigg]+\frac{\Delta x}{2}\left(p_{ij+1}^{n}-p_{ij-1}^{n}\right)\\
&=\ \Delta x\Delta y\frac{f_{v\,ij}^{n+1}+f_{v\,ij}^{n}}{2},
\end{aligned}\tag{2.9}$$

using the boundary conditions (2.5) and Remark 2.1.

(*ii*) We compute the pressure variation δp^{n+1} from :

$$\begin{aligned}
&\Delta y\frac{\delta p_{i+1j}^{n+1}-\delta p_{ij}^{n+1}}{\Delta x}+\Delta y\frac{\delta p_{i-1j}^{n+1}-\delta p_{ij}^{n+1}}{\Delta x}+\Delta x\frac{\delta p_{ij+1}^{n+1}-\delta p_{ij}^{n+1}}{\Delta y}+\Delta x\frac{\delta p_{ij-1}^{n+1}-\delta p_{ij}^{n+1}}{\Delta y}\\
&=\ \frac{2}{\Delta t}\Bigg[\Delta y\left(F_{u\,i+\frac{1}{2}j}^{n+\frac{1}{2}}-F_{u\,i-\frac{1}{2}j}^{n+\frac{1}{2}}\right)+\Delta x\left(F_{v\,ij+\frac{1}{2}}^{n+\frac{1}{2}}-F_{v\,ij-\frac{1}{2}}^{n+\frac{1}{2}}\right)\Bigg],
\end{aligned}\tag{2.10}$$

using the boundary conditions (2.6).

(*iii*) We compute the new velocities $\mathbf{u}^{n+1}$ and the new fluxes F_u^{n+1} and F_v^{n+1} from :

$$\Delta x\Delta y\frac{\mathbf{u}_{ij}^{n+1}-\mathbf{u}_{ij}^{n+\frac{1}{2}}}{\Delta t}=-\frac{1}{2}\begin{pmatrix}\frac{\Delta y}{2}\left(\delta p_{i+1j}^{n+1}-\delta p_{i-1j}^{n+1}\right)\\ \frac{\Delta x}{2}\left(\delta p_{ij+1}^{n+1}-\delta p_{ij-1}^{n+1}\right)\end{pmatrix},\tag{2.11}$$

$$\frac{F_{u\,i+\frac{1}{2}j}^{n+1}-F_{u\,i+\frac{1}{2}j}^{n+\frac{1}{2}}}{\Delta t}=-\frac{1}{2}\frac{\delta p_{i+1j}^{n+1}-\delta p_{ij}^{n+1}}{\Delta x},\tag{2.12}$$

$$\frac{F_{v\,ij+\frac{1}{2}}^{n+1}-F_{v\,ij+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta t}=-\frac{1}{2}\frac{\delta p_{ij+1}^{n+1}-\delta p_{ij}^{n+1}}{\Delta y},\tag{2.13}$$

using the boundary conditions (2.6).

In the above algorithm, using the first step we know the intermediate velocities $\mathbf{u}^{n+\frac{1}{2}}$ at the center of the cells. However, in the second and third steps, we need to know the intermediate velocity fluxes $F_u^{n+\frac{1}{2}}$ and $F_v^{n+\frac{1}{2}}$, hence, these fluxes must be interpolated from the values of the intermediate velocities at the center of the cells. As this interpolation is crucial, we devote the next subsection to explain how we compute the fluxes.

2.3 Computation of the intermediate fluxes

The way used to interpolate the intermediate fluxes is essential because it determines the coupling between the velocity and the pressure. The simplest method to compute the intermediate fluxes is to use a linear interpolation :

$$F^{n+\frac{1}{2}}_{u\,i+\frac{1}{2}j} = \frac{u^{n+\frac{1}{2}}_{i+1j} + u^{n+\frac{1}{2}}_{ij}}{2}, \tag{2.14}$$

$$F^{n+\frac{1}{2}}_{v\,ij+\frac{1}{2}} = \frac{v^{n+\frac{1}{2}}_{ij+1} + v^{n+\frac{1}{2}}_{ij}}{2}. \tag{2.15}$$

Unfortunately, this simple interpolation does not work when we reduce the viscosity ν because the pressure and the velocities are not sufficiently coupled. Let us explain what does this mean. First, if we consider that $\mathbf{u}^{n+\frac{1}{2}}$ is known, then, from (2.8), we obtain :

$$\begin{aligned}
u^{n+\frac{1}{2}}_{ij} &= \left(\frac{\Delta x\Delta y}{\Delta t} + \nu\frac{\Delta y}{\Delta x} + \nu\frac{\Delta x}{\Delta y}\right)^{-1}\left[\frac{\Delta x\Delta y}{\Delta t}u^n_{ij} + \frac{\nu}{2}\left[\frac{\Delta y}{\Delta x}u^{n+\frac{1}{2}}_{i+1j} + \frac{\Delta y}{\Delta x}u^{n+\frac{1}{2}}_{i-1j} + \frac{\Delta x}{\Delta y}u^{n+\frac{1}{2}}_{ij+1}\right.\right.\\
&+ \left.\frac{\Delta x}{\Delta y}u^{n+\frac{1}{2}}_{ij-1} + \Delta y\frac{u^n_{i+1j} - u^n_{ij}}{\Delta x} + \Delta y\frac{u^n_{i-1j} - u^n_{ij}}{\Delta x} + \Delta x\frac{u^n_{ij+1} - u^n_{ij}}{\Delta y} + \Delta x\frac{u^n_{ij-1} - u^n_{ij}}{\Delta y}\right]\\
&- \frac{3}{2}\left[\Delta yF^n_{u\,i+\frac{1}{2}j}\frac{u^n_{i+1j} + u^n_{ij}}{2} - \Delta yF^n_{u\,i-\frac{1}{2}j}\frac{u^n_{ij} + u^n_{i-1j}}{2} + \Delta xF^n_{v\,ij+\frac{1}{2}}\frac{u^n_{ij+1} + u^n_{ij}}{2}\right.\\
&- \left.\Delta xF^n_{v\,ij-\frac{1}{2}}\frac{u^n_{ij} + u^n_{ij-1}}{2}\right] + \frac{1}{2}\left[\Delta yF^{n-1}_{u\,i+\frac{1}{2}j}\frac{u^{n-1}_{i+1j} + u^{n-1}_{ij}}{2} - \Delta yF^{n-1}_{u\,i-\frac{1}{2}j}\frac{u^{n-1}_{ij} + u^{n-1}_{i-1j}}{2}\right.\\
&+ \left.\left.\Delta xF^{n-1}_{v\,ij+\frac{1}{2}}\frac{u^{n-1}_{ij+1} + u^{n-1}_{ij}}{2} - \Delta xF^{n-1}_{v\,ij-\frac{1}{2}}\frac{u^{n-1}_{ij} + u^{n-1}_{ij-1}}{2}\right] + \Delta x\Delta y\frac{f^{n+1}_{u\,ij} + f^n_{u\,ij}}{2}\right]\\
&- \left(\frac{\Delta x\Delta y}{\Delta t} + \nu\frac{\Delta y}{\Delta x} + \nu\frac{\Delta x}{\Delta y}\right)^{-1}\frac{\Delta y}{2}\left(p^n_{i+1j} - p^n_{i-1j}\right).
\end{aligned} \tag{2.16}$$

Now, considering only the last term in the right hand side of the above equation, the contribution of the pressure in the horizontal fluxes computed with the linear interpolation (2.14) is :

$$-\left(\frac{\Delta x\Delta y}{\Delta t} + \nu\frac{\Delta y}{\Delta x} + \nu\frac{\Delta x}{\Delta y}\right)^{-1}\frac{\Delta y}{4}\left(p^n_{i+2j} + p^n_{i+1j} - p^n_{ij} - p^n_{i-1j}\right).$$

This implies that in the right hand side of (2.10), the pressure contribution to $\left(F^{n+\frac{1}{2}}_{u\,i+\frac{1}{2}j} - F^{n+\frac{1}{2}}_{u\,i-\frac{1}{2}j}\right)$ is $p^n_{i+2j} - 2p^n_{ij} + p^n_{i-2j}$. This contribution shows that the pressure terms p^n_{i+1j} and p^n_{i-1j} do not play any role in the divergence of the intermediate velocity i.e. the velocity and the pressure are decoupled. Such an interpolation leads to incorrect numerical results when we increase the Reynolds number.

We have by passed this difficulty using an approach proposed in [Pat80], [PKS88]. We have modified the interpolation method for the intermediate fluxes. Let us now describe how we compute the horizontal fluxes (and the same method is used for the vertical fluxes) :

$$\begin{aligned}
F^{n+\frac{1}{2}}_{i+\frac{1}{2}j} = & \Bigg\langle \left(\frac{\Delta x \Delta y}{\Delta t} + \nu \frac{\Delta y}{\Delta x} + \nu \frac{\Delta x}{\Delta y}\right)^{-1} \Bigg[\frac{\Delta x \Delta y}{\Delta t} u^n_{ij} + \frac{\nu}{2} \Bigg[\frac{\Delta y}{\Delta x} u^{n+\frac{1}{2}}_{i+1j} + \frac{\Delta y}{\Delta x} u^{n+\frac{1}{2}}_{i-1j} + \frac{\Delta x}{\Delta y} u^{n+\frac{1}{2}}_{ij+1} \\
+ & \frac{\Delta x}{\Delta y} u^{n+\frac{1}{2}}_{ij-1} + \Delta y \frac{u^n_{i+1j} - u^n_{ij}}{\Delta x} + \Delta y \frac{u^n_{i-1j} - u^n_{ij}}{\Delta x} + \Delta x \frac{u^n_{ij+1} - u^n_{ij}}{\Delta y} + \Delta x \frac{u^n_{ij-1} - u^n_{ij}}{\Delta y} \Bigg] \\
- & \frac{3}{2} \Bigg[\Delta y F^n_{u\, i+\frac{1}{2}j} \frac{u^n_{i+1j} + u^n_{ij}}{2} - \Delta y F^n_{u\, i-\frac{1}{2}j} \frac{u^n_{ij} + u^n_{i-1j}}{2} + \Delta x F^n_{v\, ij+\frac{1}{2}} \frac{u^n_{ij+1} + u^n_{ij}}{2} \\
- & \Delta x F^n_{v\, ij-\frac{1}{2}} \frac{u^n_{ij} + u^n_{ij-1}}{2} \Bigg] + \frac{1}{2} \Bigg[\Delta y F^{n-1}_{u\, i+\frac{1}{2}j} \frac{u^{n-1}_{i+1j} + u^{n-1}_{ij}}{2} - \Delta y F^{n-1}_{u\, i-\frac{1}{2}j} \frac{u^{n-1}_{ij} + u^{n-1}_{i-1j}}{2} \\
+ & \Delta x F^{n-1}_{v\, ij+\frac{1}{2}} \frac{u^{n-1}_{ij+1} + u^{n-1}_{ij}}{2} - \Delta x F^{n-1}_{v\, ij-\frac{1}{2}} \frac{u^{n-1}_{ij} + u^{n-1}_{ij-1}}{2} \Bigg] + \Delta x \Delta y \frac{f^{n+1}_{u\, ij} + f^n_{u\, ij}}{2} \Bigg] \Bigg\rangle_{i+\frac{1}{2}j} \\
- & \left(\frac{\Delta x \Delta y}{\Delta t} + \nu \frac{\Delta y}{\Delta x} + \nu \frac{\Delta x}{\Delta y}\right)^{-1} \Delta y \left(p^n_{i+1j} - p^n_{ij}\right), \qquad (2.17)
\end{aligned}$$

where the angle brackets stand for a linear interpolation : $\langle \alpha_{ij} \rangle_{i+\frac{1}{2}j} = \dfrac{\alpha_{i+1j} + \alpha_{ij}}{2}$.
As for the linear interpolation (2.14), we study the pressure contribution in the horizontal fluxes computed with this new interpolation (2.17), and we find :

$$-\left(\frac{\Delta x \Delta y}{\Delta t} + \nu \frac{\Delta y}{\Delta x} + \nu \frac{\Delta x}{\Delta y}\right)^{-1} \Delta y \left(p^n_{i+1j} - p^n_{ij}\right).$$

This implies that in the right hand side of (2.10), the pressure contribution in $\left(F^{n+\frac{1}{2}}_{u\, i+\frac{1}{2}j} - F^{n+\frac{1}{2}}_{u\, i-\frac{1}{2}j}\right)$ is $p^n_{i+1j} - 2p^n_{ij} + p^n_{i-1j}$ i.e. the pressure and the velocity are well-coupled.

It is also important to note that the above interpolation does not increase the cost of the simulation as we do not have to solve a new linear system in order to obtain the intermediate fluxes.

3 Collocated finite volumes scheme with a splitting method for the time discretization

In their article [GS03a], Guermond and Shen introduce a new class of splitting schemes for incompressible flows, called consistent splitting schemes. Like for the projection scheme described

in the previous section, these new schemes only require to solve a set of Helmholtz-type equations for the velocity and a Poisson equation for the pressure. Moreover, the preliminary analysis and the numerical experiments realized by the authors have shown that the consistent splitting schemes are unconditionally stable and yield full order accuracy for the velocity and the pressure in both the L^2- and H^1-norms. For all these reasons, it seems interesting to study if it is possible to use their scheme with a collocated space discretization.
The scheme used here is the second-order consistent splitting scheme tested in [GS03a].

3.1 Time discretization

First, let us quickly recall how this second-order consistent splitting scheme is built.

We start by choosing a time discretization for (1.1) :

$$\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \nu\Delta\mathbf{u}^{n+1} + \hat{\mathbf{B}}\big(\mathbf{u}^n, \mathbf{u}^{n-1}\big) + 2\nabla\mathbf{p}^n - \nabla\mathbf{p}^{n-1} = \mathbf{f}^{n+1}, \qquad (3.1)$$

where $\hat{B}\big(\mathbf{u}^n, \mathbf{u}^{n-1}\big) = \Big([2\mathbf{u}^n - \mathbf{u}^{n-1}] \cdot \nabla\Big)[2\mathbf{u}^n - \mathbf{u}^{n-1}]$. From this scheme, we will be able to compute the new velocity $\mathbf{u}^{n+1}$.

Then, to obtain the pressure, we take the divergence of (1.1) and use the incompressibility condition; we find :

$$\Delta p = div\,(\mathbf{f} + \nu\Delta\mathbf{u} - (\mathbf{u}.\nabla)\mathbf{u}). \qquad (3.2)$$

The time discretization for the above equation is :

$$\Delta p^{n+1} \quad = \quad div\left(\mathbf{f}^{n+1} + \nu\Delta\mathbf{u}^{n+1} - \hat{\mathbf{B}}\big(\mathbf{u}^n, \mathbf{u}^{n-1}\big)\right). \qquad (3.3)$$

Following [GS03a], we replace in (3.3) Δ by $-\nabla \times \nabla\times$. In this way, the accuracy of our splitting scheme will be improved [GS03a], and we obtain :

$$\Delta p^{n+1} \quad = \quad div\left(\mathbf{f}^{n+1} - \nu\nabla \times \nabla \times \mathbf{u}^{n+1} - \hat{\mathbf{B}}\big(\mathbf{u}^n, \mathbf{u}^{n-1}\big)\right). \qquad (3.4)$$

Using the following identity in (3.1) :

$$\Delta\mathbf{u}^{n+1} = \nabla div\ \mathbf{u}^{n+1} - \nabla \times \nabla \times \mathbf{u}^{n+1}, \qquad (3.5)$$

we obtain :

$$\begin{aligned} &\mathbf{f}^{n+1} - \nu\nabla \times \nabla \times \mathbf{u}^{n+1} - \hat{\mathbf{B}}\big(\mathbf{u}^n, \mathbf{u}^{n-1}\big) \\ = \quad &\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \nu\nabla div\ \mathbf{u}^{n+1} + 2\nabla p^n - \nabla p^{n-1}. \end{aligned} \qquad (3.6)$$

Inserting the previous equation in (3.4), we obtain :

$$\Delta p^{n+1} \quad = \quad div\left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \nu\nabla div\ \mathbf{u}^{n+1} + 2\nabla p^n - \nabla p^{n-1}\right). \qquad (3.7)$$

Hence :

$$\triangle\left(p^{n+1}-2p^{n}+p^{n-1}+\nu div\ \mathbf{u}^{n+1}\right) = div\left(\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\Delta t}\right). \tag{3.8}$$

In summary, the time discretization obtained by following the method of Guermond and Shen [GS03a], consists in computing the velocity $\mathbf{u}^{n+1}$ with the following equation corresponding to (3.1) :

$$\begin{cases} \dfrac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\Delta t} - \nu\Delta\mathbf{u}^{n+1} + \hat{B}\left(\mathbf{u}^{n},\mathbf{u}^{n-1}\right) + 2\nabla p^{n} - \nabla p^{n-1} = f^{n+1}, \\ \text{where} \quad \hat{B}\left(\mathbf{u}^{n},\mathbf{u}^{n-1}\right) = \left([2\mathbf{u}^{n}-\mathbf{u}^{n-1}]\cdot\nabla\right)[2\mathbf{u}^{n}-\mathbf{u}^{n-1}], \\ \mathbf{u}^{n+1}\Big|_{\partial\Omega} = \mathbf{g}, \end{cases} \tag{3.9}$$

and then to compute the pressure p^{n+1} from :

$$\begin{cases} \Delta\psi^{n+1} = div\left(\dfrac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\Delta t}\right), \\ \dfrac{\partial\psi^{n+1}}{\partial\mathbf{n}} = 0, \end{cases} \tag{3.10}$$

$$p^{n+1} = \psi^{n+1} + 2p^{n} - p^{n-1} - \nu div\ \mathbf{u}^{n+1}. \tag{3.11}$$

The main advantage of this splitting scheme is that it is no longer plagued by the artificial Neumann boundary condition of p^{n+1} (see (2.4)).

