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Comptage de réseaux et rigidité entropique pour les actions de groupes sur les arbres et les immeubles

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Chapitre 1

Introduction

I. Soit (M, g) une variété riemannienne compacte sans bord à courbure négative ou nulle. L'entropie volumique $h_{vol}(g)$ de (M, g) est définie par la formule

$$h_{vol}(g) = \lim_{r \to \infty} \frac{1}{r} \log(vol(B(x, r))),$$

où $B(x, r)$ désigne la boule de rayon r centrée en un point x du revêtement universel $\tilde{M}$ de M.

Conjecture (Gromov). *Les métriques localement symétriques sur une variété à courbure négative ou nulle de dimension au moins* 3 *sont caractérisées, parmi toutes les métriques riemanniennes, par leur volume $vol_g(M)$ et leur entropie volumique $h_{vol}(g)$.*

Cette conjecture a été démontrée pour les variétés à courbure strictement négative par Besson, Courtois et Gallot dans leur travail séminal [BCG]. Leur méthode permet en outre de donner une démonstration totalement nouvelle du théorème de rigidité forte de Mostow pour les variétés localement symétriques de rang un. Une démonstration avait été précédement donnée par Katok [Ka] dans le cas spécial des métriques dans la classe conforme d'une métrique hyperbolique sur une surface compacte orientable.

Théorème (Besson-Courtois-Gallot). *Si X est de dimension ≥ 3 et possède une métrique localement symétrique g_0 de rang un, alors pour toute métrique riemannienne g vérifiant $vol(X, g) = vol(X, g_0)$ on a*

$$h_{vol}(g) \geq h_{vol}(g_0)$$

avec égalité si et seulement si g est isométrique à g_0.

La conjecture de Gromov est encore ouverte pour les métriques localement symétriques de rang supérieur. Connell et Farb l'ont démontrée dans le cas spécial d'un produit d'espaces symétriques de rang un [CF].

Soit X un graphe fini connexe dont nous notons VX et EX l'ensemble des sommets et des arêtes orientées respectivement. Pour une distance de longueur d sur X, déterminée par les longueurs des arêtes $\{l(e)\}_{e\in EX} \in (\mathbb{R}^+)^{|EX|}$, nous posons $vol_d(X) = \frac{1}{2}\sum_{e\in EX} l(e)$, que nous appelons le volume total de (X, d). À un tel graphe métrique, on associe son entropie volumique, définie par

$$h_{vol}(d) = \lim_{r\to\infty} \frac{1}{r} \log(vol(B(x,r))),$$

où le volume désigne la somme des longueurs des arêtes (ou de la partie des arêtes) contenues dans la boule $B(x,r) \subset \tilde{X}$ du revêtement universel $\overline{X}$ de X muni de la distance relevée de d, où x est un point fixé de $\overline{X}$.

Théorème 1.0.1 (Theorem 2.2.1). *Soit X un graphe fini connexe quelconque, dont tous les sommets $x \in VX$ sont de valence $k_x + 1$ au moins égale à 3. Il existe une unique distance de longueur d sur X telle que $\mathrm{vol}_d(X) = 1$, qui minimise l'entropie volumique $h_{vol}(d)$. L'entropie volumique minimale est donnée par la formule*

$$h_{\min} = \frac{1}{2}\sum_{x\in VX} (k_x + 1)\log k_x,$$

et la distance de longueur minimisant cette entropie volumique est déterminée par ses longueurs d'arêtes

$$\forall e \in EX, \quad \ell(e) = \frac{\log(k_{i(e)}k_{t(e)})}{\sum\limits_{x\in VX}(k_x+1)\log k_x}.$$

Remarquons que nous donnons une formule close à la fois pour l'entropie volumique minimale et pour la métrique réalisant ce minimum. Cette dernière est complètement déterminée localement, i.e. la longueur de chaque arête dépend uniquement de la valence des sommets qu'elle relie. Notons aussi la similitude avec la métrique combinatoire de Bourdon sur le graphe dual d'un immeuble fuchsien X à angles droit, qui atteint la dimension conforme de Pansu, i.e. qui minimise la dimension de Hausdorff de la frontière par rapport aux métriques dans la classe quasi-conforme de ∂X ([Bo2]).

Les graphes réguliers peuvent être considérés comme analogues aux variétés riemanniennes possédant une métrique localement symétrique. Dans ce cas spécial, le résultat ci-dessus a aussi été démontré indépendamment par Kapovich et Smirnova-Nagnibeda ([KN]) par une méthode différente (à l'aide de chemins aléatoires). Le même résultat dans ce cas spécial était aussi mentionné implicitement sous une forme analogue, mais non équivalente, dans un prépublication de Rivin ([Riv]). Enfin, sous l'hypothèse supplémentaire que le graphe X admette un groupe d'automorphismes hautement transitif, ce résultat apparait précédement dans Robert [Rob]. À l'aide du théorème ci-dessus, on peut déduire :

Corollaire 1.0.2 (Corollaire 2.2.3, Theorem B dans [KN]). *Considérons l'ensemble des graphes métriques finis, sans sommets de valence* 1 *ou* 2*, et dont le groupe fondamental est un groupe libre de rang* $r \geq 2$ *fixé. Alors parmi toutes les distances de longueur de volume total un, l'entropie volumique est minimisée pour tout graphe trivalent équipé de la métrique donnant à chaque arête la même longueur.*

Il est possible de généraliser encore le théorème 2.2.1 au cadre des graphes finis de groupes finis $(X, G_\bullet)$. Dans ce cas, le degré k_x+1 de chaque sommet x est défini par $k_x+1 = \sum e \in EX,\ i(e) = x\frac{|G_x|}{|G_e|}$ (i.e., la valence d'un relèvement $\tilde{x}$ de x dans l'arbre de Bass-Serre $\tilde{X}$ de $(X, G_\bullet)$), et la notion correspondante de volume total de $(X, G_\bullet, d)$ est donnée par $vol_d(X, G_\bullet) = \frac{1}{2}\sum \frac{l(e)}{|G_e|}$. Notons que cette définition coïncide, à un facteur près, avec la notion usuelle de volume total d'un graphe de groupes dans le cas d'un graphe régulier.

Théorème 1.0.3 ([L2]). *Soit* $(X, G_\bullet)$ *un graphe fini de groupes finis dont le degré* $k_x + 1$ *de chaque sommet* x *est au moins* 3 . *Parmi toutes les distances de longueur sur* X *de volume total un dans* $(X, G_\bullet)$*, il en existe une unique minimisant l'entropie volumique. Pour cette métrique, la longueur de chaque arête est proportionnelle à* $log(k_{i(e)}k_{t(e)})$ *et l'entropie minimale est*

$$h_{\min}(X, G_\bullet) = \frac{1}{2}\sum_{x \in VX} \frac{(k_x+1)\log k_x}{|G_x|}.$$

Enfin, nous montrons, à l'aide du théorème ci-dessus, que pour un revêtement à n feuillets de graphes de groupes $\phi : (Y, H_\bullet) \to (X, G_\bullet)$, on a

$$h_{vol}(Y, H_\bullet, d)vol(Y, H_\bullet, d) \geq n\ h_{vol}(X, G_\bullet, d_0)vol(X, G_\bullet, d_0),$$

avec égalité si et seulement si la métrique d sur $(Y, H_\bullet)$ réalise le minimum de l'entropie volumique (parmi les métriques de même volume total), et si l'application ϕ est un revêtement métrique de $(Y, H_\bullet, d)$ sur $(X, G_\bullet, \lambda d_0)$ pour un $\lambda > 0$. Ce résultat peut être vu comme l'analogue précis du théorème principal de [BCG] dans le cas des graphes.

La preuve de Besson, Courtois et Gallot utilise de manière essentielle la mesure de Patterson-Sullivan et l'application barycentre. L'inégalité voulue découle alors du calcul précis du jacobien d'une certaine fonctionnelle et de ses dérivées secondes.

Les mesures de Patterson-Sullivan pour les arbres ont été introduites par Coornaert et Lyons ([C], [Ly]). Étant donné qu'il ne semble pas exister d'analogue utile de la dérivée seconde pour une fonction définie sur un arbre, notre stratégie pour démontrer le théorème 1.0.5 est de n'utiliser que cette mesure de Patterson-Sullivan. Si $\widetilde{X} \to X$ est un revêtement universel du graphe X, si $\widetilde{e} \in E\widetilde{X}$ est une arête de sommet inital $i\widetilde{x}$, notons $Cyl_{\widetilde{x}}(\widetilde{e})$ l'ensemble des rayons géodésiques d'arête initial $\widetilde{e}$. Notons qu'une densité h-conforme sur $\partial\tilde{X}$ est uniquement déterminée par les $\mu_{\tilde{x}}(Cyl_{\tilde{x}}(\tilde{e}))$ pour $e \in EX$, où $\widetilde{e}$ est n'importe quel relevé de e, et $\widetilde{x} = i(\widetilde{e})$. Soit A la matrice d'adjacence des arêtes du graphe X, et posons $x_e = \mu_{\widetilde{x}}(Cyl_{\widetilde{x}}(\widetilde{e}))$. Nous montrons que les nombres positifs $(x_e)_{e\in EX}$ sont solutions du système d'équations

$$x_e = \sum_{e'\in EX, A_{ee'}=1} e^{-hl(e')} x_{e'}.$$

Notons que l'équation est obtenue dans le chapitre 3, sans la relation explicite entre les x_e's et la mesure de Patterson-Sullivan (voir [KN] pour cette interprétation). Voici une autre manière d'interpréter la méthode de démonstration du théorème 2.2.1. La matrice d'adjacence des arêtes de X code le flot géodésique du graphe combinatoire X. Lorsqu'on fait varier la métrique sur X, la matrice A code l'application de premier retour sur l'ensemble des sommets. Étant donné que, comme nous le montrons dans la proposition 2.4.3, l'entropie volumique est égale à l'entropie topologique du flot géodésique, nous sommes amenés à étudier le flot de suspension de l'application de premier retour.

Remarque. L'avantage de ce point de vue est qu'il permet d'entrevoir une généralisation de cette méthode aux immeubles hyperboliques de dimension supérieure. Par example, notons $I_{p,q}$ l'immeuble hyperbolique à angles droits de Bourdon. Soit X le graphe dual du 1-squelette de $I_{p,q}$. Fixons aussi un quotient compact $Y = \Gamma\backslash X$ de X, et considérons toutes les métriques obtenues en variant les longueurs des arêtes de Y. Chaque appartement de $I_{p,q}$ est une copie de $\mathbb{H}^2$ pavée par des p-gones à angles droits. On peut, pour simplifier les choses, prendre comme quotient Y une 2-cellule C_0 fixée.

On peut alors formuler la question ouverte dans ce cadre : trouver l'entropie volumique minimale dans l'ensemble des métriques sur Y obtenue en faisant varier C_0 tout en maintenant fixe la somme des longueurs des cotés de C_0.

Le codage du flot géodésique sur certaines surfaces compactes a été considéré par C. Series [S], en utilisant les géodésiques qui prolongent les cotés d'un domaine fondamental. Nous savons montrer que le flot géodésique sur $I_{p,q}$ est un système sofique en nous fondant sur le fait que, sous certaines hypothèses, il n'y a qu'un nombre fini de composantes connexes de la frontière de $\mathbb{H}^2$ délimitées par les sommets des murs des arbres contenant les arêtes de C_0. Cette question n'est pas contenue dans cette thèse, mais nous espérons obtenir prochainement des résultats qui généralisent ceux du théorème 3.2.1.

II. On appelle réseau tout sous-groupe discret Γ d'un groupe localement compact G qui admet un domaine fondamental de mesure finie. Si Γ est un réseau dans G, alors il existe, à une constante multiplicative près, une unique mesure de probabilité μ_G G-invariante, la mesure de Haar, sur $\Gamma\backslash G$. Si G est un groupe de Lie, alors la mesure μ_G peut être représentée par une forme de volume lisse sur la variété $\Gamma\backslash G$ (pour cette raison, on note souvent μ_G par vol). Un réseau Γ est appelé uniforme ou cocompact si la variété $\Gamma\backslash G$ est compacte. Un surréseau Γ' de Γ (d'indice n) est un réseau dans G qui contient Γ (avec indice n). Les réseaux dans les groupes de Lie ont fait l'objet d'une étude intensive depuis plus d'un siècle, et cette théorie est maintenant devenue un domaine classique des mathématiques (voir, par exemple, [Ra], [Mar]). Parmi les propriétés importantes des réseaux dans les groupes de Lie semisimples, citons l'existence d'un covolume minimal pour les réseaux :

Théorème (Kazhdan-Margulis). *Pour tout groupe de Lie semisimple G sans facteur compact, il existe une constante $\epsilon > 0$ telle que, pour tout réseau $\Gamma \subset G$, on ait $vol(G/\Gamma) > \epsilon$.*

Les groupes des automorphismes des arbres et des immeubles semblent partager beaucoup des propriétés des groupes de Lie semisimples, et un programme a été ébauché dans le but de comparer ces deux classes de groupes (voir, par exemple, [BL] et les références qui s'y trouvent). Il est connu ([BK]), néanmoins, que le théorème ci-dessus est faux dans le cas d'un groupe d'automorphisme d'un immeuble, ce qui amène naturellement à se poser la question de la vitesse de croissance du nombre de réseaux de petit covolume.

Bass et Kulkarni [BK] ont construit des exemples de tours de réseaux $\{\Gamma_1 \subset \Gamma_2 \subset \cdots\}$ dans le groupe G des automorphismes d'un arbre localement fini, sans sommet terminal, et tels que $\lim\limits_{t\to\infty} vol(G/\Gamma_i) = 0$ (rappelons qu'un arbre sans sommet terminal est un immeuble de dimension un). Bass a de plus montré que pour un réseau uniforme Γ *fixé*, il n'existe qu'un nombre fini $u_\Gamma(n)$ de surréseaux d'indice n de Γ dans G. Ceci amène naturellement à se poser la question du comportement asymptotique de $u_\Gamma(n)$. Ce problème, soulevé par Lubotsky dans [BL], peut être considéré comme le problème de croissance des "surréseaux", un pendant à la théorie importante de la croissance des sous-groupes ([Lub], [LS]).

Dans le chapitre 2, nous apportons une réponse à cette question : nous obtenons une majoration globale pour $u_\Gamma(n)$, valable pour un réseau quelconque Γ dans un arbre localement fini quelconque, et nous donnons aussi une minoration de $u_\Gamma(n)$ pour un certain type de réseaux dans les arbres réguliers.

Théorème 1.0.4 ([L1]). *Pour tout réseau Γ dans le groupe des automorphismes d'un arbre localement fini, il existe des constantes positives c et ϵ telles que*

$$\forall n \in \mathbb{N} - \{0\}, \;\; u_\Gamma(n) \leq c n^{\epsilon log^2 n}.$$

Pour certains réseaux Γ, nous construisons explicitement des tours de réseaux dont les stabilisateurs des sommets et des arêtes sont des p-groupes, et nous en déduisons une borne inférieure pour $u_\Gamma(n)$.