3.2 Finite volume discretization

To derive the collocated finite volume discretization associated with this time discretization, we integrate the equations (3.9),(3.10) and (3.11) on each control volume K_{ij}. In doing so, we keep the notations used in the previous section and also the method used to integrate each term. Concerning the boundary conditions, we have a Dirichlet boundary condition for $\mathbf{u}^{n+1}$:

$$\begin{aligned} \Delta y\frac{\mathbf{u}^{n+1}_{M+1j}-\mathbf{u}^{n+1}_{Mj}}{\Delta x} &= \Delta y\frac{\mathbf{g}_{M+\frac{1}{2}j}-\mathbf{u}^{n+1}_{Mj}}{\frac{\Delta x}{2}}, \quad \Delta y\frac{\mathbf{u}^{n+1}_{0j}-\mathbf{u}^{n+1}_{1j}}{\Delta x} = \Delta y\frac{\mathbf{g}_{\frac{1}{2}j}-\mathbf{u}^{n+1}_{1j}}{\frac{\Delta x}{2}}, \\ \Delta x\frac{\mathbf{u}^{n+1}_{iN+1}-\mathbf{u}^{n+1}_{iN}}{\Delta y} &= \Delta x\frac{\mathbf{g}_{iN+\frac{1}{2}}-\mathbf{u}^{n+1}_{iN}}{\frac{\Delta y}{2}}, \quad \Delta x\frac{\mathbf{u}^{n+1}_{i0}-\mathbf{u}^{n+1}_{i1}}{\Delta y} = \Delta x\frac{\mathbf{g}_{i\frac{1}{2}}-\mathbf{u}^{n+1}_{i1}}{\frac{\Delta y}{2}}, \end{aligned} \tag{3.12}$$

and a Neumann boundary condition for ψ^{n+1} :

$$\begin{aligned} \Delta y\frac{\psi^{n+1}_{M+1j}-\psi^{n+1}_{Mj}}{\Delta x} &= 0, \quad \Delta y\frac{\psi^{n+1}_{0j}-\psi^{n+1}_{1j}}{\Delta x} = 0, \\ \Delta x\frac{\psi^{n+1}_{iN+1}-\psi^{n+1}_{iN}}{\Delta y} &= 0, \quad \Delta x\frac{\psi^{n+1}_{i0}-\psi^{n+1}_{i1}}{\Delta y} = 0. \end{aligned} \tag{3.13}$$

Finally, the two steps of the algorithm are the following ones :

(i) We compute the new velocities $\mathbf{u}^{n+1} = \left(u^{n+1}, v^{n+1}\right)$ from :

$$\Delta x \Delta y \frac{3\mathbf{u}_{ij}^{n+1} - 4\mathbf{u}_{ij}^{n} + \mathbf{u}_{ij}^{n-1}}{2\Delta t} - \nu \left[\Delta y \frac{\mathbf{u}_{i+1j}^{n+1} - \mathbf{u}_{ij}^{n+1}}{\Delta x} + \Delta y \frac{\mathbf{u}_{i-1j}^{n+1} - \mathbf{u}_{ij}^{n+1}}{\Delta x} \right.$$
$$+ \left. \Delta x \frac{\mathbf{u}_{ij+1}^{n+1} - \mathbf{u}_{ij}^{n+1}}{\Delta y} + \Delta x \frac{\mathbf{u}_{ij-1}^{n+1} - \mathbf{u}_{ij}^{n+1}}{\Delta y} \right] + 2 \begin{pmatrix} \frac{\Delta y}{2}\left(p_{i+1j}^{n} - p_{i-1j}^{n}\right) \\ \frac{\Delta x}{2}\left(p_{ij+1}^{n} - p_{ij-1}^{n}\right) \end{pmatrix}$$
$$- \begin{pmatrix} \frac{\Delta y}{2}\left(p_{i+1j}^{n-1} - p_{i-1j}^{n-1}\right) \\ \frac{\Delta x}{2}\left(p_{ij+1}^{n-1} - p_{ij-1}^{n-1}\right) \end{pmatrix} + \hat{\mathbf{B}}\left(\mathbf{u}^{\mathbf{n}}, \mathbf{u}^{\mathbf{n-1}}\right)_{\mathbf{ij}} = \mathbf{\Delta x \Delta y}\ \mathbf{f}_{\mathbf{ij}}^{\mathbf{n+1}}, \tag{3.14}$$

where :

$$\hat{\mathbf{B}}\left(\mathbf{u}^{\mathbf{n}}, \mathbf{u}^{\mathbf{n-1}}\right)_{\mathbf{ij}} = \begin{pmatrix} \Delta y \left[2F^{n}_{u\, i+\frac{1}{2}j} - F^{n-1}_{u\, i+\frac{1}{2}j}\right] \left[2\frac{u_{i+1j}^{n} + u_{ij}^{n}}{2} - \frac{u_{i+1j}^{n-1} + u_{ij}^{n-1}}{2}\right] \\ \Delta y \left[2F^{n}_{u\, i+\frac{1}{2}j} - F^{n-1}_{u\, i+\frac{1}{2}j}\right] \left[2\frac{v_{i+1j}^{n} + v_{ij}^{n}}{2} - \frac{v_{i+1j}^{n-1} + v_{ij}^{n-1}}{2}\right] \end{pmatrix}$$
$$- \begin{pmatrix} \Delta y \left[2F^{n}_{u\, i-\frac{1}{2}j} - F^{n-1}_{u\, i-\frac{1}{2}j}\right] \left[2\frac{u_{i-1j}^{n} + u_{ij}^{n}}{2} - \frac{u_{i-1j}^{n-1} + u_{ij}^{n-1}}{2}\right] \\ \Delta y \left[2F^{n}_{u\, i-\frac{1}{2}j} - F^{n-1}_{u\, i-\frac{1}{2}j}\right] \left[2\frac{v_{i-1j}^{n} + v_{ij}^{n}}{2} - \frac{v_{i-1j}^{n-1} + v_{ij}^{n-1}}{2}\right] \end{pmatrix}$$
$$+ \begin{pmatrix} \Delta x \left[2F^{n}_{v\, ij+\frac{1}{2}} - F^{n-1}_{v\, ij+\frac{1}{2}}\right] \left[2\frac{u_{ij+1}^{n} + u_{ij}^{n}}{2} - \frac{u_{ij+1}^{n-1} + u_{ij}^{n-1}}{2}\right] \\ \Delta x \left[2F^{n}_{v\, ij+\frac{1}{2}} - F^{n-1}_{v\, ij+\frac{1}{2}}\right] \left[2\frac{v_{ij+1}^{n} + v_{ij}^{n}}{2} - \frac{v_{ij+1}^{n-1} + v_{ij}^{n-1}}{2}\right] \end{pmatrix}$$
$$- \begin{pmatrix} \Delta x \left[2F^{n}_{v\, ij-\frac{1}{2}} - F^{n-1}_{v\, ij-\frac{1}{2}}\right] \left[2\frac{u_{ij-1}^{n} + u_{ij}^{n}}{2} - \frac{u_{ij-1}^{n-1} + u_{ij}^{n-1}}{2}\right] \\ \Delta x \left[2F^{n}_{v\, ij-\frac{1}{2}} - F^{n-1}_{v\, ij-\frac{1}{2}}\right] \left[2\frac{v_{ij-1}^{n} + v_{ij}^{n}}{2} - \frac{v_{ij-1}^{n-1} + v_{ij}^{n-1}}{2}\right] \end{pmatrix}$$

Near the boundary, we use (3.12) and Remark 2.1.

(*ii*) To compute the pressure p^{n+1}, we first compute ψ^{n+1} from :

$$\Delta y\frac{\psi^{n+1}_{i+1j}-\psi^{n+1}_{ij}}{\Delta x}+\Delta y\frac{\psi^{n+1}_{i-1j}-\psi^{n+1}_{ij}}{\Delta x}+\Delta x\frac{\psi^{n+1}_{ij+1}-\psi^{n+1}_{ij}}{\Delta y}+\Delta x\frac{\psi^{n+1}_{ij-1}-\psi^{n+1}_{ij}}{\Delta y}$$

$$= \frac{1}{2\Delta t}\Bigg[\Delta y\Big[\big(3F^{n+1}_{u\,i+\frac12 j}-4F^{n}_{u\,i+\frac12 j}+F^{n-1}_{u\,i+\frac12 j}\big)-\big(3F^{n+1}_{u\,i-\frac12 j}-4F^{n}_{u\,i-\frac12 j}+F^{n-1}_{u\,i-\frac12 j}\big)\Big]$$

$$+\ \Delta x\Big[\big(3F^{n+1}_{v\,ij+\frac12}-4F^{n}_{v\,ij+\frac12}+F^{n-1}_{v\,ij+\frac12}\big)-\big(3F^{n+1}_{v\,ij-\frac12}-4F^{n}_{v\,ij-\frac12}+F^{n-1}_{v\,ij-\frac12}\big)\Big]\Bigg], \quad (3.15)$$

using the boundary conditions (3.13).
Then, we easily obtain the pressure :

$$p^{n+1}_{ij} = \psi^{n+1}_{ij}+2p^{n}_{ij}-p^{n-1}_{ij}-\frac{\nu}{\Delta x\Delta y}\Big[\Delta y\Big(F^{n+1}_{u\,i+\frac12 j}-F^{n+1}_{u\,i-\frac12 j}\Big)+\Delta x\Big(F^{n+1}_{v\,ij+\frac12}-F^{n+1}_{v\,ij-\frac12}\Big)\Big]. \quad (3.16)$$

By contrast with the projection method described in Section 2, it is not necessary to modify the fluxes, and it is enough to linearly interpolate the fluxes F^{n+1}_u and F^{n+1}_v needed when we compute the pressure and the nonlinear term.
Consequently, this method seems simpler than the previous projection method. However, we will see in the next section that there are some small spatial oscillations for the pressure appearing in our calculations. Though these oscillations do not lead to incorrect numerical results, it is desirable to remove them disappear; we intend to solve this problem in a future work.

4 Numerical results

This section presents the numerical results obtained in the standard case of the driven cavity problem, in which the fluid is enclosed in a unit square box, with an imposed constant velocity in the horizontal direction on the top (driven) boundary, and a no-slip condition on the remaining walls. At the end of the section, we give some concluding remarks.

The driven cavity problem

In the following, we call the scheme using the projection method "PMFV" and the scheme using the splitting method of Guermond and Shen "GSFV". The driven boundary conditions are given by :

$$\mathbf{g}_{iN+\frac12} = (1,0) \quad \text{for } i=1,...,M, \quad \mathbf{g}_{i\frac12}=(0,0) \quad \text{for } i=1,...,M,$$
$$\mathbf{g}_{M+\frac12 j} = (0,0) \quad \text{for } j=1,...,N, \quad \mathbf{g}_{\frac12 j}=(0,0) \quad \text{for } j=1,...,N.$$

First, in order to validate our schemes, we compare the velocity profiles of "PMFV" and "GSFV" with the velocity profiles of Ghia [GGS82]. Figure 4.1 shows the u-velocity along the vertical line passing through the center and the v-velocity along the horizontal line passing through the center. The computations have been done for several Reynolds numbers, Re = 100, 1000, 3200 and 5000, with 128 × 128 control volumes. The velocity profiles of the schemes "PMFV" and "GSFV" are similar to the results of Ghia.

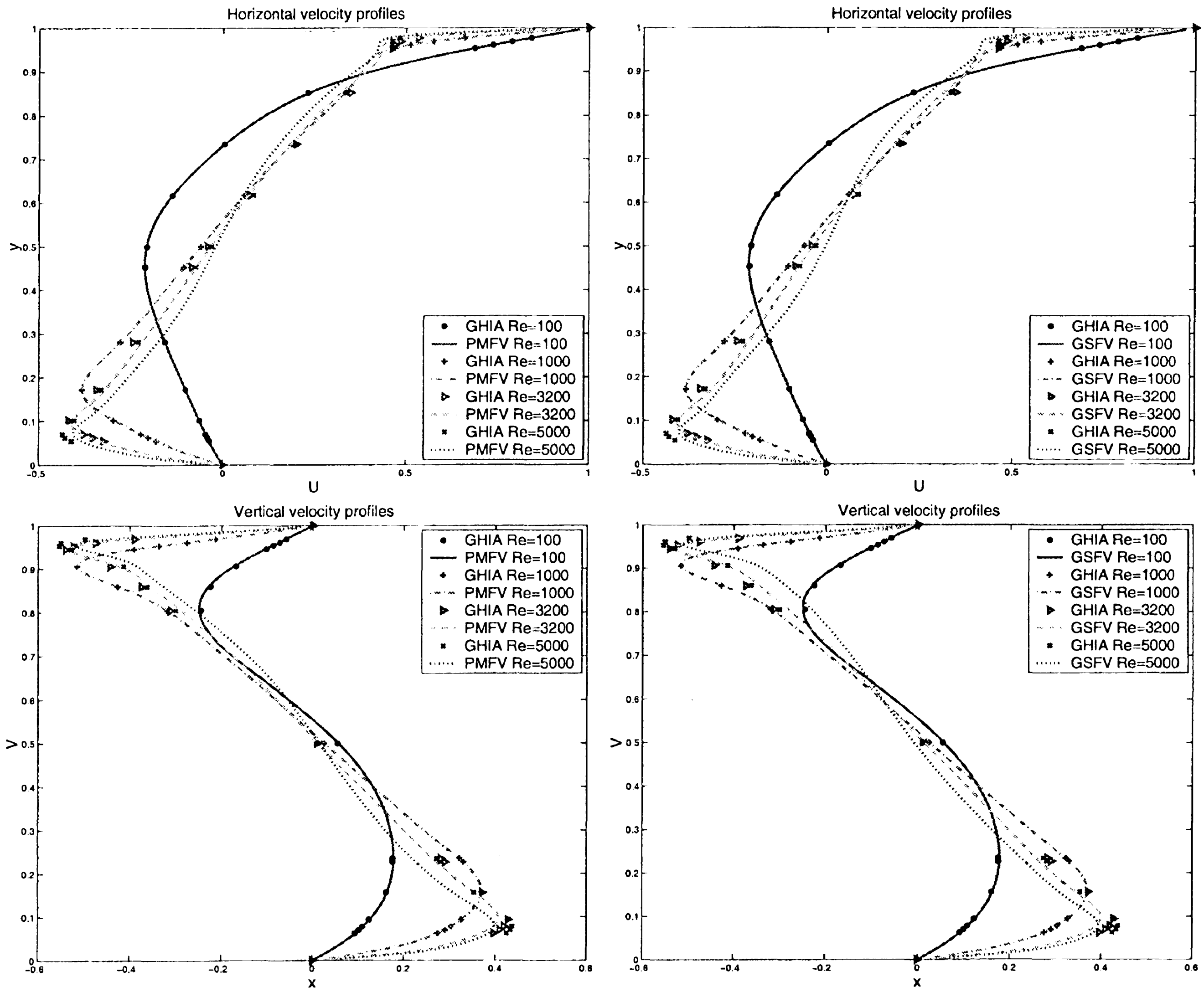

FIG. 4.1 – Comparison of the velocity profiles.

Then, on the next figures, we have drawn the contours of the vorticity, streamfunction and pressure for each scheme. Figure 4.2 to Figure 4.5 show these contours for a Reynolds number respectively equal to 100, 1000, 3200 and 5000. As we can see, the results obtained for the two schemes are similar than these in [GGS82]. However, it is possible to see some small oscillations on the pressure when the scheme "GSFV" is used. They are still present when we decrease the space step but they do not increase when the Reynolds number increase. These oscillations do not disturb the numerical results but it would be desirable to avoid them. In a future work we will see if with some different interpolations of the fluxes could cancel these oscillations. Indeed, we believe that, these oscillations are due to a small decoupling between the pressure and the velocity due to the linear interpolation of the fluxes.

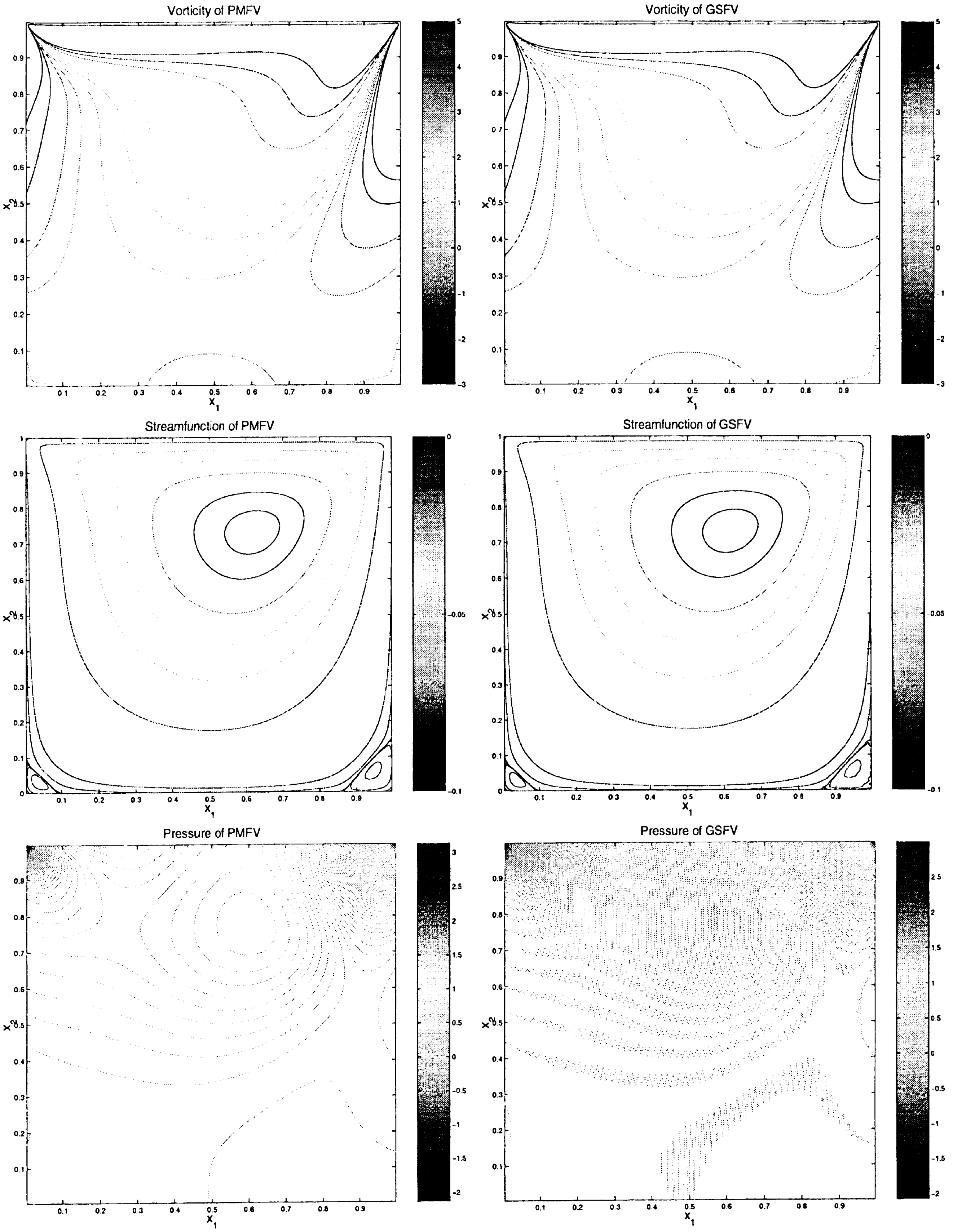

FIG. 4.2 – Vorticity, streamfunction and pressure contours for $Re = 100$.