Théorème 1.0.5 ([L1]). *Soit p un nombre premier. Si Γ est un réseau sans inversions d'arêtes dans un arbre $2p$-régulier dont le graphe de groupes quotient est isomorphe à une boucle donc le groupe de sommet est un groupe cyclique d'ordre p, et le groupe d'arête est trivial, alors pour tout N, il existe $n > N$ tel que*

$$u_\Gamma(n) \geq n^{\frac{1}{50}\log_p n - 4}.$$

Le chapitre 2 s'appuie principalement sur la théorie de Bass-Serre des graphes de groupes ([Ba], [Se]). Nous définissons une notion correcte d'isomorphisme de revêtements de graphes de groupes, et nous établissons une bijection naturelle entre l'ensemble des surréseaux d'un réseau Γ donné, et l'ensemble des classes d'isomorphismes de revêtements du graphe de groupes $T//\Gamma$ correspondant. Nous utilisons certains résultats profonds de la théorie des groupes finis dus à Pyber [P] sur le nombre de classes d'isomorphismes de groupes d'un ordre donné, ainsi que sur le nombre minimal de générateurs d'un tel groupe [Luc], [Gur], et nous en déduisons une borne supérieure pour $u_\Gamma(n)$.

Pour obtenir une borne inférieure dans le cas décrit plus-haut, nous classifions tous les revêtements fidèles de graphes de groupes dont le graphe quotient est une boucle d'indices (p,p), et dont les groupes de sommet et d'arête sont tous des p-groupes, par le graphe de groupes isomorphe à une boucle donc le groupe de sommet est un groupe cyclique d'ordre p, et le groupe d'arête est trivial.

Nous décrivons ensuite précisément la structure de ces groupes de sommet et d'arête, ainsi que celle des morphismes locaux définissant le revêtement.

Notons que ces résultats sont, à ce jour, les seuls connus décrivant le comportement asymptotique du nombre de surréseaux, en dehors du cas extrême traité par Goldschmitt en 1980 [Go] (classification des $(3-3)$ amalgames (rappelons qu'un $(3,3)$-amalgame est un graphe de groupes dont le graphe muni des indices des groupes d'arêtes dans les groupes de sommets est réduit à une arête avec les deux indices des deux arêtes (opposés, donc) égaux à 3), qui entraine que $u_\Gamma(n) = 0$ pour n grand pour certains réseaux dans un arbre 3-régulier).

La théorie des graphes de groupes de Bass-Serre a été généralisée à une théorie des complexes de groupes par Haefliger ([H1], [BH]), dans l'optique de coder les actions de groupes sur des complexes polyhédraux. Cette théorie est développée dans le cadre des petites catégories sans boucles ("scwol") dont remplacent l'espace sous-jacent. À chaque action d'un groupe G sur un scwol χ est associé un complexe de groupes $G(\mathcal{Y})$ sur le scwol quotient $\mathcal{Y} = G\backslash\mathcal{X}$.

Prenons pour χ un scwol simplement connexe, par exemple obtenu par subdivision barycentrique d'un immeuble hyperbolique ou euclidien. Soit Γ un réseau dans $Aut(\chi)$, et notons encore $u_\Gamma(n)$ le nombre de surréseaux de Γ dans $Aut(\chi)$ d'indice n.

Dans le chapitre 4, nous étudions le comportement asymptotique de $u_\Gamma(n)$. Il y a plusieurs difficultés à surmonter : tout d'abord, la théorie de Haefliger restreinte aux scwol de dimension un n'est pas équivalente à la théorie de Bass-Serre (ceci répond à une question posée dans [BH]) ; ensuite, tous les complexes de groupes ne proviennent pas de l'action d'un groupe sur un scwol. On dit d'un complexe de groupes pouvant s'obtenir de cette manière qu'il est développable.

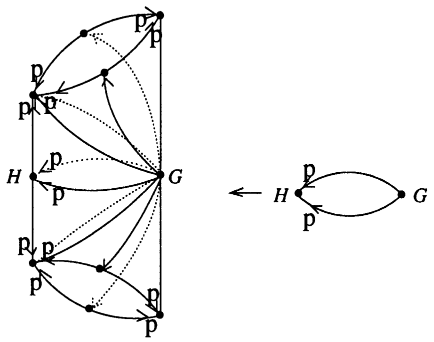
Soit Γ un réseau uniforme du groupe des automorphisms d'un scwol simplement connexe localement fini, nous introduisons d'abord la notion de fidélité d'un complexe de groupes, analogue à celle de graphe de groupes donnée par Bass, dans la Section 4.3.3. Nous établissons une correspondance bijective entre l'ensemble des surréseaux de Γ d'indice n et l'ensemble des classes d'isomorphisme de revêtements à n feuillets d'un complexe de groupes développable, par $\Gamma\backslash\backslash\mathcal{X}$. Notons que cette correspondance est valable dans un cadre très général, plus large que celui des immeubles.

Théorème 1.0.6 (Théoreme 4.1.4). *Pour tout réseau uniforme Γ dans le groupe des automorphismes d'un scwol $\mathcal{X}$ simplement connexe et localement fini, il existe deux constantes positives ϵ et* *c telles que*

$$\forall n \in \mathbb{N}, \quad u_\Gamma(n) \leq c n^{\epsilon log^2 n}.$$

Dans le cas d'un réseau Γ d'un immeuble hyperbolique dont le scwol associé est noté $\mathcal{X}$, nous démontrons que la propriété d'être développable pour un complexe de groupes revêtu par $\mathcal{X}$, est une conséquence de la courbure négative de l'immeuble, et nous avons également une borne inférieure pour $u_\Gamma(n)$, déduite de celle construite pour le cas des arbres dans [L1] pour certains Γ.

Nous fixons un complexe de groupe quotient $G(\mathcal{Y})$ d'un immeuble hyperbolique n'ayant qu'une cellule de dimension 2, dont le stabilisateur de la cellule de dimension 2 est trivial, et dont le scwol indexé sous-jacent est comme ci-dessus. Nous fixons également un graphe de groupes $G(\mathcal{X})$ dont le graphe indexé sous-jacent est une boucle d'indices (p, p).



Pour tout revêtement de graphes de groupes $G(\mathcal{X}) \to H(\mathcal{X})$ nous construisons de manière fonctorielle, par une méthode inspirée de [Th], un plongement fidèle de $H(\mathcal{X})$ dans un complexe de groupes $H(\mathcal{Y})$, lui-même revêtement de $G(\mathcal{Y})$.

Théorème 1.0.7 (Theorem 4.1.5, [LT]). *Pour tout réseau Γ (à stabilisateur de face trivial) dans le groupe des automorphismes d'un immeuble hyperbolique de Bourdon à angle droit, dont le complexe de groupes quotient a un scwol indexé sous-jacent comme ci-dessus, il existe $c_1, c_2 > 0$ tels que, pour tout $N > 0$, il existe $n > N$ vérifiant*

$$u_\Gamma(n) \geq c_1 n^{c_2 log\, n}.$$

Les immeubles hyperboliques ont fait l'objet de nombreuses recherches de la part de Tits, Bourdon, Cartwright, Gaboriau, Haglund, Paulin, et autres. On trouve des exemples en dimension deux dans les travaux de Bourdon ([Bo], [Bo2]), et en dimension trois dans ceux de Haglund et Paulin

[HP]. Nous avons l'intention d'étudier le comportement asymptotique de $u_\Gamma(n)$ pour ces exemples d'immeubles hyperboliques.

Ce mémoire est organisé comme suit. Le chapitre 2 porte sur l'entropie volumique minimale pour les graphes, qui est partiellement tiré de notre article [L2], avec deux sections supplémentaires à la fin (Section 2.4. et 2.5) décrivant quelques autres caractérisations de l'entropie volumique pour les graphes. Dans le chapitre 3, nous traitons du problème du comptage des surréseaux dans les groupes d'automorphismes d'arbres localement finis. Cette partie est plus ou moins la version “mise à jour” de notre premier article ([L1]). Dans le chapitre 4, nous traitons des problème de comptage des surréseaux, cette fois dans les groupes d'automorphismes de scwols localement finis et simplement connexes. Nous donnons une borne supérieure universelle pour le cas général, ainsi qu'une borne inférieure pour certains groupes agissant sur certains immeubles hyperboliques. Même si le résultat concernant la borne supérieure du chapitre 4 implique le résultat analogue du chapitre 2, les notions de revêtement de graphes de groupes et de revêtement de complexes de groupes sont différentes, et donc les deux preuves du théorème central, établissant la correspondance bijective entre l'ensemble des surréseaux d'une part et l'ensemble des classes d'isomorphismes de revêtements de l'autre, sont différentes. Cette différence subtile est expliquée dans la dernière section du chapitre 4. Les résultats de ce dernier chapitre sont le fruit d'une recherche en commun avec Anne Thomas [LT].

Chapitre 2

Volume entropy for graphs

2.1 Volume entropy and path growth

Let us consider a nonempty connected unoriented finite graph X without any terminal vertex. We will denote the set of vertices by VX and the set of oriented edges of X by EX. We denote again by X the geometric realization of X. For every edge e, let us denote by $i(e)$ and $t(e)$ the initial and the terminal vertex of e, respectively. We define a *length distance* d on X by assigning a positive real number $\ell(e) = \ell(\bar{e})$ for each unoriented edge $\{e, \bar{e}\}$ of X, and by letting $d = d_\ell : X \times X \to [0, \infty[$ be the maximal distance which makes each half-edge of an edge e containing a vertex, isometric to $[0, \frac{\ell(e)}{2}]$. For a length distance d_ℓ, let $l_{\max} = \max_{e \in EX} \ell(e)$ and $l_{\min} = \min_{e \in EX} \ell(e)$. Define the volume of X by

$$\mathrm{Vol}(X, d) = \frac{1}{2} \sum_{e \in EX} \ell(e),$$

i.e., the sum of the lengths of the unoriented edges. We denote by $\Delta(X)$ the set of all length distances $d = d_\ell$ on X normalized so that $\mathrm{Vol}(X, d) = 1$.

For a fixed length distance d, let us consider a universal covering tree $\tilde{X} \to X$ equipped with the lifted distance $\tilde{d}$ of d. For any connected subset S of $\tilde{X}$, let us denote by $\ell(S)$ the sum of the lengths of (the maximal pieces of) the edges in S. We define the *volume entropy* $h_{\mathrm{vol}}(d) = h_{\mathrm{vol}}(X, d)$ as

$$h_{\mathrm{vol}}(d) = \limsup_{r \to \infty} \frac{1}{r} \log \ell(B(x_0, r)),$$

where $B(x_0, r) = B_d(x_0, r)$ is the ball of radius r with center a fixed vertex x_0 in $(\tilde{X}, \tilde{d})$. The entropy $h_{\mathrm{vol}}(d)$ does not depend on the base point x_0, and we may sum either on the oriented or on the non-oriented edges. Note also the homogeneity property

$$h_{\mathrm{vol}}(d_{\alpha\ell}) = \frac{1}{\alpha} h_{\mathrm{vol}}(d_\ell), \tag{2.1}$$

for every $\alpha > 0$. Remark that $h_{\text{vol}}(X,d)\text{vol}(X,d)$ is invariant under dilations, therefore to minimize the entropy with constant volume, it suffices to consider the length metrics of volume 1.

If $\pi_1 X$ is not cyclic, or equivalently (as X has no terminal vertices) if X is not reduced to one cycle, then $h_{\text{vol}} = h_{\text{vol}}(d)$ is strictly positive, which we will assume from now on (see for instance [Bo]). It was shown by Roblin ([Robl]) that the upper limit above is in fact a limit. This implies that as $r \to \infty$,

$$\ell(B(x_0, r)) = e^{h_{\text{vol}(d)}r+o(r)}.$$

By a *metric path of length r in X*, we mean the image of a local isometry $f : [0, r] \to X$. Note that the endpoint of a metric path is not necessarily a vertex. By a *combinatorial n-path of length r in X*, we mean a path $\underline{p} = e_1e_2 \cdots e_n$ of consecutive edges in X without backtracking such that $\sum_{j=1}^{n-1} \ell(e_j) < r \le \sum_{j=1}^{n} \ell(e_j)$. A combinatorial path is a combinatorial n-path for some n in $\mathbb{N}$.

Lemma 2.1.1. *Let $N_r(x_0)$ be the cardinality of the set of combinatorial paths of length r in $\tilde{X}$ starting at $x_0 \in V\tilde{X}$. Then the number $N_r(x_0)$ satisfies*

$$\limsup_{r\to\infty} \frac{\log N_r(x_0)}{r} = \lim_{r\to\infty} \frac{\log N_r(x_0)}{r} = h_{\text{vol}}.$$

Proof. It follows directly from $\ell(B(x_0,r)) = e^{(h_{\text{vol}}+o(1))r}$ that for any $l > 0$,

$$\limsup_{r\to\infty} \frac{\log \ell(B(x_0,r) - B(x_0,r-l))}{r} = \lim_{r\to\infty} \frac{\log \ell(B(x_0,r) - B(x_0,r-l))}{r} = h_{\text{vol}}.$$

Now let $N'_r(x_0)$ be the cardinality of the set of metric paths of length r starting at x_0. As $\tilde{X}$ has no terminal vertices, for any $\epsilon > 0$,

$$\epsilon N'_{r-\epsilon}(x_0) \le \ell(B(x_0,r) - B(x_0,r-\epsilon)) \le \epsilon N'_r(x_0).$$

Therefore

$$\limsup_{r\to\infty} \frac{\log N'_r(x_0)}{r} = \lim_{r\to\infty} \frac{\log N'_r(x_0)}{r} = h_{\text{vol}}.$$

It is clear that we get a combinatorial path of length r by continuing a metric path of length r until it meets a vertex. Also, two distinct combinatorial paths of length r cannot be extensions of one metric path of length r by the strict inequality in the definition of a combinatorial path. It follows that $N_r(x_0) = N'_r(x_0)$, thus $N_r(x_0)$ has the same exponential growth rate as $N'_r(x_0)$, which is h_{vol}. $\square$

Let $A = A(X)$ be the *edge adjacency matrix of X*, i.e. a $|EX| \times |EX|$ matrix such that A_{ef} has value 1 if ef is a combinatorial 2-path, i.e. if $t(e) = i(f)$ and $\bar{e} \ne f$, and value 0 otherwise. It is easy to see that the entry A^n_{ef} of the matrix A^n is nonzero if and only if there is a combinatorial $(n+1)$-path starting with e and ending with f. (Note that the definition of A_{ef} implies that such a path does not have backtracking.)

It is easy to show that for any connected graph without any terminal vertex, which is not a cycle, the matrix A is irreducible. Recall that a nonnegative matrix M is *irreducible* if for every i, j, there exists an integer $n > 0$ such that $(A^n)_{ij} > 0$. Let us give a detailed proof for completeness. By connectedness, it is sufficient to show that for every edge $e \in EX$, there exists an edge path without backtracking from e to $\bar{e}$. Since the graph of not a cycle and has no terminal vertex, there are at least two cycles C, C' containing e. Let x, y be two vertices which are the end points of $C \cap C'$. Let $[x, y]_C$ and $[x, y]_{C'}$ be the two disjoint paths in the set $C - C'$ and $C' - C$. Now the edge path starting at e, following $C \cap C'$, until say x, then following $[x, y]_C$ until it arrives at y, and then following $[x, y]_{C'}$ until it arrives at x, then following C' without backtracking, clearly passes through $\bar{e}$.

Now consider the matrix $A' = A'(d, h)$ defined by $A'_{ef} = A_{ef}e^{-h\ell(f)}$, depending on h and the length distance d_ℓ on X. The matrix A' is clearly irreducible since A is irreducible.