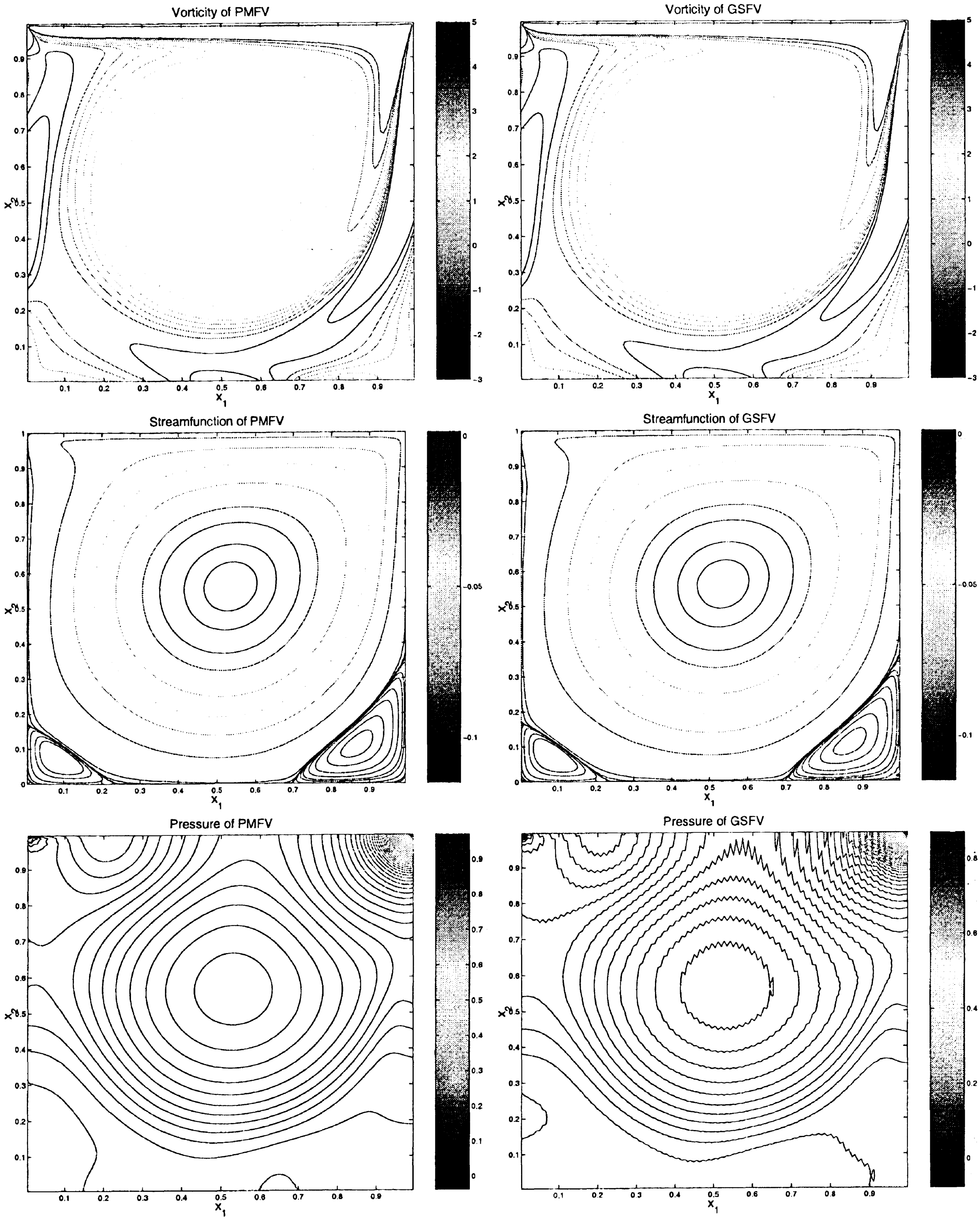

FIG. 4.3 – Vorticity, streamfunction and pressure contours for $Re = 1000$.

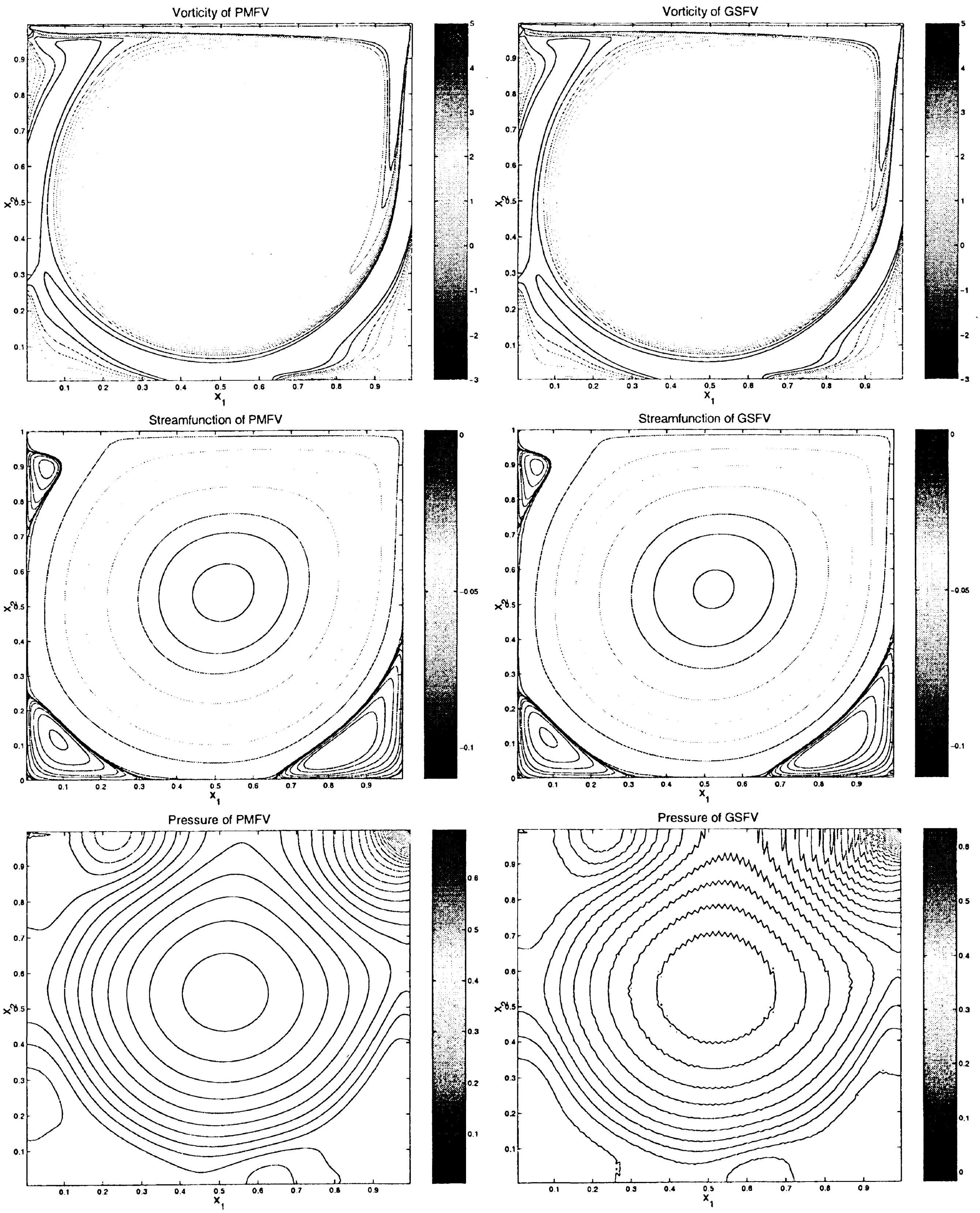

FIG. 4.4 – Vorticity, streamfunction and pressure contours for $Re = 3200$.

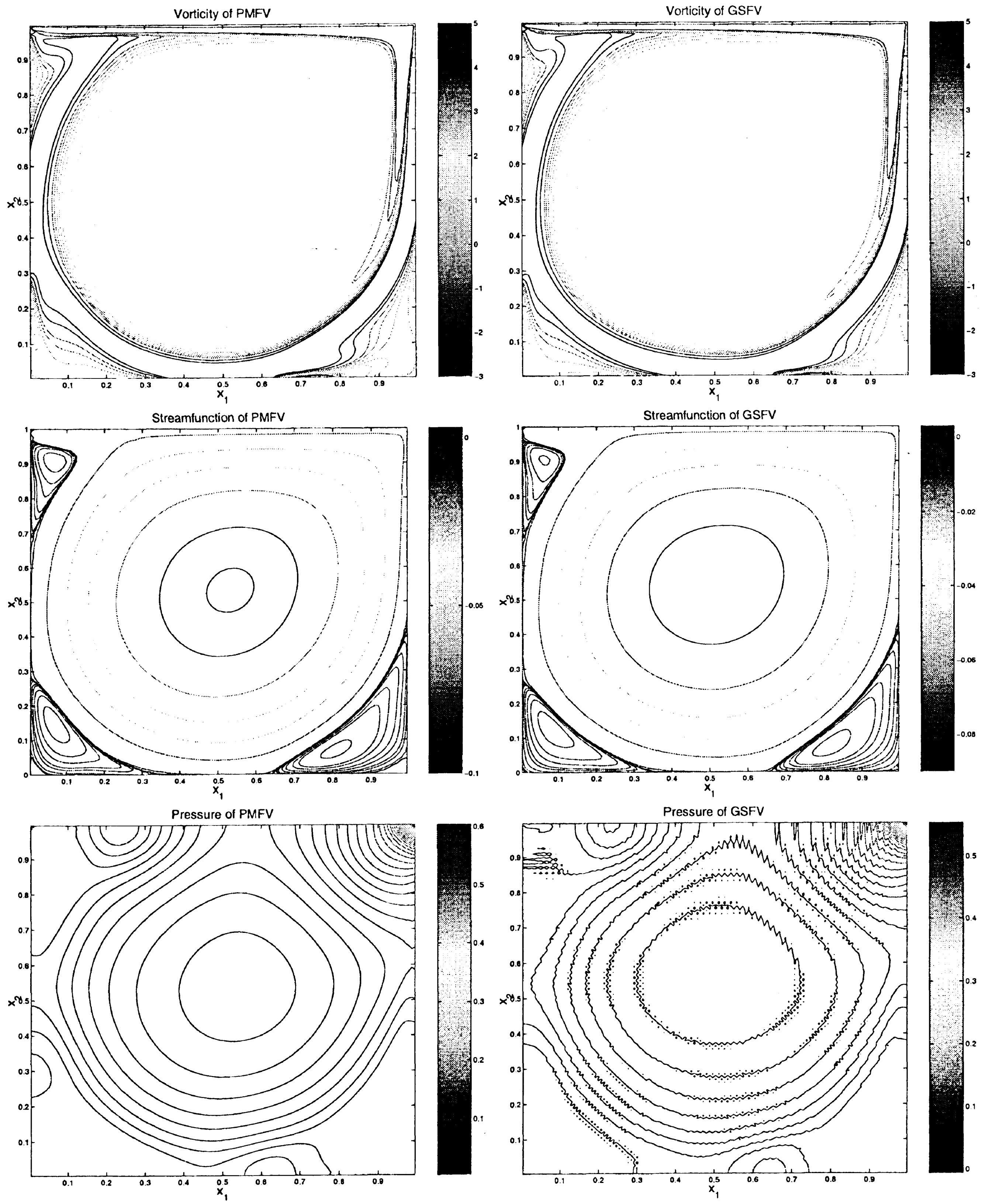

FIG. 4.5 – Vorticity, streamfunction and pressure contours for $Re = 5000$.

Finally, Figure 4.6 shows the divergence of the velocities i.e. how the incompressibility condition is satisfied. The divergence is of the order of 10^{-18} for the scheme "PMFV" and of order 10^{-10} for the scheme "GSFV". We are not surprised by this results as it is well-known that for a projection method, the divergence is equal to the computer accuracy, but this is not the case for a splitting method.

We have also observed that this difference between the divergences of the velocity is independent of the Reynolds number.

Though the divergence of the velocities is not of the order of the computer accuracy for the scheme "GSFV", since it is smaller than the scheme error (i.e. $\mathcal{O}(\Delta x^2 + \Delta y^2)$ and $\mathcal{O}(\Delta t^2)$), it is therefore acceptable.

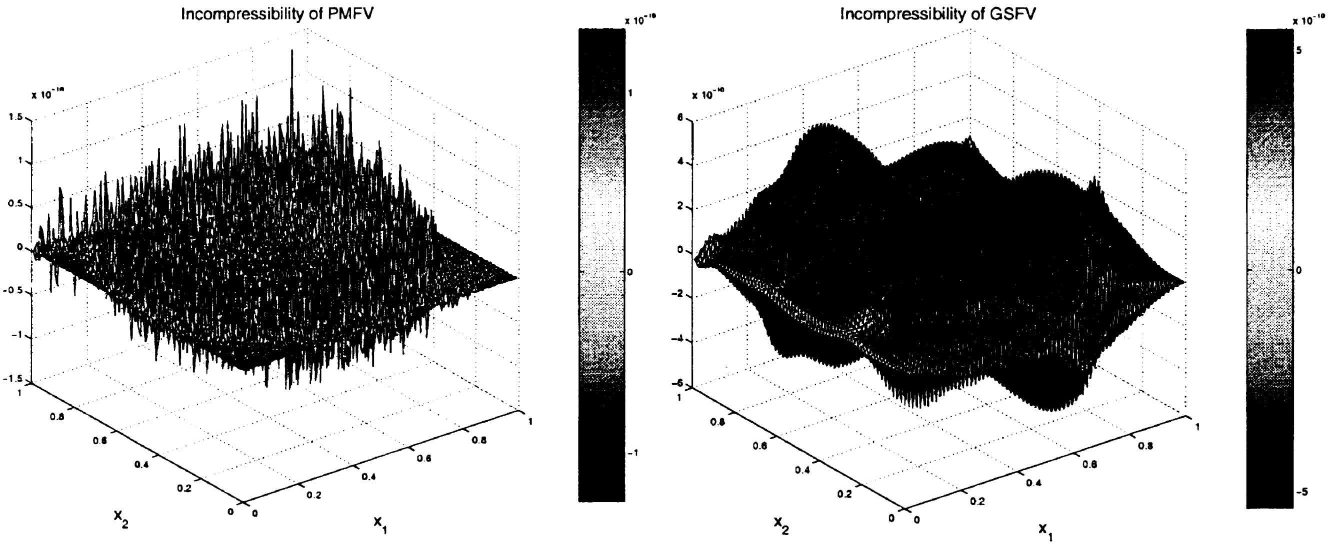

FIG. 4.6 – Incompressibilty for $Re = 5000$.

Concluding remarks

We have presented in this paper two collocated finite volume schemes for incompressible flows. We have seen that the choice of the time discretization is essential and determines the method used to interpolate the fluxes. The numerical results obtained with the scheme using a projection method are correct. The numerical results obtained by the splitting method are also correct but we have noticed some small oscillations on the pressure. We believe that these oscillations are due to a small decoupling between the pressure and the velocities. Consequently, it should be interesting to try more elaborate interpolation methods in order to compute the velocity fluxes. By this way, we hope to obtain a better coupling between the velocities and the pressure for this scheme, and thus, to fully benefit from the accuracy advantages of the splitting method of Guermond and Shen.

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Chapitre 4

Finite volume discretization and multilevel methods in flow problems

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Finite volume discretization and multilevel methods in flow problems

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Abstract : This article is intended as a preliminary report on the implementation of a finite volume multilevel scheme for the discretization of the incompressible Navier-Stokes equations. As is well known the use of staggered grids (e.g. MAC grids, [PKS88]) is a serious impediment for the implementation of multilevel schemes in the context of finite differences. This difficulty is circumvented here by the use of a collocated finite volume discretization ([FT03], [PKS88]), for which the algebra of multilevel methods is much simpler than in the context of MAC type finite differences.
The general ideas and the numerical simulations are presented in this article in the simplified context of a two-dimensional Burgers equations; the two and three dimensional Navier-Stokes equations introducing new difficulties related to the incompressibility condition and the time discretization, will be considered elsewhere (see also [FT03]).

1 Introduction

Multilevel numerical methods are necessary to accurately describe and simulate phenomena involving a large number of scales, such as turbulence phenomena or the equations of nonhomogeneous solids. Thanks to the increase of the power of computers in terms of memory capacity and calculation speed, such methods are becoming accessible and more attention is now devoted to them. See e.g. the *Encyclopedia of Computational Mechanics* [SBH], and, in the mathematical literature, the new journal *Multiscale Modeling and Simulation* [Hou]. This article is aimed at exploring the implementation of multilevel methods for finite volume discretizations of the

Navier-Stokes equations.

Incremental Unknowns (IUs) were introduced in [Tem90] as a mean to approximate inertial manifolds and define multilevel schemes in a finite difference context. Several articles have been devoted to finite difference IUs. For elliptic or simple parabolic problems, see e.g. [Che98], [CT91], [DJT99], [Tem94], [Tem96] and, [CDL98], [CDGT01], [LZ03], in the context of finite element discretizations, pseudo spectral methods and wavelets.
For the incompressible Navier-Stokes equations, the recommended use of a staggered grid renders rather involved the algebra of incremental unknowns although this is not out of reach, see [Gar00]. Indeed, when we want to solve the incompressible Navier-Stokes equations with finite difference methods, we usually use a MAC scheme [HW65] or another similar scheme using staggered grids : the pressure is defined at the center and the velocities at the mid-edges of the cells. This implies that we have to define two kinds of IUs, one for the pressure and one for the velocities, thus significantly increasing the amount of implementation work.
To circumvent this difficulty a different approach is proposed in this article : classical finite differences are replaced by collocated finite volume discretizations which are classically used in the literature (see e.g. [LL94a], [LL94b], [PKS88], [SCV93] and, [RT03], in the context of geophysical fluid dynamics). For collocated finite volume discretizations, the pressure variable as well as the components of the velocity are defined at the center of the cell (called the control volume in the finite volume literature, see e.g. [EGH00]), thus simplifying the implementation of multilevel finite difference methods.
In this preliminary report, we restrict ourselves to the multilevel discretization of the two-dimensional Burgers equations. For the Navier-Stokes equations, additional difficulties occur which will be considered elsewhere, namely the handling of the pressure and the time discretization. We intend to address this question using the projection method [Cho68], [Tem69], [VK86], and the tools considered in [FT03].

Note also that a two-level method is described in this article; however the implementation of a multilevel method by repeating the procedure described here would not raise any major new problem from the theoretical or implementation points of view.

Let Ω be a bounded domain of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider, as a model of nonlinear evolution equation, the two-dimensional Burgers problem. These equations can be interpreted as a simplified formulation of the incompressible Navier-Stokes equations for which the pressure gradient is provided. They constitute a model of viscous flows which incorporates the diffusive viscous terms and the nonlinear convection processes but the pressure gradient terms and the incompressibility condition are removed. We use these equations as a model for the development of algorithms for the Navier-Stokes equations. The problem studied is the following :

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \times [0,T], \tag{1.1}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\mathbf{u}\Big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \tag{1.3}$$

where $\mathbf{u} = \big(u(x_1,x_2,t), v(x_1,x_2,t)\big)$ is the (unknown) velocity, and $\mathbf{f}(x_1,x_2,t)$, $\mathbf{g}(x_1,x_2,t)$, and

$\mathbf{u}_0(x_1, x_2)$ are given; we will also denote by $\mathbf{x} = (x_1, x_2)$ a point in Ω.

The concept of Incremental Unknowns consists in decomposing the unknown $\mathbf{u}$ in several components, that is, for two levels, the large scale component $\mathbf{y}$ and the small scale component $\mathbf{z}$, $\mathbf{u} = \mathbf{y} + \mathbf{z}$. By comparison with a partial differential equation, the introduction of this decomposition yields a number of new schemes by combining time explicit/implicit discretizations for $\mathbf{y}$ and $\mathbf{z}$. In the following and for the sake of simplicity, the nonlinear term will be always computed with time explicit schemes; therefore, we will concentrate our discussion on the choice of explicit vs implicit discretizations for the $\mathbf{y}$ and $\mathbf{z}$ components of the linear term.
The basic observation here is that the CFL stability condition leads to a smaller time step being required on the fine grid than on the coarse grid. Our proposed methods consist in using an explicit scheme on the coarse grid and an implicit method on the fine grid. More precisely we will use a time explicit scheme for the large scales $\mathbf{y}$ while freezing the small scales $\mathbf{z}$, say for I_0 time steps. Then, from time to time, we compute the whole solution $\mathbf{u}$ on the fine grid, for say for I_1 time steps; in this case, we use a time implicit scheme which is always stable (in the nonlinear case), and therefore there is no restriction on the time step for these steps. The general aim is, as much as possible, to obtain an accuracy close to that of the fine grid, with a computational cost close to that of the coarse grid. In fact, and as indicated above, we choose here to always compute the nonlinear terms with an explicit scheme, so that the previous comments only apply to the linear (diffusive) parts. Hence there is here further flexibility in the algorithm, whose study is left to future work. Also as indicated above, we only develop in this article a two level scheme; however the methodology easily extends to a multilevel strategy which would compound the savings in computing time.
From the theoretical point of view, we discuss the stability condition of our scheme in the linear case, leaving discussion for the whole equation for future work.
From the implementation point of view, the issue is to properly choose I_0 and I_1 and the strategy to alternate integrations on the coarse and fine grids. As we said, a thorough discussion is left for future work; see however [DJT99] for various implementations of this approach, and [Tem96] for a simplified description and numerical analysis. As shown in [Pou98], the error produced by freezing the small scales can be controlled; furthermore, it is not necessary to freeze the $\mathbf{z}$ component while advancing the $\mathbf{y}$ component, and some simplified advancement procedures for $\mathbf{z}$ can produce a good improvement [Pou98].