Theorem 2.1.2. *Let X be a connected finite graph without any terminal vertex, which is not a cycle, endowed with a length distance $d = d_\ell$. The volume entropy h_{vol} is the only positive constant h such that the following system of linear equations with unknowns $(x_e)_{e \in EX}$ has a solution with $x_e > 0$ for every $e \in EX$.*

$$x_e = \sum_{f \in EX} A_{ef} e^{-h\ell(f)} x_f, \tag{2.2}$$

for every $e \in EX$.

Proof. By the assumption on the graph, for every $h > 0$, we can apply Perron-Frobenius theorem (see [Gan] for example) to the irreducible nonnegative matrix $A' = (A_{ef}e^{-h\ell(f)})$, which says that the spectral radius of the matrix $A'(h)$ is a positive eigenvalue $\lambda(h)$, which is simple, with an eigenvector $(x_e = x_e(h))$ whose entries are all positive. The function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is clearly a continuous function of h since the characteristic function of the matrix A' is a polynomial in $\{e^{-h\ell(e)} : e \in EX\}$, and $\lambda(0) \geq 1$ since $\lambda(0)$ is the spectral radius of an irreducible nonzero matrix $A'(0)$ of nonnegative integer coefficients. Also, $\lambda(h) \to 0$ as $h \to \infty$, since the coefficients of $A'(h)$ tends to 0 as $h \to \infty$. By the mean value theorem, there exists an h satisfying $\lambda(h) = 1$.

Now assume that $h > 0$ satisfies (2.2) for some positive x_e's. Fix an arbitrary edge $e \in EX$, and choose a vertex x_0 in a universal cover $\tilde{X}$ of X which is an initial vertex of a fixed lift $\tilde{e}$ of e in $\tilde{X}$.

Let us fix a positive constant $r \geq \ell_{\max}$.

Let $P_r(e)$ be the set of combinatorial paths of length r in X starting with e. We will denote a combinatorial path in X by $\underline{p} = e_1 e_2 \cdots e_n$, its terminal edge by $t(\underline{p}) = e_n$ and its *metric length* by $\ell(\underline{p}) = \sum_{i=1}^{n} \ell(e_i)$. For $n \geq 2$, denote by $\mathcal{P}_n(e)$ (resp. $\mathcal{P}'_n(e)$) the set of combinatorial k-paths of length r with $k \leq n$ (resp. combinatorial n-paths of m-length strictly less than r) in X starting with e. Remark that $\mathcal{P}_n(e) \cap \mathcal{P}'_n(e) = \emptyset$ and if n is large enough, $\mathcal{P}_n(e) = P_r(e)$ and $\mathcal{P}'_n(e) = \emptyset$.

Let us rewrite the equation (2.2)

$$e^{-h\ell(e)} x_e = \sum_{\underline{p} \in \mathcal{P}_2(e) \cup \mathcal{P}'_2(e)} e^{-h\ell(\underline{p})} x_{t(\underline{p})}.$$

Let us replace each $x_{t(\underline{p})}$ in the above equation by $\sum\limits_{f \in EX} \rho_{t(\underline{p})f} e^{-h\ell(f)} x_f$ whenever $\ell(\underline{p}) < r$, i.e. when $\underline{p} \in \mathcal{P}'_2(e)$. The resulting equation is

$$e^{-h\ell(e)} x_e = \sum_{\underline{p} \in \mathcal{P}_3(e) \cup \mathcal{P}'_3(e)} e^{-h\ell(\underline{p})} x_{t(\underline{p})}.$$

Repeat this process : at each step, for each $\underline{p} \in \mathcal{P}'_n(e)$, replace $x_{t(\underline{p})}$ on the right hand side of the previous equation by $\sum\limits_{f \in EX} A_{t(\underline{p})f} e^{-h\ell(f)} x_f$, to get

$$e^{-h\ell(e)} x_e = \sum_{\underline{p} \in \mathcal{P}_{n+1}(e) \cup \mathcal{P}'_{n+1}(e)} e^{-h\ell(\underline{p}) x_{t(\underline{p})}}.$$

For n large enough, the resulting equation is

$$e^{-h\ell(e)} x_e = \sum_{\underline{p} \in P_r(e)} e^{-h\ell(\underline{p})} x_{t(\underline{p})}.$$

(In the case when the lengths of the edges are all equal to 1 and r is a positive integer, we continue until we get the equation $\underline{x} = A^{r-1} \underline{x}$.) For more formal proof, see Lemma 2.1.4 at the end of this section.

Then in the resulting equation, the number of times each x_f appears on the right hand side is exactly the number $N_r(e, f)$ of combinatorial paths of length r in $\tilde{X}$ with initial edge $\tilde{e}$ and terminal edge some lift of f in $\tilde{X}$. Note also that the metric length of such a path is at least r and less than $r + l_{\max}$. Thus

$$\sum_{f \in EX} N_r(e, f) e^{-h(r + l_{\max})} x_f \leq e^{h\ell(e)} x_e \leq \sum_{f \in EX} N_r(e, f) e^{-hr} x_f.$$

Now for integers r, by multiplying by $e^{hr-h\ell(e)}x_e^{-1}$ and taking the r-th root and the log on each part of the equation above, we deduce that

$$\frac{1}{r}\log\left(N_r(e,f)e^{-h(l_{\max}+\ell(e))}\frac{x_f}{x_e}\right)\leq h\leq\frac{1}{r}\log\left(\sum N_r(e,f)e^{-h\ell(e)}\frac{x_f}{x_e}\right),$$

thus

$$\lim_{r\to\infty}\frac{1}{r}\log\left(\sum N_r(e,f)e^{-h(l_{\max}+\ell(e))}\frac{x_f}{x_e}\right)\leq h\leq\lim_{r\to\infty}\frac{1}{r}\log\left(\sum N_r(e,f)e^{-h\ell(e)}\frac{x_f}{x_e}\right).$$

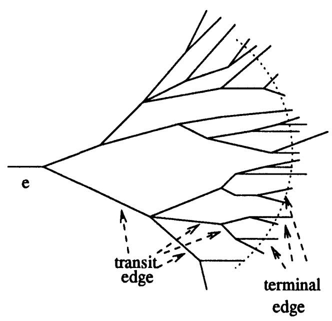
Now since $\leq N_r(x_0)e^{-hr}\sum x_f$ and $N_r(x_0)$ has exponential growth rate h_{vol} by the Lemma 2.1.1, the right hand side is bounded above by h_{vol}. As $N_r(x_0)=\sum_{e,f\in EX,\ i(e)=\pi(x_0)} N_r(e,f)$, where $\pi:\tilde{X}\to X$ is the universal covering map, there exist some e and f, depending on r, such that $N_r(e,f)e^{-hr}\geq\frac{1}{|EX|^2}N_r(x_0)e^{-hr}$. Therefore the left hand side is bounded below by h_{vol} as well. $\square$

Remark. Hersonsky and Hubbard showed in [HH] that the Hausdorff dimension of the limit set of a Schottky subgroup of the automorphism group of a simplicial tree satisfies similar systems of equations.

Supplement for the proof of 2.1.2 Let us recall that $\widetilde{X}$ is a tree without terminal vertex. Let $U\subset\tilde{X}$ be the closure of the connected component of $\widetilde{X}-\{o(\widetilde{e})\}$ containing $\tilde{e}$.

Definition 2.1.3. *A good subtree is a connected subtree $K\subset U$ containing $\tilde{e}$, satisfying the following property :*

for any edge $e\in EK$, the valency of $t(e)$ in K either is one, in which case e is called a terminal edge, or equals the valency of $t(e)$ in $\widetilde{X}$, in which case e is called a transit edge.



The radius of K is the maximal number of edges in a path in K going from $o(\tilde{e})$ to a terminal vertex.

For a terminal vertex f of K, let $p(f)$ be the unique path in $K \cup \{\tilde{e}\}$ going from $o(\tilde{e})$ to f.