In Section 2, we set the notations used for the meshes and the unknowns. Section 3 is devoted to the definition of the Finite Volume Incremental Unknowns (FVIU). Then, in Section 4, we describe the multilevel algorithm and the numerical schemes. Finally, in Section 5, we present the numerical results.

2 Notations

For simplicity, we consider a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$ which is discretized by two levels of rectangular finite volume meshes as shown in Figure 2.1. The control volumes for the fine mesh $\mathcal{M}_1$ have all the same dimension : $\Delta x_1 \times \Delta x_2$ with $3N_{x_1}\Delta x_1 = L_1$, $3N_{x_2}\Delta x_2 = L_2$, where N_{x_1}, N_{x_2} are given integers ; there are thus $9N_{x_1}N_{x_2}$ volume controls corresponding to the fine mesh. The proposed control volumes for the coarse mesh $\mathcal{M}_0$ have dimension : $3\Delta x_1 \times 3\Delta x_2$. These two meshes are clearly two admissible finite volume meshes in the sense of [EGH00].

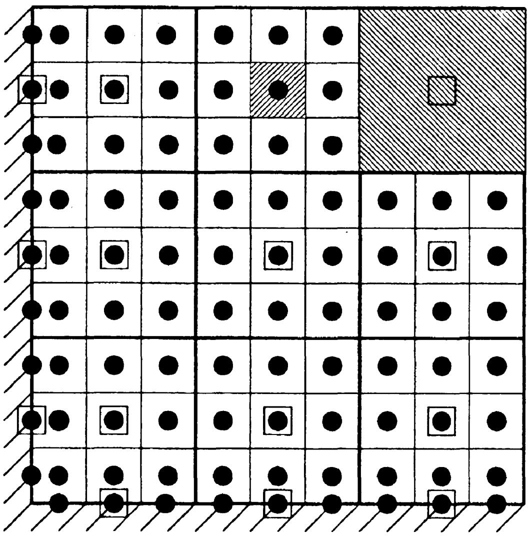

FIG. 2.1 – Space discretization with two meshes : the • are the centers of the control volumes of the fine mesh $\mathcal{M}_1$ and the □ are the centers of the control volumes of the coarse mesh $\mathcal{M}_0$.

Remark 2.1. Traditionally, for example in a finite difference context (e. g. for multigrid methods), the grid coarsening factor is 2. Here, due to the choice of the finite volume construction, this grid coarsening factor becomes equal to 3 in order to make sure that the fine grid is properly and *simply* imbedded in the coarse one.

It is convenient to use the *NSWE* stencil as in Figure 2.2 to name the unknowns ; we use also small letters for the fine mesh and capital letters for the coarse mesh. A control volume of the fine mesh will be denoted by K_m and a control volume of the coarse mesh will be denoted by K_M, see Figure 2.3.

For a given control volume K_m centered at m, which belongs to the fine mesh $\mathcal{M}_1$, we define the velocity unknowns by :

$$\mathbf{u}_m \simeq \frac{1}{\Delta x_1 \Delta x_2}\int_{K_m} \mathbf{u}\; dx_1 dx_2.$$

Moreover, we introduce the following notation for the velocities at the interfaces :

$$\begin{aligned} \mathbf{u}_{ms} &\simeq \frac{1}{\Delta x_1}\int_{\Gamma_{ms}} \mathbf{u}\; dx_1, \quad \mathbf{u}_{me} \simeq \frac{1}{\Delta x_2}\int_{\Gamma_{me}} \mathbf{u}\; dx_2, \\ \mathbf{u}_{mn} &\simeq \frac{1}{\Delta x_1}\int_{\Gamma_{mn}} \mathbf{u}\; dx_1, \quad \mathbf{u}_{mw} \simeq \frac{1}{\Delta x_2}\int_{\Gamma_{mw}} \mathbf{u}\; dx_2. \end{aligned}$$

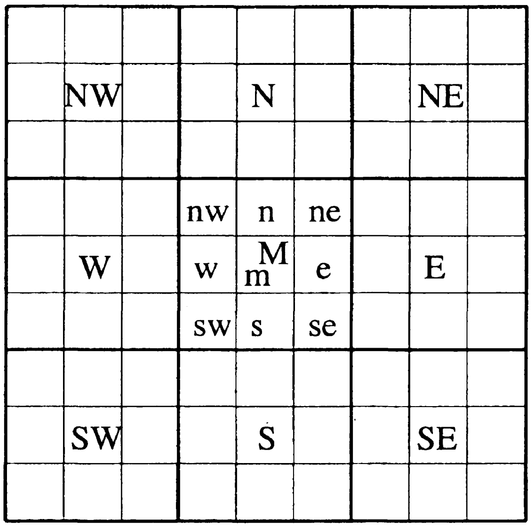

FIG. 2.2 – The $NSWE$ stencil on a coarse and a fine volume.

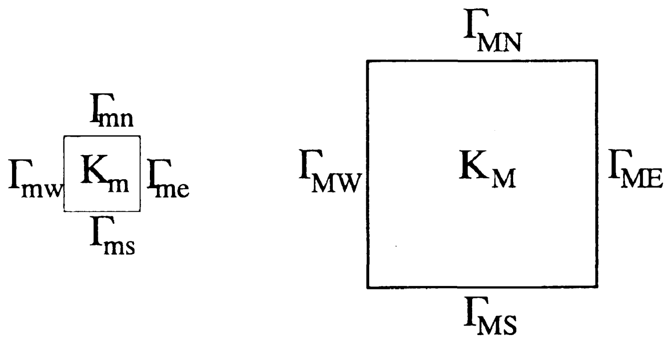

FIG. 2.3 – Edges of a control volume K_m of the fine mesh, and of a control volume K_M of the coarse mesh.

In practice, as explained below, these interface values will be linear interpolations of the values at the centers of the adjacent volumes.

3 Definition of the Finite Volume Incremental Unknowns (FVIUs)

In this section, we define the Finite Volume Incremental Unknowns (FVIUs) for the first component u of the velocity $\mathbf{u} = (u, v)$; the second component v of the velocity (and the pressure for the Navier-Stokes equations) is handled in the same way. The FVIUs are associated with the fine mesh volume as shown in Figure 3.1.

Definition 3.1 (FVIU). *We suppose known the velocity on the fine mesh. For the control volume K_M, we define the large scale component y_M (which corresponds to the velocity on the*

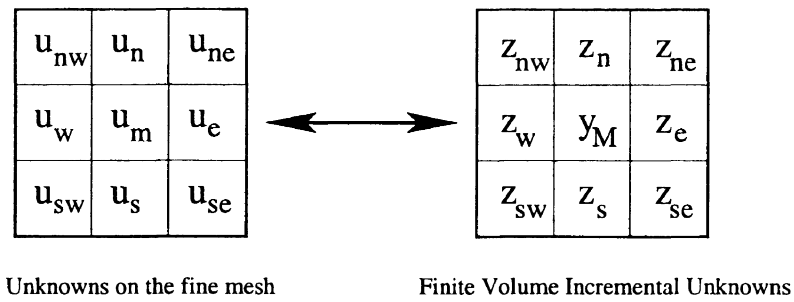

FIG. 3.1 – Finite Volume Incremental Unknowns.

coarse mesh) and the small scale components z_s, z_e, z_n, z_w, z_{sw}, z_{se}, z_{nw}, z_{ne}, *as follows :*

$$\begin{aligned}
&y_M = u_m,\\
&z_s = u_s - \frac{1}{3}\left(y_S + 2y_M\right), \; z_e = u_e - \frac{1}{3}\left(y_E + 2y_M\right),\\
&z_n = u_n - \frac{1}{3}\left(y_N + 2y_M\right), \; z_w = u_w - \frac{1}{3}\left(y_W + 2y_M\right),\\
&z_{sw} = u_{sw} - \frac{1}{3}\left(y_S + y_M + y_W\right), \; z_{se} = u_{se} - \frac{1}{3}\left(y_S + y_M + y_E\right),\\
&z_{nw} = u_{nw} - \frac{1}{3}\left(y_N + y_M + y_W\right), \; z_{ne} = u_{ne} - \frac{1}{3}\left(y_N + y_M + y_E\right).
\end{aligned}$$

Remember that $y_S = u_S$, $y_W = u_W$, $y_E = u_E$ and $y_N = u_N$ are the velocities on the adjacent cells of the coarse mesh as shown on Figure 2.2.
The incremental unknowns z's are defined as a linear combination of the u's at the neighboring cells, and, by Taylor's formula (see below), the z's are of order Δx_1^2 $(\Delta x_1 \sim \Delta x_2)$.

Remark 3.1. Near the boundary of Ω, we have to slightly adapt the definition of the FVIUs. For example, in the configuration of Figure 3.2, we define z_s, z_w, z_{sw}, z_{se}, z_{nw}, as follows :

$$\begin{aligned}
z_s &= u_s - \frac{1}{3}\left(2u_{\bar S} + y_M\right), \quad z_w = u_w - \frac{1}{3}\left(2u_{\bar W} + y_M\right),\\
z_{sw} &= u_{sw} - \frac{1}{3}\left(2u_{\bar S} - y_M + 2u_{\bar W}\right), \quad z_{se} = u_{se} - \frac{1}{3}\left(2u_{\bar S} + y_E\right),\\
z_{nw} &= u_{nw} - \frac{1}{3}\left(y_N + 2u_{\bar W}\right),
\end{aligned}$$

where $u_{\bar S}$, $u_{\bar W}$ are given by the Dirichlet boundary conditions.

Remark 3.2. Due to Definition 3.1 and Remark 3.1, given a solution u computed on the fine mesh, we can deduce the small components z and the large component y. Reciprocally, it is clear that when we know the component y which is defined on the coarse mesh and the components z, we can compute u on the fine mesh.

After these definitions, we want to verify that the small components z are smaller than the large component y. We write $h = (\Delta x_1, \Delta x_2)$ and we show the following lemma :

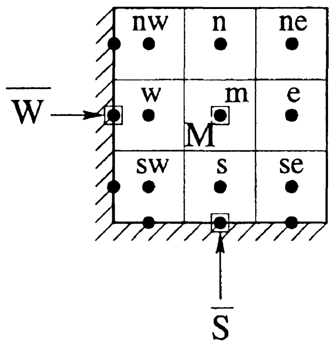

FIG. 3.2 – Boundary Finite Volume Incremental Unknowns.

Lemma 3.1. *All the small components z_s, z_e, z_n, z_w, z_{sw}, z_{se}, z_{nw} and z_{ne} are of order h^2.*

Proof :
By Taylor's formula it is easy to check that the small components z_s, z_e, z_n, z_w, z_{sw}, z_{se}, z_{nw} and z_{ne} are of order h^2. For z_s we have :

$$\begin{aligned} z_s &= u_s - \frac{1}{3}\left(y_S + 2y_M\right), \\ &= u_s - \frac{1}{3}\left(u_S + 2u_M\right), \\ &= u_s - \frac{1}{3}\left(u_s - 2\Delta x_2 \partial_y u_s + \mathcal{O}(\Delta x_2^2) + 2\left(u_s + \Delta x_2 \partial_{x_2} u_s + \mathcal{O}(\Delta x_2^2)\right)\right), \\ &= \mathcal{O}(\Delta x_2^2). \end{aligned}$$

Following the same procedure we easily see that :

$$z_n = \mathcal{O}(\Delta x_2^2),\ z_w = \mathcal{O}(\Delta x_1^2),\ z_e = \mathcal{O}(\Delta x_1^2).$$

Then, for z_{sw} we have :

$$\begin{aligned} z_{sw} &= u_{sw} - \frac{1}{3}\left(y_S + y_M + y_W\right), \\ &= u_{sw} - \frac{1}{3}\left(u_S + u_M + u_W\right), \\ &= u_{sw} - \frac{1}{3}\big(u_{sw} + \Delta x_1 \partial_{x_1} u_{sw} - 2\Delta x_2 \partial_{x_2} u_{sw} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2) \\ &+ u_{sw} + \Delta x_1 \partial_{x_1} u_{sw} + \Delta x_2 \partial_{x_2} u_{sw} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2) \\ &+ u_{sw} - 2\Delta x_1 \partial_{x_1} u_{sw} + \Delta x_2 \partial_{x_2} u_{sw} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)\big), \\ &= \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2). \end{aligned}$$

Similarly, we have :

$$\begin{aligned} z_{se} &= \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2), \\ z_{ne} &= \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2), \\ z_{nw} &= \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2). \end{aligned}$$

Finally, for a boundary cell as, for example, in the configuration of Figure 3.2, we have for z_s :

$$\begin{aligned}
z_s &= u_s - \frac{1}{3}\left(2u_{\bar{S}} + y_M\right), \\
&= u_s - \frac{1}{3}\left(2u_{\bar{S}} + u_M\right), \\
&= u_s - \frac{1}{3}\left(2\left(u_s - \frac{\Delta x_2}{2}\partial_{x_2}u_s + \mathcal{O}(\Delta x_2^2)\right) + u_s + \Delta x_2\partial_{x_2}u_s + \mathcal{O}(\Delta x_2^2)\right), \\
&= \mathcal{O}(\Delta x_2^2),
\end{aligned}$$

hence also $z_w = \mathcal{O}(\Delta x_1^2)$.

For z_{se}, we obtain similarly

$$\begin{aligned}
z_{se} &= u_{se} - \frac{1}{3}\left(2u_{\bar{S}} + y_E\right), \\
&= u_{se} - \frac{1}{3}\left(2u_{\bar{S}} + u_E\right), \\
&= u_{se} - \frac{1}{3}\left(2\left(u_{se} - \Delta x_1\partial_{x_1}u_{se} - \frac{\Delta x_2}{2}\partial_{x_2}u_{se} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)\right)\right. \\
&+ \left. u_{se} + \Delta x_2\partial_{x_2}u_{se} + 2\Delta x_1\partial_{x_1}u_{se} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)\right), \\
&= \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2),
\end{aligned}$$

and again $z_{nw} = \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)$.

For z_{sw}, we obtain as well :

$$\begin{aligned}
z_{sw} &= u_{sw} - \frac{1}{3}\left(2u_{\bar{S}} - y_M + 2u_{\bar{W}}\right), \\
&= u_{sw} - \frac{1}{3}\left(2u_{\bar{S}} - u_M + 2u_{\bar{W}}\right), \\
&= u_{sw} - \frac{1}{3}\left(2\left(u_{sw} + \Delta x_1\partial_{x_1}u_{sw} - \frac{\Delta x_2}{2}\partial_{x_2}u_{sw} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)\right)\right. \\
&+ 2\left(u_{sw} + \Delta x_2\partial_{x_2}u_{sw} - \frac{\Delta x_1}{2}\partial_{x_1}u_{sw} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)\right) \\
&- \left.\left(u_{sw} + \Delta x_1\partial_{x_1}u_{sw} + \Delta x_2\partial_{x_2}u_{sw} + \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2)\right)\right), \\
&= \mathcal{O}(\Delta x_1^2) + \mathcal{O}(\Delta x_2^2).
\end{aligned}$$

In conclusion, all the small components z are of order h^2. □

4 Multilevel algorithm and schemes

Using the results of the previous section and Definition 3.1 for each component of the velocity $\mathbf{u} = (u, v)$, we are now able to decompose $\mathbf{u}$ into its large component $\mathbf{y} = (y_u, y_v)$ and its small components $\mathbf{z} = (z_u, z_v)$.

Moreover, for the time discretization of problem (1.1)-(1.2), let $T > 0$ be fixed, and let the time step be $\Delta t = T/N_t$ where N_t is an integer. For $k = 0, ..., N_t$, we define $\mathbf{u}^k$ as the approximate value of $\mathbf{u}$ at time $t_k = k\Delta t$. In the following we will use a time finite difference scheme to approximate the time derivative, i.e. we will not use a space-time finite volume discretization but rather a space finite volume discretization with a time finite difference scheme. This choice allows us to use an implicit time discretization scheme for the diffusive terms.

4.1 Computation of $\mathbf{u}^{k+1}$ on the fine mesh

To compute $\mathbf{u}^{k+1}$, we choose a second order time discretization which consists of an implicit Crank-Nicolson scheme for the diffusive terms and an Adams-Bashforth scheme for the nonlinear convective terms. Consequently, we do not have a diffusive stability condition. The scheme is the following :

$$\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} = \frac{\nu}{2}\Delta\Big(\frac{\mathbf{u}^{k+1} + \mathbf{u}^k}{2}\Big) - \frac{3}{2}(\mathbf{u}^k.\nabla)\mathbf{u}^k + \frac{1}{2}(\mathbf{u}^{k-1}.\nabla)\mathbf{u}^{k-1} + \frac{\mathbf{f}^{k+1} + \mathbf{f}^k}{2}.$$

Then, we integrate the above equation over each control volume K_m of the fine mesh. For each of the terms we obtain the following :

- For the time derivative :

$$\int_{K_m} \frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} dx_1 dx_2 \simeq \Delta x_1 \Delta x_2 \frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\Delta t}.$$

- For the diffusive term :

$$\begin{aligned}
\int_{K_m} \Delta\mathbf{u}\, dx_1 dx_2 &= \int_{\partial K_m} \frac{\partial \mathbf{u}}{\partial n}\, d\Gamma, \\
&= \int_{\Gamma_{me}} \nabla\mathbf{u}.\mathbf{n}\, dx_2 + \int_{\Gamma_{mw}} \nabla\mathbf{u}.\mathbf{n}\, dx_2 + \int_{\Gamma_{mn}} \nabla\mathbf{u}.\mathbf{n}\, dx_1 \\
&+ \int_{\Gamma_{ms}} \nabla\mathbf{u}.\mathbf{n}\, dx_1, \\
&\simeq \Delta x_2 \frac{\mathbf{u}_e - \mathbf{u}_m}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w - \mathbf{u}_m}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n - \mathbf{u}_m}{\Delta x_2} + \Delta x_1 \frac{\mathbf{u}_s - \mathbf{u}_m}{\Delta x_2}.
\end{aligned}$$

– For the nonlinear term :

$$\int_{K_m} (\mathbf{u}.\nabla)\mathbf{u}\, dx_1dx_2 = \begin{pmatrix} \int_{K_m} \left[u\partial_{x_1}u + v\partial_{x_2}u\right] dx_1dx_2 \\ \int_{K_m} \left[u\partial_{x_1}v + v\partial_{x_2}v\right] dx_1dx_2 \end{pmatrix},$$

$$\simeq \begin{pmatrix} u_m \int_{K_m} \partial_{x_1}u dx_1dx_2 + v_m \int_{K_m} \partial_{x_2}u dx_1dx_2 \\ u_m \int_{K_m} \partial_{x_1}v dx_1dx_2 + v_m \int_{K_m} \partial_{x_2}v dx_1dx_2 \end{pmatrix},$$

$$\simeq \begin{pmatrix} u_m \int_{\partial K_m} u n_{x_1} d\Gamma + v_m \int_{\partial K_m} u n_{x_2} d\Gamma \\ u_m \int_{\partial K_m} v n_{x_1} d\Gamma + v_m \int_{\partial K_m} v n_{x_2} d\Gamma \end{pmatrix},$$

$$\simeq \begin{pmatrix} \Delta x_2 u_m \left(u_{me} - u_{mw}\right) + \Delta x_1 v_m \left(u_{mn} - u_{ms}\right) \\ \Delta x_2 u_m \left(v_{me} - v_{mw}\right) + \Delta x_1 v_m \left(v_{mn} - v_{ms}\right) \end{pmatrix}.$$

– For the $\mathbf{f}$ term :

$$\int_{K_m} \frac{\mathbf{f}^{k+1} + \mathbf{f}^k}{2}\, dx_1dx_2 \simeq \Delta x_1 \Delta x_2 \frac{\mathbf{f}_m^{k+1} + \mathbf{f}_m^k}{2}.$$

This leads us to propose the following finite volume scheme that we will use to compute the velocities $\mathbf{u}^{k+1}$ on the fine mesh :

$$\begin{aligned} &\Delta x_1 \Delta x_2 \frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\Delta t} - \frac{\nu}{2}\Bigg(\Delta x_2 \frac{\mathbf{u}_e^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_2} \\ &+ \Delta x_1 \frac{\mathbf{u}_s^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_2} + \Delta x_2 \frac{\mathbf{u}_e^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^k - \mathbf{u}_m^k}{\Delta x_2} + \Delta x_1 \frac{\mathbf{u}_s^k - \mathbf{u}_m^k}{\Delta x_2}\Bigg) \\ &= \Delta x_1 \Delta x_2 \frac{\mathbf{f}_m^{k+1} + \mathbf{f}_m^k}{2} - \frac{3}{2} \begin{pmatrix} \Delta x_2 u_m^k \left(u_{me}^k - u_{mw}^k\right) + \Delta x_1 v_m^k \left(u_{mn}^k - u_{ms}^k\right) \\ \Delta x_2 u_m^k \left(v_{me}^k - v_{mw}^k\right) + \Delta x_1 v_m^k \left(v_{mn}^k - v_{ms}^k\right) \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} \Delta x_2 u_m^{k-1} \left(u_{me}^{k-1} - u_{mw}^{k-1}\right) + \Delta x_1 v_m^{k-1} \left(u_{mn}^{k-1} - u_{ms}^{k-1}\right) \\ \Delta x_2 u_m^{k-1} \left(v_{me}^{k-1} - v_{mw}^{k-1}\right) + \Delta x_1 v_m^{k-1} \left(v_{mn}^{k-1} - v_{ms}^{k-1}\right) \end{pmatrix}, \end{aligned} \tag{4.1}$$

with a linear interpolation for the interface velocities :

$$\mathbf{u}_{me} = \frac{\mathbf{u}_e + \mathbf{u}_m}{2},\ \mathbf{u}_{mw} = \frac{\mathbf{u}_w + \mathbf{u}_m}{2},$$
$$\mathbf{u}_{mn} = \frac{\mathbf{u}_n + \mathbf{u}_m}{2},\ \mathbf{u}_{ms} = \frac{\mathbf{u}_s + \mathbf{u}_m}{2}.$$

When K_m is a boundary cell, we have to adapt the previous scheme in accordance with the Dirichlet boundary conditions. For the nonlinear term, we do not need to interpolate the interface velocities, we use :

$$\mathbf{u}_{mw} = \mathbf{g}_{\bar{w}}, \quad \mathbf{u}_{me} = \mathbf{g}_{\bar{e}},$$
$$\mathbf{u}_{ms} = \mathbf{g}_{\bar{s}}, \quad \mathbf{u}_{mn} = \mathbf{g}_{\bar{n}}.$$

For the diffusive term, we use :

$$\Delta x_2 \frac{\mathbf{u}_e - \mathbf{u}_m}{\Delta x_1} = \Delta x_2 \frac{2\mathbf{g}_{\bar{e}} - 2\mathbf{u}_m}{\Delta x_1}, \quad \Delta x_2 \frac{\mathbf{u}_w - \mathbf{u}_m}{\Delta x_1} = \Delta x_2 \frac{2\mathbf{g}_{\bar{w}} - 2\mathbf{u}_m}{\Delta x_1},$$
$$\Delta x_1 \frac{\mathbf{u}_n - \mathbf{u}_m}{\Delta x_2} = \Delta x_1 \frac{2\mathbf{g}_{\bar{n}} - 2\mathbf{u}_m}{\Delta x_2}, \quad \Delta x_1 \frac{\mathbf{u}_s - \mathbf{u}_m}{\Delta x_2} = \Delta x_1 \frac{2\mathbf{g}_{\bar{s}} - 2\mathbf{u}_m}{\Delta x_2}.$$

4.2 Computation of $\mathbf{y}^{k+1}$ on the coarse mesh

Now, we describe the scheme for the large scale component $\mathbf{y}^{k+1}$. This scheme will be deduced from a scheme written for $\mathbf{u}^{k+1}$ on the fine mesh. Here, we choose an Euler explicit time discretization for its simplicity but an implicit time discretization as in Section 4.1 can also be envisioned :

$$\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} = \nu \Delta \mathbf{u}^k - (\mathbf{u}^k.\nabla)\mathbf{u}^k + \mathbf{f}^k.$$

We will see that the stability condition of the multilevel scheme derived from this explicit scheme is less restrictive than the stability condition obtained if we use the same explicit scheme to compute all the components of $\mathbf{u}$ on the fine mesh.

Indeed, as in the previous subsection, we integrate the above equation over each control volume K_m of the fine mesh, and we obtain :

$$\begin{aligned} \Delta x_1 \Delta x_2 \frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\Delta t} &= \nu \left(\Delta x_2 \frac{\mathbf{u}_e^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^k - \mathbf{u}_m^k}{\Delta x_2} + \Delta x_1 \frac{\mathbf{u}_s^k - \mathbf{u}_m^k}{\Delta x_2} \right) \\ &- \begin{pmatrix} \Delta x_2 u_m^k \left(u_{me}^k - u_{mw}^k \right) + \Delta x_1 v_m^k \left(u_{mn}^k - u_{ms}^k \right) \\ \Delta x_2 u_m^k \left(v_{me}^k - v_{mw}^k \right) + \Delta x_1 v_m^k \left(v_{mn}^k - v_{ms}^k \right) \end{pmatrix} + \Delta x_1 \Delta x_2 \mathbf{f}_m^k, \end{aligned} \tag{4.2}$$

Using a linear interpolation for the interface velocities,

$$\mathbf{u}_{me} = \frac{\mathbf{u}_e + \mathbf{u}_m}{2}, \quad \mathbf{u}_{mw} = \frac{\mathbf{u}_w + \mathbf{u}_m}{2},$$
$$\mathbf{u}_{mn} = \frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \quad \mathbf{u}_{ms} = \frac{\mathbf{u}_s + \mathbf{u}_m}{2},$$

the scheme (4.2) can be rewritten as follows :

$$\begin{aligned}
\Delta x_1 \Delta x_2 \frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\Delta t} &= \nu \left(\Delta x_2 \frac{\mathbf{u}_e^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^k - \mathbf{u}_m^k}{\Delta x_2} + \Delta x_1 \frac{\mathbf{u}_s^k - \mathbf{u}_m^k}{\Delta x_2} \right) \\
&- \begin{pmatrix} \frac{\Delta x_2}{2} u_m^k \left(u_e^k - u_w^k \right) + \frac{\Delta x_1}{2} v_m^k \left(u_n^k - u_s^k \right) \\ \frac{\Delta x_2}{2} u_m^k \left(v_e^k - v_w^k \right) + \frac{\Delta x_1}{2} v_m^k \left(v_n^k - v_s^k \right) \end{pmatrix} + \Delta x_1 \Delta x_2 \mathbf{f}_m^k. \qquad (4.3)
\end{aligned}$$

In a second step, we replace in (4.3) the unknown $\mathbf{u}$ by the FVIU $\mathbf{y}$ and $\mathbf{z}$ as defined in Definition 3.1. For each term, we obtain :

– For the time derivative term :

$$\Delta x_1 \Delta x_2 \frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\Delta t} = \Delta x_1 \Delta x_2 \frac{\mathbf{y}_M^{k+1} - \mathbf{y}_M^k}{\Delta t}.$$

– For the diffusive term :

$$\begin{aligned}
&\nu \left(\Delta x_2 \frac{\mathbf{u}_e^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^k - \mathbf{u}_m^k}{\Delta x_2} + \Delta x_1 \frac{\mathbf{u}_s^k - \mathbf{u}_m^k}{\Delta x_2} \right) \\
= \;& \nu \left(\frac{\Delta x_2}{\Delta x_1} \left(\mathbf{z}_e^k + \mathbf{z}_w^k \right) + \frac{\Delta x_1}{\Delta x_2} \left(\mathbf{z}_n^k + \mathbf{z}_s^k \right) \right) \\
+ \;& \frac{\nu}{3} \left(\Delta x_2 \frac{\mathbf{y}_E^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{y}_W^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{y}_N^k - \mathbf{y}_M^k}{\Delta x_2} + \Delta x_1 \frac{\mathbf{y}_S^k - \mathbf{y}_M^k}{\Delta x_2} \right).
\end{aligned}$$

– For the nonlinear term :

$$\begin{aligned}
&- \begin{pmatrix} \frac{\Delta x_2}{2} u_m^k \left(u_e^k - u_w^k \right) + \frac{\Delta x_1}{2} v_m^k \left(u_n^k - u_s^k \right) \\ \frac{\Delta x_2}{2} u_m^k \left(v_e^k - v_w^k \right) + \frac{\Delta x_1}{2} v_m^k \left(v_n^k - v_s^k \right) \end{pmatrix} \\
= \;& - \begin{pmatrix} \frac{\Delta x_2}{2} y_{u\,M}^k \left(z_{u\,e}^k - z_{u\,w}^k \right) + \frac{\Delta x_1}{2} y_{v\,M}^k \left(z_{u\,n}^k - z_{u\,s}^k \right) \\ \frac{\Delta x_2}{2} y_{u\,M}^k \left(z_{v\,e}^k - z_{v\,w}^k \right) + \frac{\Delta x_1}{2} y_{v\,M}^k \left(z_{v\,n}^k - z_{v\,s}^k \right) \end{pmatrix} \\
- \;& \begin{pmatrix} \frac{\Delta x_2}{6} y_{u\,M}^k \left(y_{u\,E}^k - y_{u\,W}^k \right) + \frac{\Delta x_1}{6} y_{v\,M}^k \left(y_{u\,N}^k - u_{u\,S}^k \right) \\ \frac{\Delta x_2}{6} y_{u\,M}^k \left(y_{v\,E}^k - y_{v\,W}^k \right) + \frac{\Delta x_1}{6} y_{v\,M}^k \left(y_{v\,N}^k - y_{v\,S}^k \right) \end{pmatrix}.
\end{aligned}$$

– For the $\mathbf{f}$ term :

$$\Delta x_1 \Delta x_2 \mathbf{f}_m^k = \Delta x_1 \Delta x_2 \mathbf{f}_m^k.$$

The multilevel strategy used here (see also Section 4.3) consists in fixing the small components $\mathbf{z}$ during the time steps computing on the coarse mesh. So, for a given solution $\mathbf{u}^{k_0}$ on the fine mesh, we consider that the small components $\mathbf{z}$ are fixed during the next I_0 time-steps to their value $\mathbf{z}^{k_0}$ deduced from $\mathbf{u}^{k_0}$, and we compute the solution $\mathbf{y}^{k+1}$ on the coarse mesh using the following scheme :

$$\begin{aligned}
\Delta x_1 \Delta x_2 \frac{\mathbf{y}_M^{k+1} - \mathbf{y}_M^k}{\Delta t} &= \frac{\nu}{3}\Bigg(\Delta x_2 \frac{\mathbf{y}_E^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{y}_W^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{y}_N^k - \mathbf{y}_M^k}{\Delta x_2} \\
+ & \quad \Delta x_1 \frac{\mathbf{y}_S^k - \mathbf{y}_M^k}{\Delta x_2} \Bigg) + \nu \left(\frac{\Delta x_2}{\Delta x_1}\left(\mathbf{z}_e^{k_0} + \mathbf{z}_w^{k_0}\right) + \frac{\Delta x_1}{\Delta x_2}\left(\mathbf{z}_n^{k_0} + \mathbf{z}_s^{k_0}\right)\right) + \Delta x_1 \Delta x_2 \mathbf{f}_m^k \\
- & \begin{pmatrix} \frac{\Delta x_2}{2} y_{u\,M}^k \left(z_{u\,e}^{k_0} - z_{u\,w}^{k_0}\right) + \frac{\Delta x_1}{2} y_{v\,M}^k \left(z_{u\,n}^{k_0} - z_{u\,s}^{k_0}\right) \\ \frac{\Delta x_2}{2} y_{u\,M}^k \left(z_{v\,e}^{k_0} - z_{v\,w}^{k_0}\right) + \frac{\Delta x_1}{2} y_{v\,M}^k \left(z_{v\,n}^{k_0} - z_{v\,s}^{k_0}\right) \end{pmatrix} \\
- & \begin{pmatrix} \frac{\Delta x_2}{6} y_{u\,M}^k \left(y_{u\,E}^k - y_{u\,W}^k\right) + \frac{\Delta x_1}{6} y_{v\,M}^k \left(y_{u\,N}^k - u_{u\,S}^k\right) \\ \frac{\Delta x_2}{6} y_{u\,M}^k \left(y_{v\,E}^k - y_{v\,W}^k\right) + \frac{\Delta x_1}{6} y_{v\,M}^k \left(y_{v\,N}^k - y_{v\,S}^k\right) \end{pmatrix} \quad \text{for } k = k_0, ..., k_0 + I_0 - 1
\end{aligned} \tag{4.4}$$

The above scheme is proposed away from the boundary of Ω. For a boundary cell, the scheme for $\mathbf{y}^{k+1}$ is obtained in a similar way using the modified definition of the FVIUs (see Remark 3.1). For example, in the situation of Figure 3.2 the scheme for $\mathbf{y}_M^{k+1}$ becomes :

$$\begin{aligned}
\Delta x_1 \Delta x_2 \frac{\mathbf{y}_M^{k+1} - \mathbf{y}_M^k}{\Delta t} &= \frac{\nu}{3}\Bigg(\Delta x_2 \frac{\mathbf{y}_E^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_2 \frac{2\mathbf{y}_{\bar{W}}^k - 2\mathbf{y}_M^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{y}_N^k - \mathbf{y}_M^k}{\Delta x_2} \\
+ & \quad \Delta x_1 \frac{2\mathbf{y}_{\bar{S}}^k - 2\mathbf{y}_M^k}{\Delta x_2} \Bigg) + \nu \left(\frac{\Delta x_2}{\Delta x_1}\left(\mathbf{z}_e^{k_0} + \mathbf{z}_w^{k_0}\right) + \frac{\Delta x_1}{\Delta x_2}\left(\mathbf{z}_n^{k_0} + \mathbf{z}_s^{k_0}\right)\right) + \Delta x_1 \Delta x_2 \mathbf{f}_m^k \\
- & \begin{pmatrix} \frac{\Delta x_2}{2} y_{u\,M}^k \left(z_{u\,e}^{k_0} - z_{u\,w}^{k_0}\right) + \frac{\Delta x_1}{2} y_{v\,M}^k \left(z_{u\,n}^{k_0} - z_{u\,s}^{k_0}\right) \\ \frac{\Delta x_2}{2} y_{u\,M}^k \left(z_{v\,e}^{k_0} - z_{v\,w}^{k_0}\right) + \frac{\Delta x_1}{2} y_{v\,M}^k \left(z_{v\,n}^{k_0} - z_{v\,s}^{k_0}\right) \end{pmatrix} \\
- & \begin{pmatrix} \frac{\Delta x_2}{6} y_{u\,M}^k \left(y_{u\,E}^k + y_{u\,M}^k - 2y_{u\,\bar{W}}^k\right) + \frac{\Delta x_1}{6} y_{v\,M}^k \left(y_{u\,N}^k + y_{u\,M}^k - 2y_{u\,\bar{S}}^k\right) \\ \frac{\Delta x_2}{6} y_{u\,M}^k \left(y_{v\,E}^k + y_{v\,M}^k - 2y_{v\,\bar{W}}^k\right) + \frac{\Delta x_1}{6} y_{v\,M}^k \left(y_{v\,N}^k + y_{v\,M}^k - 2y_{v\,\bar{S}}^k\right) \end{pmatrix}.
\end{aligned} \tag{4.5}$$

4.3 Multilevel algorithm

The simple multilevel algorithm used in this article is described for two levels.
At level 1, we work on the fine mesh $\mathcal{M}_1$ and compute the velocities $\mathbf{u}^{k+1}$ with the scheme (4.1).
At level 0, we work on the coarse mesh $\mathcal{M}_0$ and compute the velocities $\mathbf{y}^{k+1}$ with the scheme (4.4), $\mathbf{z}$ being frozen at its value $\mathbf{z}^{k_0}$ ($k \geq k_0$).

The multilevel algorithm is the following : from $t = 0$ and until the final time, we alternate I_1 time steps at level 1 and I_0 time steps at level 0. Figure 4.1 shows an example of multilevel algorithm where $I_1 = 5$ and $I_0 = 10$.

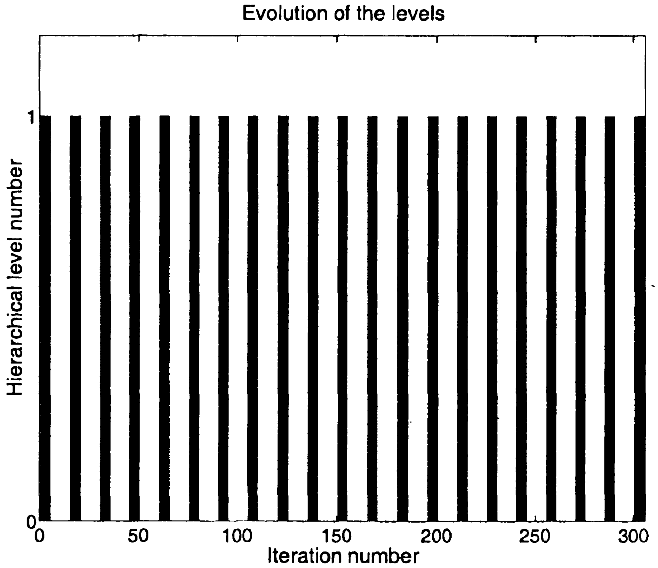

FIG. 4.1 – Time evolution of the levels in the multilevel algorithm ($I_1 = 5$ and $I_0 = 10$).

4.4 Linear stability analysis of the multilevel algorithm

The stability analysis in the nonlinear case and when the scheme for y^{k+1} derives from a Crank-Nicholson and Adams-Bashforth scheme, will be considered in a future work. Here, we only study the numerical stability in the linear case of the multilevel algorithm defined in Section 4.3. For the sake of simplicity, we conduct the stability analysis in the space periodic case, using the von Neumann spectral stability analysis.