Lemma 2.1.4. *Let K be a good subtree, and let TEK be the set of terminal edges of K. Then*

$$e^{-h\ell(e)}x_e = \sum_{f \in TEK} e^{-h\ell(p(f))}x_{\pi(f)}, \tag{*}$$

where π is the natural projection from $\widetilde{X}$ to X.

Démonstration. We proceed by induction on the radius $\mathrm{Rad}(K)$ of K. If $\mathrm{Rad}(K) = 0$, then $VK = \{i(\tilde{(e)}), t(\tilde{e})\}$ and $EK = \{\tilde{e}\}$, and the left hand side and the right hand side of the equation (*) are identical.

Let $n > 0$. Let us assume that (*) is true for any good subtree of radius n. Let K be of radius $n+1$. Let $TEK_{\leq n}$ (TEK_{n+1}) be the set of terminal edges f of K such that $p(f)$ contains at most n edges (respectively $n+1$ edges). Let FEK_n be the set of transit edges w for which $p(w)$ contains exactly n edges.

Let $K' \subset K$ be the subtree obtained by deleting all terminal edges in TEK_{n+1}. Then K' is good and $TEK' = TEK_{\leq n} \coprod FEK_n$. By induction hypothesis,

$$\begin{aligned} e^{-h\ell(\tilde{e})} &= \sum_{\tilde{f} \in TEK'} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})} \\ &= \sum_{\tilde{f} \in TEK_{\leq n}} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})} + \sum_{\tilde{f} \in FEK_n} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})}. \end{aligned}$$

Any terminal edge e in TEK_{n+1} is adjacent to exactly one transit edge in FEK_n, and any edge in U following a transit edge in FEK_n belongs to TEK_{n+1}. Using $x_f = \sum\limits_{g \in EX} \rho_{fg}e^{-h\ell(g)}x_g$, and $\ell(p(\tilde{g})) = \ell(p(\tilde{f})) + \ell(\pi(\tilde{g}))$ if $\tilde{g} \in TEK_{n+1}$ and $\tilde{f} \in FEK_n$ satisfy $i(\tilde{g}) = t(\tilde{f})$, we have

$$\begin{aligned} \sum_{\tilde{f} \in FEK_n} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})} &= \sum_{\tilde{f} \in FEK_n} e^{-h\ell(p(\tilde{f}))} \sum_{\tilde{g} \in E\tilde{X}, i(\tilde{g}) = t(\tilde{f})} e^{-h\ell(\pi(\tilde{g}))}x_{\pi(\tilde{g})} \\ &= \sum_{\tilde{g} \in TEK_{n+1}} e^{-h\ell(p(\tilde{g}))}x_{\pi(\tilde{g})}. \end{aligned}$$

Since $TEK = TEK_{\leq n} \coprod TEK_{n+1}$, we conclude that

$$\begin{aligned} e^{-h\ell(\tilde{e})} &= \sum_{\tilde{f} \in TEK_{\leq n}} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})} + \sum_{\tilde{f} \in TEK_{n+1}} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})} \\ &= \sum_{\tilde{f} \in TEK} e^{-h\ell(p(\tilde{f}))}x_{\pi(\tilde{f})}, \end{aligned}$$

thus the equation (*) holds for K as well. □

2.2 Minimal volume entropy

In this section, we prove the main theorem announced in the introduction, using Theorem 2.1.2.

Theorem 2.2.1. *Let X be a finite connected graph such that the valency at each vertex x, which we denote by k_x+1, is at least 3. Then there is a unique d in $\Delta(X)$ minimizing the volume entropy $h_{\mathrm{vol}}(d)$. The minimal volume entropy is*

$$h_{\min}(X) = \frac{1}{2}\sum_{x\in VX}(k_x+1)\log k_x,$$

and the entropy minimizing length distance $d = d_\ell$ is characterized by

$$\ell(e) = \frac{\log k_{i(e)}k_{t(e)}}{\sum\limits_{x\in VX}(k_x+1)\log k_x}, \quad \forall e\in EX.$$

Remark. Since we can eliminate all the vertices of valency two without changing the entropy, the existence of d in $\Delta(X)$ minimizing the volume entropy, with minimal value given by the same formula, holds for any graph who does not have a terminal vertex and is not isometric to a circle. What is uniquely defined at such a minimum is the length of each connected component of X where the vertices of valency at least three are removed.

Proof. By assumption, $k_x \geq 2$ for every $x \in VX$. By Theorem 2.2.1, the volume entropy $h = h_{\mathrm{vol}}$ satisfies

$$x_e = \sum_{f\in EX} A_{ef}e^{-h\ell(f)}x_f,$$

for each edge $e \in EX$ for some positive x_e's. Set $y_e = e^{-h\ell(e)}x_e > 0$ for each edge e. Then the above equations implies

$$e^{h\ell(e)}y_e = \sum_{f\in EX} A_{ef}y_f \geq k_{t(e)} \prod_{f\in EX,\, A_{ef}=1} y_f^{1/k_{i(f)}}. \tag{2.3}$$

The last inequality is simply the inequality between the arithmetic mean and the geometric mean of y_f's, since there are exactly $k_{t(e)} = k_{i(f)}$ edges f such that $A_{ef}=1$. Multiplying over all the edges, we get

$$\prod_{e\in EX} e^{h\ell_e}y_e \geq \prod_{e\in EX}(k_{t(e)} \prod_{f\in EX,\, A_{ef}=1} y_f^{1/k_{i(f)}}).$$

On the right hand side of the equation, each term $y_f^{1/k_{i(f)}}$ appears exactly $k_{i(f)}$ times, since each edge f follows exactly $k_{i(f)}$ edges with terminal vertex $i(f)$. Canceling $\prod\limits_{e\in EX} y_e >$ 0 from each side, we get

$$e^{2h} \geq \prod_{e \in EX} k_{t(e)} = \prod_{x \in VX} k_x^{(k_x+1)}, \tag{2.4}$$

since $\sum_{e \in EX} \ell(e) = 2$. The equality holds if and only if equality in the inequality (2.3) holds for each $e \in EX$, i.e. the y_f's, for $f \in EX$ following e, are all equal.

Suppose that the equality in the inequality (2.4) holds. In particular,

$$h = \frac{1}{2} \sum_{x \in VX} (k_x + 1) \log k_x.$$

Since the valency at each vertex is at least 3, we can choose another edge $g \neq f$ followed by e and conclude that y_f depends only on the initial vertex $i(f)$ of f. Let $z_{i(f)} = y_f > 0$. Then the equation (2.2) in Theorem 2.1.2 amounts to

$$e^{h\ell(e)} z_{i(e)} = \sum_{f \in EX} A_{ef} z_{i(f)} = k_{t(e)} z_{t(e)}.$$

Since $\ell(e) = \ell(\bar{e})$, we also have $e^{h\ell(e)} z_{t(e)} = k_{i(e)} z_{i(e)}$. Thus $z_{i(e)}/z_{t(e)} = k_{t(e)}/e^{h\ell(e)} = e^{h\ell(e)}/k_{i(e)}$ and

$$e^{h\ell(e)} = \sqrt{k_{i(e)} k_{t(e)}},$$

so that

$$\ell(e) = \frac{\log k_{i(e)} k_{t(e)}}{\sum_{x \in VX} (k_x + 1) \log k_x}. \tag{2.5}$$

In particular, ℓ is uniquely defined by this formula. The length distance defined by the formula (2.5) clearly satisfies the equations (2.2), with

$$h = \frac{1}{2} \sum_{x \in VX} (k_x + 1) \log k_x,$$

and x_e's defined, uniquely up to constant, by setting

$$x_e = \sqrt{k_{t(e)}}.$$

By uniqueness in Theorem 2.1.2, the positive number h given above is the volume entropy of the given length distance, and it is the minimal entropy of the graph. □