Lemma 4.1 (Stability condition). *In the linear case, the following is a sufficient stability condition for the multilevel algorithm :*

$$\frac{\nu \Delta t}{3}\left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2}\right) \leq \frac{1}{2}. \tag{4.6}$$

Proof :

To perform the stability analysis, we consider one cycle of the multilevel algorithm i.e. I_1 time steps on the fine grid followed by I_0 time steps on the coarse grid.

The first step of this proof consists in the stability analysis of the scheme (4.1) which is used during the first I_1 time steps in order to compute $\mathbf{u}^{k+1}$ on the fine mesh. Removing the nonlinear term and the source term in (4.1), this yields :

$$\Delta x_1 \Delta x_2 \frac{\mathbf{u}_m^{k+1} - \mathbf{u}_m^k}{\Delta t} - \frac{\nu}{2}\left(\Delta x_2 \frac{\mathbf{u}_e^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_2} \right.$$
$$\left. + \Delta x_1 \frac{\mathbf{u}_s^{k+1} - \mathbf{u}_m^{k+1}}{\Delta x_2} + \Delta x_2 \frac{\mathbf{u}_e^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{u}_w^k - \mathbf{u}_m^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{u}_n^k - \mathbf{u}_m^k}{\Delta x_2} + \Delta x_1 \frac{\mathbf{u}_s^k - \mathbf{u}_m^k}{\Delta x_2} \right) = 0. \tag{4.7}$$

Here, the stability analysis for the schemes considered is made using the von Neumann spectral stability analysis. This analysis is local and it ignores the boundary conditions. Namely we look for the amplification factor χ_1 for the scheme (4.7). For that purpose, we replace the $\mathbf{u}$ terms in (4.7) by :

$$\begin{array}{lll} \mathbf{u}_m^{k+1} = \chi_1, & \mathbf{u}_m^k = 1, & \mathbf{u}_s^k = \exp(-i\vartheta_2), \\ \mathbf{u}_e^k = \exp(i\vartheta_1), & \mathbf{u}_n^k = \exp(i\vartheta_2), & \mathbf{u}_w^k = \exp(-i\vartheta_1), \end{array}$$

we easily obtain that :

$$\chi_1(\vartheta_1, \vartheta_2) = -\frac{2\nu \frac{\Delta x_2}{\Delta x_1} \sin^2 \frac{\vartheta_1}{2} + 2\nu \frac{\Delta x_1}{\Delta x_2} \sin^2 \frac{\vartheta_2}{2}}{\frac{\Delta x_1 \Delta x_2}{\Delta t} + 2\nu \frac{\Delta x_2}{\Delta x_1} \sin^2 \frac{\vartheta_1}{2} + 2\nu \frac{\Delta x_1}{\Delta x_2} \sin^2 \frac{\vartheta_2}{2}}. \tag{4.8}$$

This amplification factor means that $\mathbf{u}_m^{k+1} = \chi_1(\vartheta_1, \vartheta_2)\, \mathbf{u}_m^k$. Now, let us see what this amplification factor means for the large scales $\mathbf{y}$ and for the small scales $\mathbf{z}$. For a control volume K_m of the fine grid which is also located at the center of a control volume K_M of the coarse mesh, we note that, using Definition 3.1 :

$$\begin{aligned} \mathbf{y}_M^{k+1} &= \mathbf{u}_m^{k+1} = \chi_1\, \mathbf{u}_m^k = \chi_1\, \mathbf{y}_M^k, \\ \mathbf{z}_s^{k+1} &= \mathbf{u}_s^{k+1} - \frac{1}{3}\left(\mathbf{y}_S^{k+1} + 2\mathbf{y}_M^{k+1}\right), \\ &= \chi_1 \left(\mathbf{u}_s^k - \frac{1}{3}\left(\mathbf{y}_S^k + 2\mathbf{y}_M^k\right)\right), \\ &= \chi_1\, \mathbf{z}_s^k, \end{aligned}$$

and following the same way, we have $\mathbf{z}_{sw}^{k+1} = \chi_1\, \mathbf{z}_{sw}^k$, $\mathbf{z}_{se}^{k+1} = \chi_1\, \mathbf{z}_{se}^k$, $\mathbf{z}_w^{k+1} = \chi_1\, \mathbf{z}_w^k$, $\mathbf{z}_e^{k+1} = \chi_1\, \mathbf{z}_e^k$, $\mathbf{z}_{nw}^{k+1} = \chi_1\, \mathbf{z}_{nw}^k$, $\mathbf{z}_n^{k+1} = \chi_1\, \mathbf{z}_n^k$, $\mathbf{z}_{ne}^{k+1} = \chi_1\, \mathbf{z}_{ne}^k$.
Hence, we have shown that, during this first I_1 time steps, the amplification factor for $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{z}$ is the same, that χ_1. We easily notice that $\left|\chi_1(\vartheta_1, \vartheta_2)\right| < 1$ for all $(\vartheta_1, \vartheta_2)$ in $(-\pi, \pi)$.

Then, in the second step of this proof, we study the stability of the scheme (4.9) used during the following I_0 time steps in order to compute the large scales $\mathbf{y}^{k+1}$ on the coarse grid. For the stability analysis of this part of the multilevel algorithm, we remove in (4.9) the nonlinear term, the source term and the term in $\mathbf{z}$ since the small scales $\mathbf{z}$ are frozen during these I_0 time steps; we find :

$$\Delta x_1 \Delta x_2 \frac{\mathbf{y}_M^{k+1} - \mathbf{y}_M^k}{\Delta t} - \frac{\nu}{3}\left(\Delta x_2 \frac{\mathbf{y}_E^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_2 \frac{\mathbf{y}_W^k - \mathbf{y}_M^k}{\Delta x_1} + \Delta x_1 \frac{\mathbf{y}_N^k - \mathbf{y}_M^k}{\Delta x_2} \right.$$
$$\left. + \quad \Delta x_1 \frac{\mathbf{y}_S^k - \mathbf{y}_M^k}{\Delta x_2} \right) = 0. \tag{4.9}$$

Replacing in (4.9) the $\mathbf{y}$ terms by :

$$\begin{array}{lll} \mathbf{y}_M^{k+1} = \chi_0, & \mathbf{y}_M^k = 1, & \mathbf{y}_S^k = \exp(-3i\vartheta_2), \\ \mathbf{y}_E^k = \exp(3i\vartheta_1), & \mathbf{y}_N^k = \exp(3i\vartheta_2), & \mathbf{y}_W^k = \exp(-3i\vartheta_1), \end{array}$$

we obtain the following amplification factor for the large component $\mathbf{y}$:

$$\begin{aligned} \chi_0(\vartheta_1, \vartheta_2) &= 1 - \frac{\nu \Delta t}{3 \Delta x_1 \Delta x_2} \left(\frac{\Delta x_2}{\Delta x_1} (2 - \exp(3i\vartheta_1) - \exp(-3i\vartheta_1)) \right. \\ &+ \left. \frac{\Delta x_1}{\Delta x_2} (2 - \exp(3i\vartheta_2) - \exp(-3i\vartheta_2)) \right), \end{aligned}$$

$$\chi_0(\vartheta_1, \vartheta_2) = 1 - \frac{4\nu\Delta t}{3\Delta x_1^2} \sin\left(\frac{3\vartheta_1}{2}\right)^2 - \frac{4\nu\Delta t}{3\Delta x_2^2} \sin\left(\frac{3\vartheta_2}{2}\right)^2. \tag{4.10}$$

Hence, during the I_0 time steps spent on the coarse grid, we have $\mathbf{y}^{k+1} = \chi_0 \mathbf{y}^k$, and of course $\mathbf{z}^{k+1} = \mathbf{z}^k$.

Finally, for the whole cycle with $I_0 + I_1$ steps, the stability condition for the small scales $\mathbf{z}$ is $|\chi_1|^{I_1} < 1$ which is always satisfied, and the stability condition for the large scales $\mathbf{y}$ is $|\chi_0^{I_0} \chi_1^{I_1}| < 1$. If we neglect the improvement to the stability condition due to $|\chi_1| < 1$, we obtain the stability condition $|\chi_0| < 1$. Since we want $|\chi_0| < 1$ for all possible values of ϑ_1 and ϑ_2 in $(-\pi, \pi)$, we are lead to condition (4.6).

□

Remark 4.1. Condition (4.6) allows a time step three times larger than if we were to use an explicit scheme for $\mathbf{u}$ (i.e. for both $\mathbf{y}$ and $\mathbf{z}$).
If we take into account the fact that $|\chi_1| < 1$, then instead of (4.6), we obtain :

$$\frac{\nu \Delta t}{3} \left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2} \right) \le \underbrace{\frac{1}{4} + \frac{1}{4} \left(1 + \left[2\nu\Delta t \left(\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2} \right) \right]^{-1} \right)^{I_1/I_0}}_{> \frac{1}{2}}. \tag{4.11}$$

Remark 4.2. The stability analysis above must be reconsidered when the nonlinear terms are reintroduced and when the boundary conditions are not periodic anymore. The conclusions should not be significantly different; see [Tem93], [Tem94] for similar situations.

Remark 4.3. We have to remember also that the proposed algorithm can be easily extended to several levels (more than two); there is then the flexibility offered by the use of time explicit vs implicit schemes at each level, thus possibly increasing the factor 3 by which the size of Δt can be augmented. Note also that even if an implicit scheme is used on the coarse grid(s), there is no (CFL) stability condition on Δt, but there is still a gain of computing time, since the matrices to invert to compute the $\mathbf{y}$'s are smaller than those we need to invert to compute all of $\mathbf{u}$.

5 Numerical results and concluding remarks

5.1 Numerical results

In this section, we present the numerical results obtained by using the multilevel algorithm presented in the previous section.

Let the domain Ω be the square $[0,1] \times [0,1]$. We consider a fine mesh $\mathcal{M}_1$ with a space step $h = 1/180$ ($h = \Delta x_1 = \Delta x_2$ in this section), and the space step of the coarse mesh is $3h = 1/60$. We are interested in the Burgers flow from $T = 0$ to $T = 4$.

We solve the Burgers problem with the initial condition :

$$\mathbf{u}(x_1, x_2, 0) = \begin{pmatrix} \cos(\pi x_1)\cos(\pi x_2) \\ (x_1 + x_2)/4 \end{pmatrix}, \tag{5.1}$$

and with the boundary condition :

$$\mathbf{u}(x_1, x_2, t)\Big|_{\partial\Omega} = \mathbf{u}(x_1, x_2, 0)\Big|_{\partial\Omega} \quad \text{for all } t \in [0, T], \tag{5.2}$$

and with the following source term :

$$\mathbf{f}(x_1, x_2, t) = \begin{pmatrix} \sin(3\pi x_1)\sin(4\pi x_2) + 3\sin(16\pi x_1)\sin(8\pi x_2) \\ \sin(4\pi x_1)\sin(2\pi x_2) + 2\sin(10\pi x_1)\sin(16\pi x_2) \end{pmatrix}. \tag{5.3}$$

The Reynolds number used for all the numerical results presented here is $Re = 100$, the time step is $\Delta t = 10^{-3}$. The stability condition (4.6) is satisfied.

This numerical test has already been studied in [CLT97] and [Pou98]. The time independent source force is imposed in order to obtain a Burgers flow sufficiently perturbed. This forcing is a substitute for the energy transfer phenomena, displayed in the energy spectrum, from the low frequencies towards the high frequencies. These phenomena which are an important feature of the Navier-Stokes flow, are probably absent in the Burgers flow but are parametrized by this forcing.

Since the source force which generates the "turbulent" behavior of the flow is time independent, we can stop the simulation at $T = 4$. After this time, the flow has almost totally lost its "turbulent" behavior.

As we does not know an exact solution of this problem, we compute a reference solution $\mathbf{u}_{ref}$ of this problem which will use in the following to compare different possible multilevel algorithms. To obtain it, we use the time and space second order scheme (4.1) on a reference mesh $\mathcal{M}_{ref}$ with a space step $h_{ref} = 1/540$ and with a time step $\Delta t_{ref} = 10^{-4}$.

In a first time, we want to numerically study the behavior of the small scale components $\mathbf{z}$ and of the large scale components $\mathbf{y}$. Consequently, we compute the solution $\mathbf{u}$ on the fine mesh $\mathcal{M}_1$ in using the scheme (4.1) and we plot the time evolution of the following discrete ℓ_2 norms of the FVIUs :

$$|\mathbf{u}^n|_2 = \left[\sum_m \Delta x_1 \Delta x_2 \left((u_m^n)^2 + (u_m^n)^2\right)\right]^{\frac{1}{2}},$$

$$|\mathbf{y}^n|_2 = \left[\sum_M 9\Delta x_1 \Delta x_2 \left((y_{u_M}^n)^2 + (y_{v_M}^n)^2\right)\right]^{\frac{1}{2}},$$

$$|\mathbf{z}^n|_2 = \left[\sum_M \Delta x_1 \Delta x_2 \left(\sum_{m\in\{sw,s,se,w,e,nw,n,ne\}} (z_{u_m}^n)^2 + (z_{v_m}^n)^2\right)\right]^{\frac{1}{2}},$$

and the time evolution of their variations :

$$|d\mathbf{u}^n|_2 = \left[\sum_m \Delta x_1 \Delta x_2 \left((u_m^n - u_m^{n-1})^2 + (u_m^n - u_m^{n-1})^2\right)\right]^{\frac{1}{2}},$$

$$|d\mathbf{y}^n|_2 = \left[\sum_M 9\Delta x_1 \Delta x_2 \left((y_{u_M}^n - y_{u_M}^{n-1})^2 + (y_{v_M}^n - y_{v_M}^{n-1})^2\right)\right]^{\frac{1}{2}},$$

$$|d\mathbf{z}^n|_2 = \left[\sum_M \Delta x_1 \Delta x_2 \left(\sum_{m\in\{sw,s,se,w,e,nw,n,ne\}} (z_{u_m}^n - z_{u_m}^{n-1})^2 + (z_{v_m}^n - z_{v_m}^{n-1})^2\right)\right]^{\frac{1}{2}},$$

where $\sum_m$ (respectively $\sum_M$) means the sum on all the control volumes of the fine mesh (respectively of the coarse mesh).

From Figure 5.1, we see that the behavior of the FVIUs is very similar to the behavior of the finite differences IUs [Pou98]. As expected, the norm of $\mathbf{y}$ is greater than the norm of $\mathbf{z}$ and the time variation of the large scales $\mathbf{y}$ is similar to the variation of $\mathbf{u}$. Furthermore, we note that the time variation of the small scales $\mathbf{z}$ is always smaller than the time step $\Delta t = 10^{-3}$, from this we can justify our choice to froze this small scales during I_0 time steps in the multilevel algorithm. Of course, more the time variation of the small scales $\mathbf{z}$ is close to Δt, more the error commited in freezing this small scales is important.

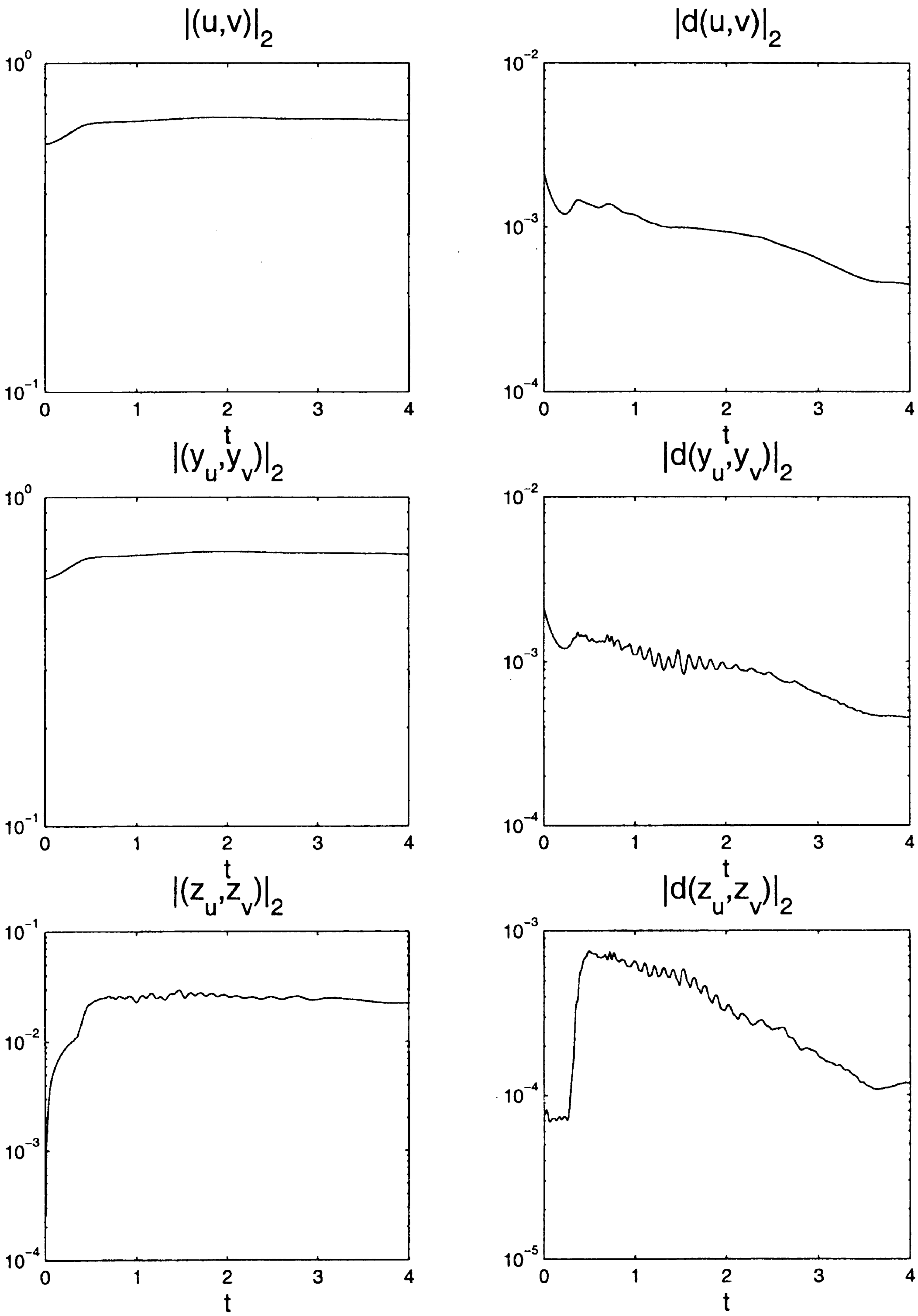

FIG. 5.1 – Time evolution of the discrete L^2 norms of the FVIUs and of their variations.

Then, Figure 5.2 shows the evolution of the velocity field computed with the multilevel method for $t = 1, 2, 3, 4$. We notice a steep gradient for the first component of the solution, into the lower part of the domain.

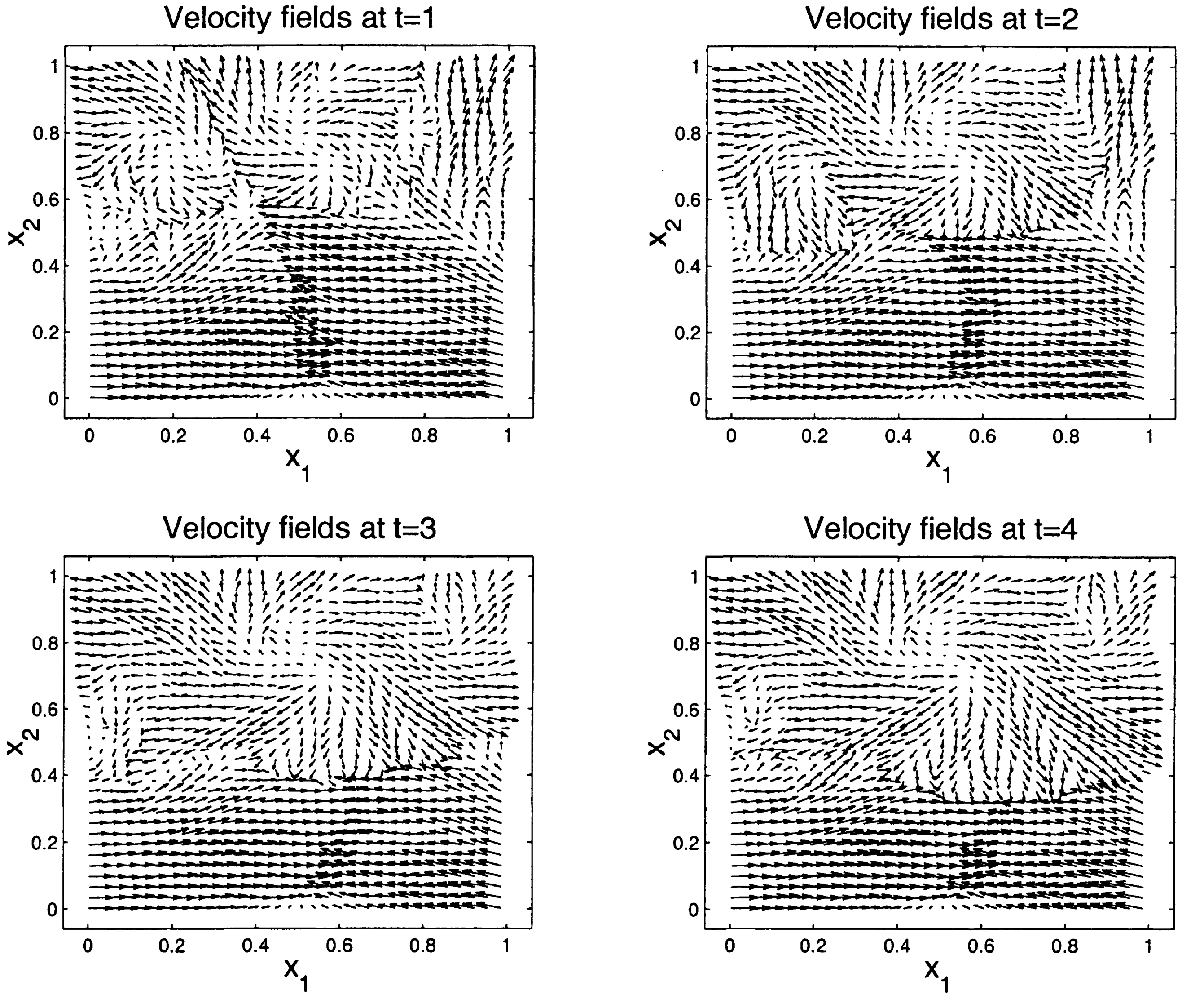

FIG. 5.2 – Velocity fields evolution when the multilevel algorithm is used.

In a third step, we compare the solution $\mathbf{u}$ obtained with different parameters I_0, I_1 in the multilevel schemes and the solution $\mathbf{u}_{ref}$ obtained with a one level calculation on the reference mesh. Figure 5.3 shows the time evolution of the L^2 norm of the difference between the two solutions $|\mathbf{u} - \mathbf{u}_{ref}|_{L^2}$ for each choice of parameters. We notice that this difference is strongly correlated to the "turbulent" behavior of the flow. Indeed, the difference decreases when the "turbulent" behavior of the flow decreases i.e. after $t = 2$. More elaborate multilevel algorithms, correlated with the turbulence, should reduce the difference [DJT99] but we can nevertheless note that the choices of I_0 and I_1 presented here seems reasonable for a first test of the multilevel algorithm. Indeed, their corresponding graphs are under the graph obtained with a Fully Implicit Scheme on the Coarse Grid (FISCG) and they are above the graph obtained with a Fully Implicit Scheme on the Fine Grid (FISFG).

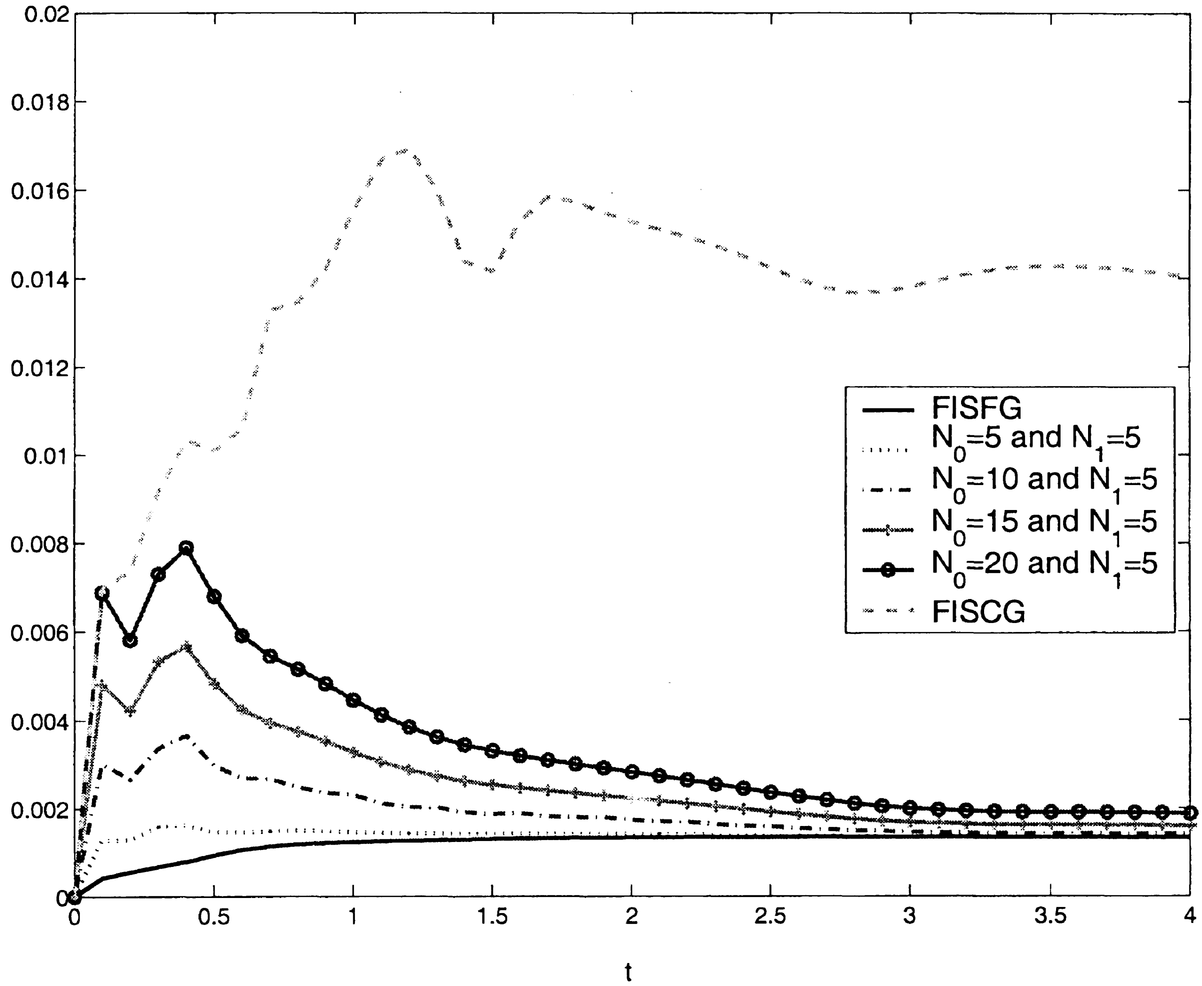

FIG. 5.3 - Time evolution of $|\mathbf{u} - \mathbf{u}_{ref}|_{L^2}$ with different choices of the parameters I_0 and I_1. The extreme cases correspond to the Fully Implicit Scheme on the Coarse Grid (FISCG) and the Fine Grid (FISFG).

More about the choice of the parameters I_0 and I_1. We have chosen 6 points in the domain : $P_1 = (0.320, 0.394)$, $P_2 = (0.605, 0.375)$, $P_3 = (0.929, 0.745)$, $P_4 = (0.488, 0.492)$, $P_5 = (0.746, 0.488)$ and $P_6 = (0.683, 0.242)$, the same as in [DJT99]. Figure 5.4 and Figure 5.5 show the time evolution of the velocity at these points. Figure 5.6 and Figure 5.7 show the time evolution of the difference between the velocities $\mathbf{u}$ and $\mathbf{u}_{ref}$ at these points. We also note here that the choices tested in this article for the parameters I_0 and I_1 are suitable for a first validation of our multilevel algorithm.

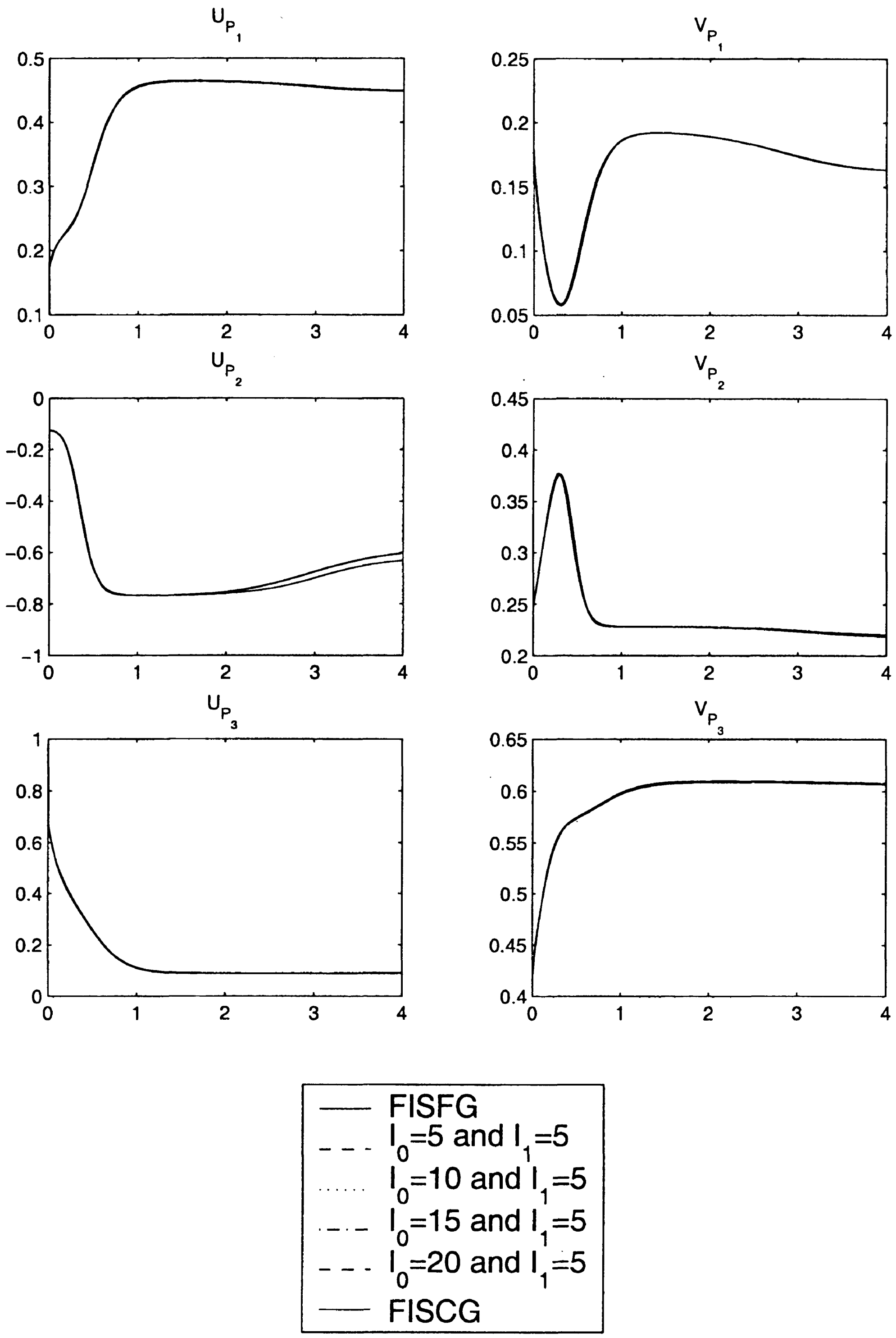

FIG. 5.4 – Time evolution of the velocity at the points P_1, P_2 and P_3 for different choices of the parameters I_0 and I_1. The extreme cases correspond to the Fully Implicit Scheme on the Coarse Grid (FISCG) and the Fine Grid (FISFG).

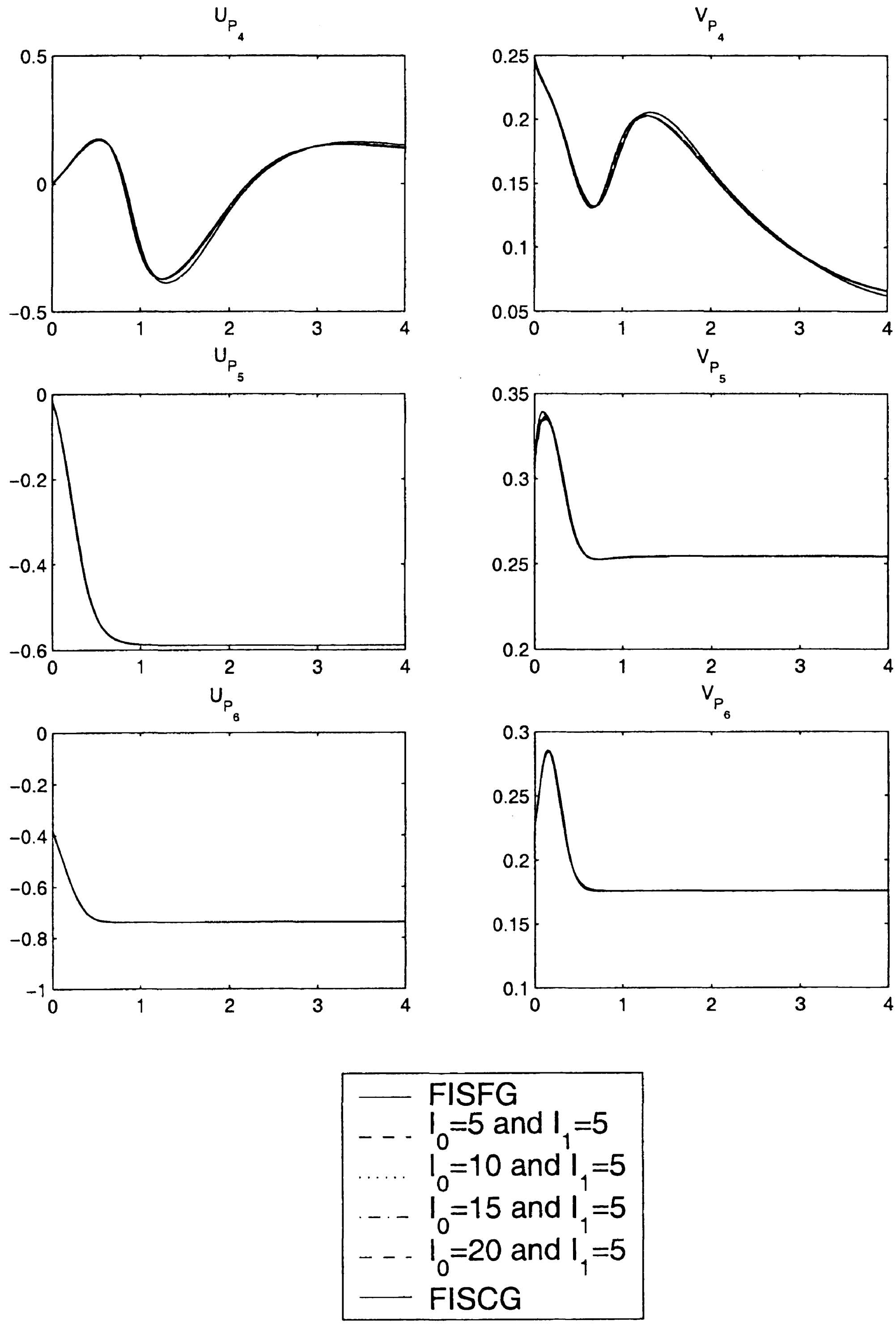

FIG. 5.5 – Time evolution of the velocity at the points P_4, P_5 and P_6 for different choices of the parameters I_0 and I_1. The extreme cases correspond to the Fully Implicit Scheme on the Coarse Grid (FISCG) and the Fine Grid (FISFG).

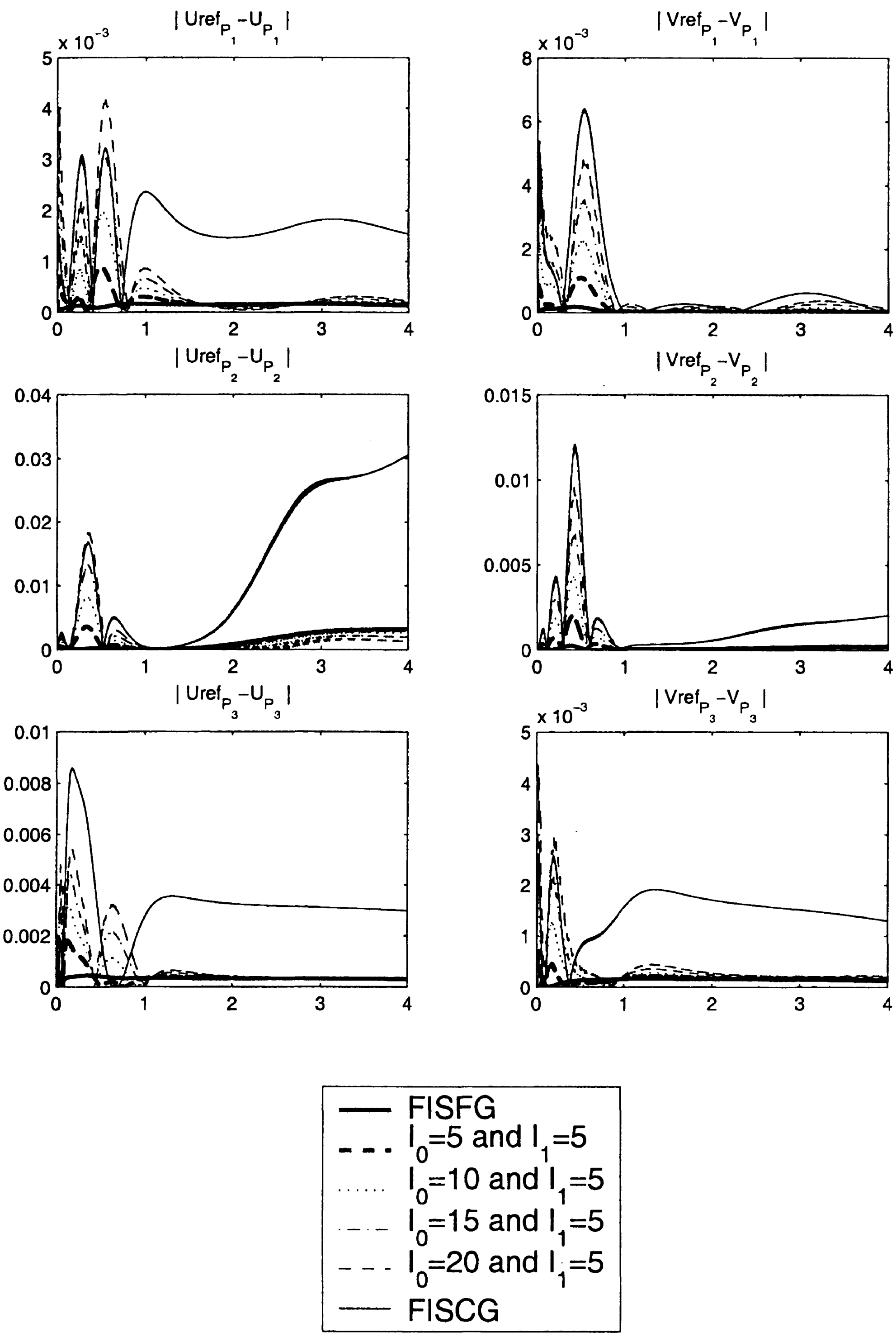

FIG. 5.6 – Time evolution of $|\mathbf{u} - \mathbf{u}_{ref}|$ at the points P_1, P_2 and P_3 for different choices of the parameters I_0 and I_1. The extreme cases correspond to the Fully Implicit Scheme on the Coarse Grid (FISCG) and the Fine Grid (FISFG).