Corollary 2.2.2. *If X is a $(k_1 + 1, k_2 + 1)$-biregular graph, with $k_1 > 1, k_2 > 1$, then the volume entropy of the normalized length distances on X is minimized exactly when the lengths of the edges are all equal, and the minimal volume entropy is $\frac{|EX|}{4} \log(k_1 k_2)$.*

Proof. Suppose that X a (k_1+1, k_2+1)-biregular graph, i.e. $k_{i(e)}k_{t(e)} = k_1k_2$ for any edge e. Let $d = d_\ell \in \Delta(X)$ be the entropy-minimizing length distance. Then $\ell(e) = \frac{1}{2h}\log(k_1k_2)$ does not depend on e, thus $\ell(e) = \frac{2}{|EX|}$. From $e^{h\ell(e)} = \sqrt{k_{i(e)}k_{t(e)}}$, the volume entropy of this length distance is $h = \frac{|EX|}{4}\log(k_1k_2)$. □

Corollary 2.2.3. *If X is a $(k+1)$-regular graph, with $k>1$, then the volume entropy of the normalized length distances on X is minimized exactly when the lengths of the edges are all equal, and the minimal volume entropy is $\frac{|EX|}{2}\log k$.*

Proof. This is a special case of the above corollary with $k_1 = k_2 = k$. □

Remark. The last corollary appears implicitly in a preprint of I. Rivin ([Riv]). There he considers graphs with weights given on the vertices rather than the edges. The directed line graph $L(X)$ of a graph X is an oriented graph defined so that $VL(X) = EX$ and $EL(X) = \{(a,b) \in EX^2 : t(a) = i(b), a \neq \bar{b}\}$. To a given set of weights on the edges $\{\ell(e)\}_{EX}$, is associated a set of weights $\{\ell'(x)\}_{VL(X)}$ on the vertices of $L(X)$. One can sees that paths on X without backtracking correspond to paths with backtracking on $L(X)$, see [Riv] page 14. The minimum of volume entropy of the graph $L(X)$ with vertex weights $h((\ell'(x)))_{VL(X)}$ (computed by I. Rivin) lies in the image of the map $(\ell(e)) \mapsto (\ell'(x))$ only when the graph is regular. It seems that for general graphs, one result cannot be deduced from the other.

Remark. Corollary 2.2.3 was also shown independently by I. Kapovich and T. Nagnibeda [KN] by a different method (using random walks). Note that one of their main results, on the minimal entropy among all graphs having a fixed fundamental group, can be deduced from Theorem 2.2.1 as in the following corollary. A special case when the graph has a highly transitive automorphism group had been shown earlier by G. Robert ([Rob]).

Corollary 2.2.4. *([KN] Theorem B) Consider the set of all finite metric graphs without a vertex of valency one or two, whose fundamental group is a free group of given rank $r \geq 2$. Then among volume 1 length metrics, the volume entropy is minimized by any (regular) trivalent graph in this set, with the metric assigning the same length for every edge.*

Proof. Let (X,d) be such a graph. Suppose that there is a vertex x of valency k_x+1 strictly greater than three, with outgoing edges $e_1, \dots, e_{k_x+1}$. Let us introduce a new vertex y and a new edge f, and replace x and its outgoing edges $e_1, \cdots, e_{k_x+1}$, by two vertices x and y, with outgoing edges

$f, e_3, \cdots, e_{k_x+1}$ and $e_1, e_2, \bar{f}$, respectively. Repeat the operation on x, until the valency of x reduces to three, to get a new graph X'. The graph X' has $k_x - 2$ more vertices than X, all with valency three.

Let d_0 and d_0' be the unique normalized entropy-minimizing length distances on X and X', respectively. By the formula in Theorem 2.2.1, since for $t \geq 3$, $(t+1)\log t > (t-1)3\log 2$, it follows that

$$\begin{aligned} h_{\mathrm{vol}}(X, d) \geq h_{\mathrm{vol}}(X, d_0) &= \frac{1}{2} \sum_{z \in VX - \{x\}} (k_z+1)\log k_z + (k_x+1)\log k_x \\ &> \frac{1}{2} \sum_{z \in VX - \{x\}} (k_z+1)\log k_z + (k_x-1)3\log 2 = h_{\mathrm{vol}}(X', d_0'). \end{aligned}$$

Repeat the operation until we get a regular trivalent graph. Now by Corollary 2.2.3, the volume entropy is minimized when all the edges have the same length. □

2.3 Entropy for graphs of groups

As another corollary of Theorem 2.2.1, let us show the analogous result of Theorem 2.2.1 for graphs of groups. Let $(X, G_\bullet)$ be any finite connected graph of finite groups. (Basic references for graphs of groups are [Se] and [Ba].) Let T be a (Bass-Serre) universal covering tree of $(X, G_\bullet)$ and let $p : T \to X$ be the canonical projection. The *degree of a vertex x of* $(X, G_\bullet)$ is defined by

$$\sum_{e \in EX, i(e)=x} \frac{|G_x|}{|G_e|}.$$

Note that this is usually different from the valency of x in the graph X. It is easy to see that it is equal to the valency of any lift of x in VT, and we will denote it again by $k_x + 1$. We define a *length distance d_ℓ on* $(X, G_\bullet)$ as a length distance d_ℓ on the underlying graph X. The *volume of* $(X, G_\bullet, d_\ell)$ for a given length distance d_ℓ on $(X, G_\bullet)$, is defined by

$$\mathrm{Vol}_\ell(X, G_\bullet) = \frac{1}{2} \sum_{e \in EX} \frac{\ell(e)}{|G_e|}.$$

Note that in the case where $\ell(e)$ is equal to 1 for every edge e and T is k-regular, the volume $\mathrm{Vol}_\ell(X, G_\bullet)$ is $k/2$ times the usual definition of the volume $\sum_{x \in VX} 1/|G_x|$ of a graph of groups since $k = \sum_{e \in EX, i(e)=x} |G_x|/|G_e|$. The *volume entropy* $h_{\mathrm{vol}}(X, G_\bullet, d_\ell)$ *of* $(X, G_\bullet, d_\ell)$ is defined to be the exponential growth of the balls in T for the lifted metric as in the case of graphs.

Proposition 2.3.1. *Let $(X, G_\bullet)$ be a finite graph of finite groups such that the degree at each vertex x of $(X, G_\bullet)$ is at least three. Among the normalized (i.e. volume one) length distances on $(X, G_\bullet)$, there exists a unique normalized length distance minimizing the volume entropy. At this minimum, the length of each edge is proportional to $\log(k_{i(e)}k_{t(e)})$ and the minimal volume entropy is*

$$h_{\min}(X, G_\bullet) = \frac{1}{2}\sum_{x \in VX} \frac{(k_x + 1)\log k_x}{|G_x|}.$$

Proof. Let Γ be a fundamental group of the graph of groups $(X, G_\bullet)$. There exists a free normal subgroup Γ' of Γ of finite index (see [Se]), say m. The group Γ' acts freely on T, hence the quotient graph $X' = \Gamma' \backslash T$ is a finite connected graph. It is easy to see that each x in VX (resp. e in EX) has $\frac{m}{|G_x|}$(resp. $\frac{m}{|G_e|}$) lifts in VX' (resp. EX') by the canonical map $\pi : X' \to X$, since

$$m = [\Gamma : \Gamma'] = \frac{\sum_{x' \in VX'} 1}{\sum_{x \in VX} 1/|G_x|}$$

(see [Ba] for example). It is clear that for every y in EX', the valency $k_y + 1$ is equal to the degree $k_{\pi(y)} + 1$. Any length distance d_ℓ of volume one on $(X, G_\bullet)$ can be lifted to X' to define a length distance $d_{\ell'}$ normalized so that $\ell'(e) = \ell(\pi(e))$ for every e in EX', and

$$\mathrm{Vol}_{\ell'}(X') = \frac{1}{2}\sum_{e \in EX'} \ell'(e) = \frac{1}{2}\sum_{e \in EX} \frac{m}{|G_e|}\ell(e) = m.$$