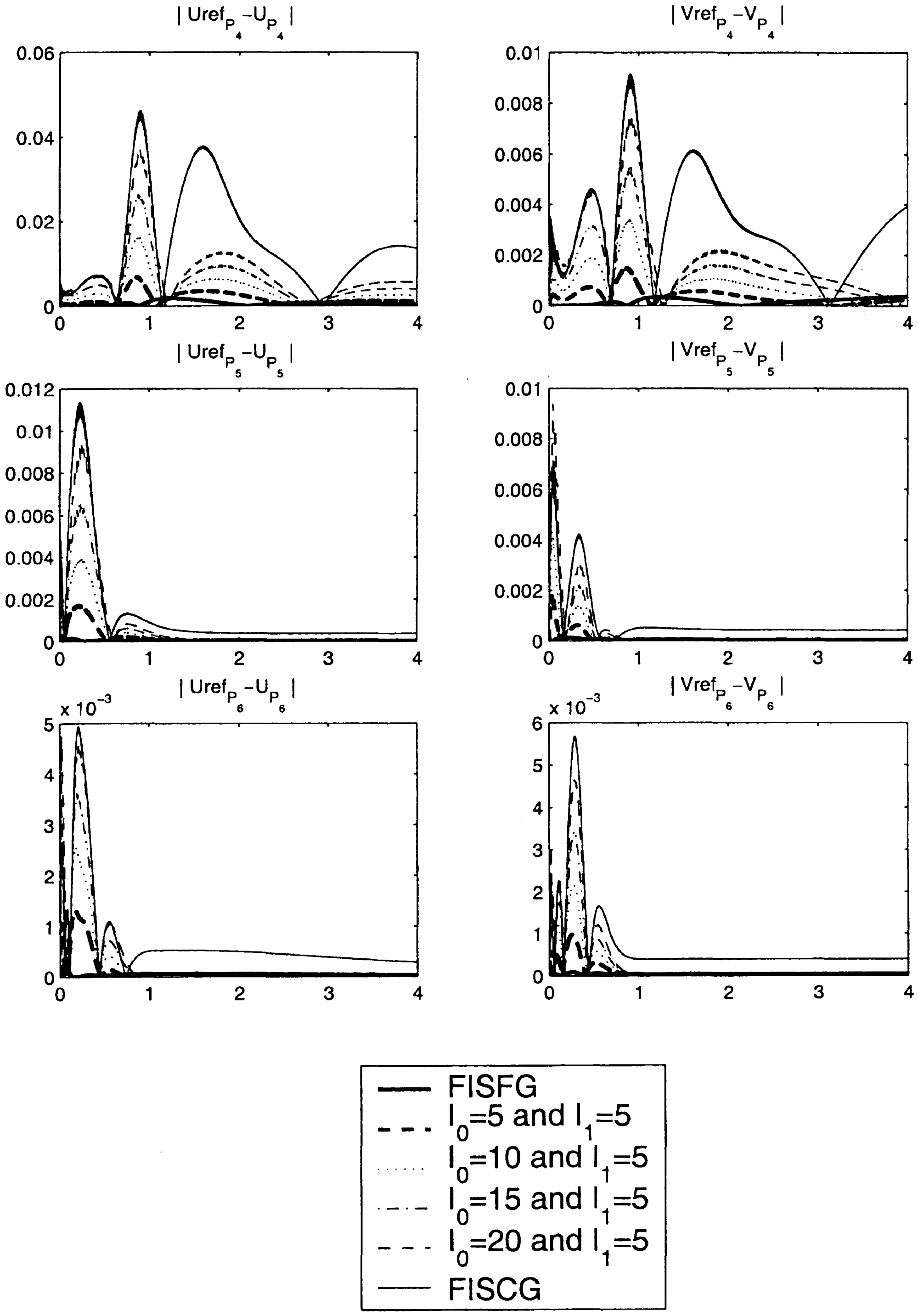

FIG. 5.7 – Time evolution of $|\mathbf{u} - \mathbf{u}_{ref}|$ at the points P_4, P_5 and P_6 for different choices of the parameters I_0 and I_1. The extreme cases correspond to the Fully Implicit Scheme on the Coarse Grid (FISCG) and the Fine Grid (FISFG).

Finally, for a reasonable choice $I_0 = 10$, $I_1 = 5$, we represent on Figures 5.8 and 5.9 the differences $u - u_{ref}$ and $v - v_{ref}$. These figures show the localization in space of the error on the computed solution. We notice that the error is more important where the gradient is steep. With a more elaborate multilevel algorithm which use a test in order to decide if the next calculation must be made at level 0 or at level 1 (see in a different context [DJT99], [CLT97]), it is possible to greatly improve these results in controlling these differences.

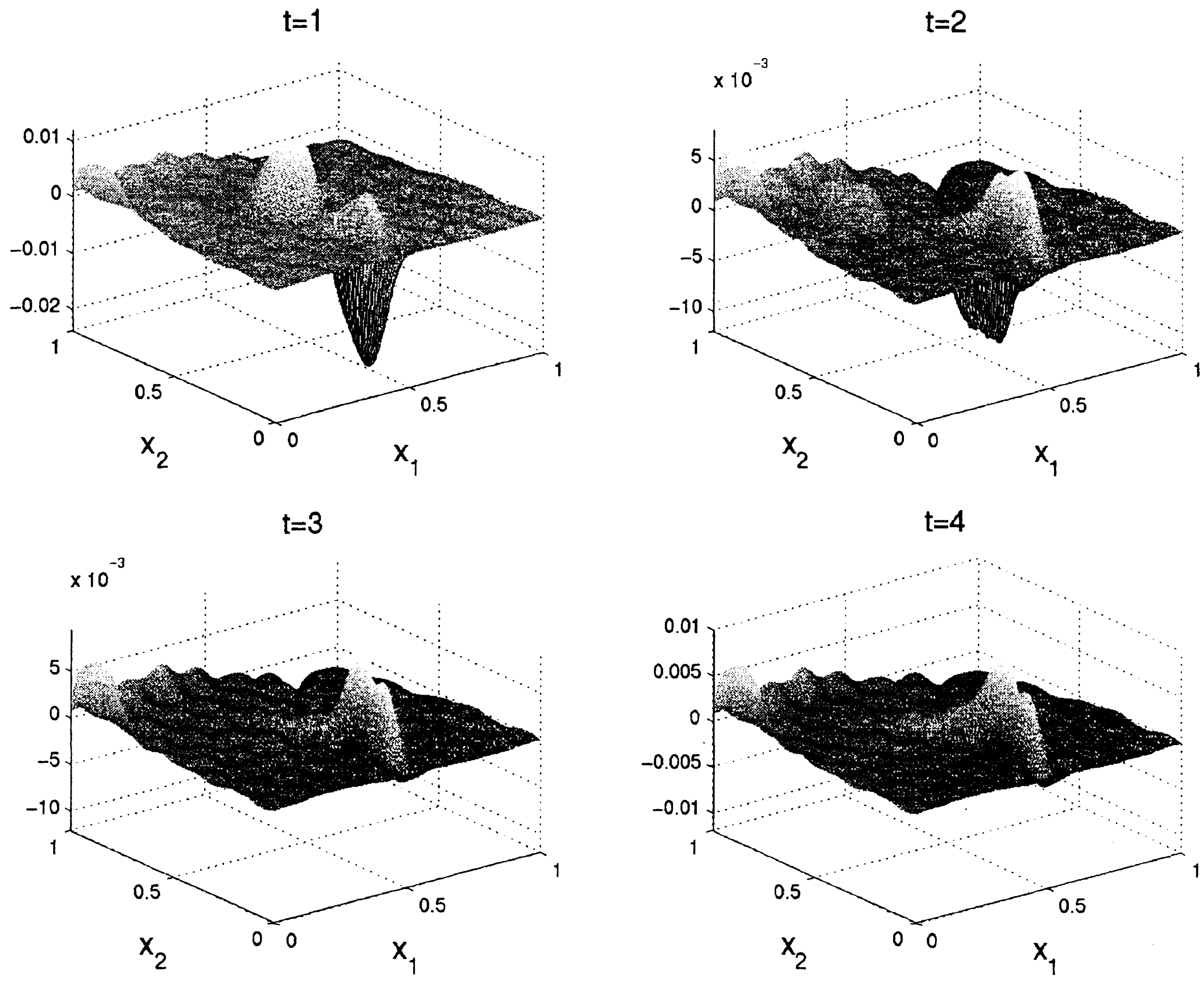

FIG. 5.8 – Difference between the horizontal velocities : $u - u_{ref}$ ($I_0 = 10$ and $I_1 = 5$).

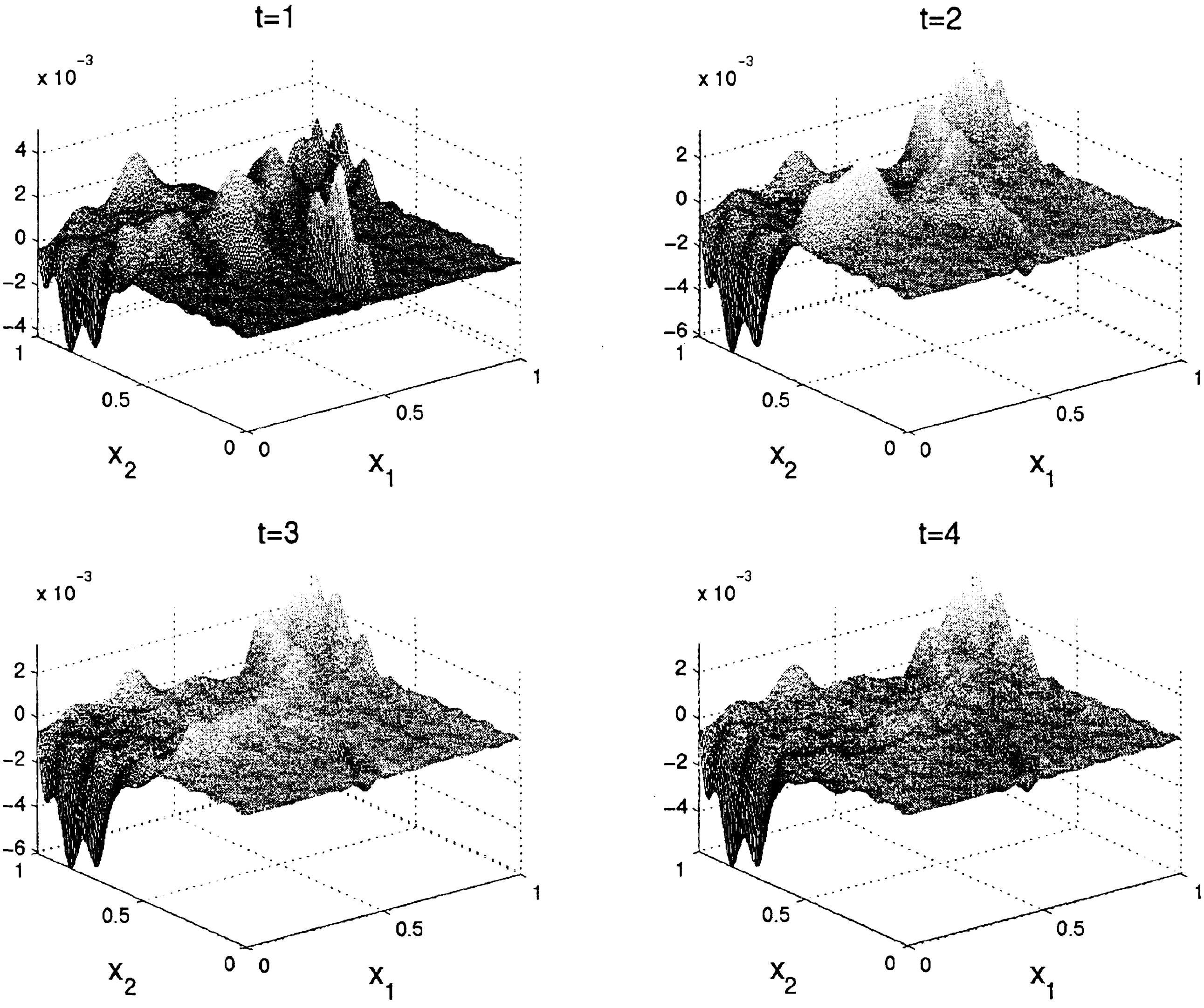

FIG. 5.9 – Difference between the vertical velocities : $v - v_{ref}$ ($I_0 = 10$ and $I_1 = 5$).

Finally, Table 5.1 shows in a table the different CPU-times in function of choice of the parameters I_0 and I_1. As expected, the CPU time decrease when the resolution on the coarse grid increase. For example, if we choose $I_0 = 10$ and $I_1 = 5$ the gain in CPU time is equal to 55%.

Parameters	CPU time
FISFG	$2136s$
$I_0 = 5$ and $I_1 = 5$	$1277s$
$I_0 = 10$ and $I_1 = 5$	$922s$
$I_0 = 15$ and $I_1 = 5$	$709s$
$I_0 = 20$ and $I_1 = 5$	$605s$
FISCG	$120s$

TAB. 4.1 – CPU time in function of the choice of the parameters I_0 and I_1. The extreme cases correspond to the Fully Implicit Scheme on the Coarse Grid (FISCG) and the Fine Grid (FISFG).

5.2 Concluding remarks

In this article, we have introduced a new type of incremental unknowns derived from finite volume resolutions. These unknowns seem to behave in a way similar to the finite difference incremental unknowns and we believe that they are fully adapted to the simulation of incompressible flows. In a subsequent article, we will use these unknowns to solve the Navier-Stokes equations; problems in geophysical fluid mechanics could also be considered [LTW92a],[LTW92b]. As indicated before, an advantage of the finite volume resolution is that it can be collocated, and therefore, it is easier to use the incremental unknowns with this kind of discretization than with a staggered method such as a MAC scheme.

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Conclusion

Nous avons tout d'abord présenté un schéma numérique permettant de résoudre l'équation de la chaleur non linéaire avec des conditions aux bords de type Robin. Dans ce problème les termes non linéaires qui interviennent vérifient des hypothèses de monotonie. Comme discrétisation spatiale, nous avons utilisé une méthode volumes finis : on a donc intégré l'équation sur chaque volume de contrôle. Concernant maintenant la discrétisation temporelle, nous avons utilisé un schéma différences finies de type Crank-Nicholson. Notre premier travail fût de démontrer l'existence de la solution approchée obtenue à l'aide du schéma numérique. Ensuite, nous avons démontré, grâce à une méthode de monotonie, que la solution approchée convergeait vers l'unique solution du problème continu.

Dans un deuxième temps, nous nous sommes interessés à l'élaboration de schémas volumes finis à variables colocalisées (les vitesses et la pression sont calculées aux même endroit, par opposition aux schémas de type MAC) permettant de résoudre les équations de Navier-Stokes dans le cas d'un fluide incompressible. La principale difficulté de ce type de schémas est inhérente à la colocalisation des inconnues qui engendre un découplage des vitesses et de la pression. Il faut donc, en utilisant par exemple une méthode particulière pour interpoler les flux aux arêtes, trouver un moyen de rétablir un couplage correct entre les vitesses et la pression. L'intérêt d'arriver à obtenir de tels schémas est grand car ils permettent une hiérarchisation spatiale plus aisée et ils sont également facilement adaptables à des géométries complexes.
Nous avons testé deux types de discrétisation temporelle : une méthode de projection et une méthode de "splitting". Nous avons ainsi compris l'importance de la discrétisation temporelle lorsque l'on utilise des méthodes volumes finis à variables colocalisées. En effet, les techniques nécessaires pour résoudre le problème de découplage des vitesses et de la pression, sont fortement dépendantes de la méthode de discrétisation temporelle choisie.
D'un point de vue plus théorique, nous avons étudié la stabilité du schéma dont la discrétisation temporelle est assurée par une méthode de projection.

Possédant donc deux schémas volumes finis à variables colocalisées permettant de résoudre les équations de Navier-Stokes incompressible, la dernière partie de ces travaux fût donc consacrée au développement d'une méthode multi-niveaux adaptée aux méthodes volumes finis.
Nous avons choisi de nous inspirer d'une décomposition des inconnues en Inconnues Incrémentales comme cela se fait en différences finies. Cette décomposition est basée sur les propriétés du

développement de Taylor et est donc mathématique, contrairement aux méthodes spectrales où la décomposition se fait par troncature du développement en série de Fourier de la solution.
De plus, afin de ne pas affronter simultanément les difficultés liées à la résolution des équations de Navier-Stokes et celles liées à la mise au point d'une nouvelle méthode multi-niveaux, nous avons appliqué notre méthode sur le problème de Burgers. Ce problème servant traditionnellement de problème modèle pour tester de telles méthodes avant de passer aux équations de Navier-Stokes, nous avons pu voir que nos résultats étaient semblables à ceux existant dans la littérature.

Du point de vue des perspectives ouvertes par ces travaux, ils nous paraît raisonnablement envisageable de développer un algorithme utilisant les méthodes volumes finis à variables co-localisées et la méthode multi-niveaux décrites précédemment afin de résoudre les équations de Navier-Stokes incompressible. En effet, les vitesses et la pression étant calculées au même endroit, nous pouvons leur appliquer la même hiérarchisation spatiale ce qui simplifie significativement l'écriture des schémas numériques.
Pour le bon fonctionnement d'un tel algorithme multi-niveaux, il faudra également définir des critères indiquant sur quels niveaux les calculs doivent être effectués, le but étant bien entendu de rester le plus longtemps possible sur le maillage le plus grossier sans que l'erreur commise ne devienne trop importante. De plus, nous pourrons également étudier si il est possible faire cela de façon locale i.e. de travailler sur un maillage grossier là où les vitesses varient peu et sur un maillage plus fin là elles varient beaucoup. Outre cela, nous pouvons envisager une parallélisation de l'algorithme, l'ajout d'un terme stochastique afin de simuler des phénomènes très turbulents, ou encore de travailler sur des domaines à géométrie complexe. Les perspectives d'une telle méthode sont donc nombreuses et variées.