The volume entropy of (X', d'_ℓ) is equal to the volume entropy of $(X, G_\bullet, d_\ell)$ as they have the same universal covering metric tree. By the homogeneity property 2.1, we can apply Theorem 2.2.1 to conclude that among the length distances of volume m on X', there exists a unique entropy-minimizing length distance $d'_0 = d_{\ell'}$ on X'. By uniqueness in Theorem 2.2.1, the length distance d'_0 is invariant under the group Γ/Γ'. In particular, there is a normalized length distance $d_0 = d_\ell$ on $(X, G_\bullet)$ whose lift to X' defines d'_0. The minimal volume entropy of $(X, G_\bullet)$ is clearly the volume entropy of (X', d'_0) since for any length distance d on $(X, G_\bullet)$,

$$h_{\mathrm{vol}}(X, G_\bullet, d) = h(X', d') \geq h(X', d'_0) = h_{\mathrm{vol}}(X, G_\bullet, d_0),$$

where d' is defined as the lift of d on X'. Since the length $\ell'(e)$ of an edge e is proportional to $\log(k_{i(e)}k_{t(e)}) = \log(k_{\pi(i(e))}k_{\pi(t(e))})$ for every edge e in EX', so is true for every edge e in EX. Since each vertex x in VX appears $\frac{m}{|G_x|}$ times in X' and the degree $k_x + 1$ is equal to the valency $k_{x'} + 1$

of any lift $x' \in \pi^{-1}(x)$ of x in X', the minimal volume entropy of $(X, G_\bullet)$ is

$$\begin{aligned} h_{d_0}(X, G_\bullet) &= h(X', d_0') = \frac{1}{m} h(X', \frac{1}{m} d_0') = \frac{1}{2m} \sum_{x' \in VX'} (k_x' + 1) \log k_x' \\ &= \frac{1}{2m} \sum_{x \in VX} \frac{m}{|G_x|} (k_x + 1) \log k_x = \frac{1}{2} \sum_{x \in VX} \frac{(k_x + 1) \log k_x}{|G_x|}. \end{aligned}$$

□

Now we want to consider a more general situation than in Proposition 2.3.1. The main theorem in [BCG] says that if $f : (Y, g) \to (X, g_0)$ is a continuous map of non-zero degree between compact connected n-dimensional Riemannian manifolds and g_0 is a locally symmetric metric with negative curvature, then

$$h^n(Y, g)\mathrm{vol}(Y, g) \geq |\deg f| h^n(X, g_0)\mathrm{vol}(X, g_0),$$

and the equality holds if and only if f is homotopic to a Riemannian covering.

Let $(X, G_\bullet, d_0 = d_\ell)$ be a finite (connected) graph of finite groups endowed with the normalized length distance minimizing the volume entropy. Let $(Y, H_\bullet, d)$ be another finite graph of finite groups with a length distance. Let $\phi = (\phi, \phi_\bullet, \gamma_\bullet) : (Y, H_\bullet) \to (X, G_\bullet)$ be a (Bass-Serre) covering of graphs of groups (see [Ba]). The value

$$n := \sum_{y \in \phi^{-1}(x)} \frac{|G_x|}{|H_y|} = \sum_{f \in \phi^{-1}(e)} \frac{|G_e|}{|H_f|}$$

does not depend on the vertex x nor on the edge e of X since the graph X is connected, and it is an integer. A covering graph of groups with the above n is said to be *n-sheeted* (see [L1]).

When $(Y, H_\bullet)$ and $(X, G_\bullet)$ are graphs (of trivial groups), the next corollary can be considered as an analog of the main theorem in [BCG].

Corollary 2.3.2. *Let $\phi : (Y, H_\bullet) \to (X, G_\bullet)$ be a n-sheeted covering of graphs of groups and let d_0 be the entropy-minimizing length distance on $(X, G_\bullet)$ of volume one. Suppose that the degree at each vertex of $(X, G_\bullet)$ and $(Y, H_\bullet)$ is at least three. Then there holds*

$$h_{\mathrm{vol}}(Y, H_\bullet, d) vol(Y, H_\bullet, d) \geq n\ h_{\mathrm{vol}}(X, G_\bullet, d_0) vol(X, G_\bullet, d_0).$$

The equality holds if and only if the length distance d on $(Y, H_\bullet)$ is a length distance minimizing entropy among the length distances of the same volume, and in that case the map ϕ is a metric covering from $(Y, H_\bullet, d)$ to $(X, G_\bullet, \lambda d_0)$, for some $\lambda > 0$.

Proof. By the homogeneity property (2.1), we may assume that $\mathrm{vol}(Y, H_\bullet, d) = 1$. Applying Proposition 2.3.1 to $(Y, H_\bullet)$ and $(X, G_\bullet)$, it follows that there exists a unique length distance $d_0' = d_{\ell'}$ on Y minimizing the volume entropy and that

$$\begin{aligned} h_{\mathrm{vol}}(Y, H_\bullet, d) \geq h_{\min}(Y, H_\bullet) &= \frac{1}{2}\sum_{y\in VY}\frac{(k_y+1)\log k_y}{|H_y|} = \frac{1}{2}\sum_{x\in VX}\sum_{y\in\phi^{-1}(x)}\frac{(k_x+1)\log k_x}{|H_y|} \\ &= \frac{1}{2}n\sum_{x\in VX}\frac{(k_x+1)\log k_x}{|G_x|} = nh_{\min}(X, G_\bullet) = nh_{\mathrm{vol}}(X, G_\bullet, d_0). \end{aligned}$$

By Proposition 2.3.1, the equality holds if and only if $d = d_0'$. In that case, the length of each edge e in EY is proportional to $\log(k_{i(e)}k_{t(e)}) = \log(k_{i(\phi(e))}k_{t(\phi(e))})$, thus proportional to the length of the edge $\phi(e)$. More precisely, let $\ell'(e) = c'\log(k_{i(e)}k_{t(e)})$ for every $e \in EY$ and let $\ell(e) = c\log(k_{i(e)}k_{t(e)})$ for every $e \in EX$. From the assumption $\mathrm{vol}_\ell(X, G_\bullet) = \mathrm{vol}_{\ell'}(Y, H_\bullet) = 1$, it follows that

$$1 = \frac{1}{2}\sum_{g\in EY}\frac{c'\log(k_{i(g)}k_{t(g)})}{|H_g|} = \frac{1}{2}\sum_{e\in EX}\sum_{g\in\phi^{-1}(e)}\frac{c'\log(k_{i(g)}k_{t(g)})}{|H_g|} = \frac{1}{2}\sum_{e\in EX}\frac{nc'\log(k_{i(e)}k_{t(e)})}{|G_e|},$$

and therefore

$$c' = \frac{1}{\frac{n}{2}\sum\limits_{e\in EX}\frac{\log(k_{i(e)}k_{t(e)})}{|G_e|}} = \frac{c}{n},$$

in other words, $\ell'(e) = \ell(e)/n$.

We conclude that for any length distance d on $(Y, H_\bullet)$, there holds

$$h_{\mathrm{vol}}(Y, H_\bullet, d)vol(Y, H_\bullet, d) \geq n\, h_{\mathrm{vol}}(X, G_\bullet)vol(X, G_\bullet, d_0).$$

By Proposition 2.3.1 the equality holds if and only if d is proportional to d_0', say $d = \lambda n d_0'$ for some $\lambda > 0$. Then the length of each edge e in $(Y, H_\bullet, d)$ is $\lambda\ell(\phi(e))$, and the map ϕ is a metric covering from (Y, d_0') to $(X, \lambda\, d_0)$. $\square$

2.4 Volume entropy and topological entropy of geodesic flows

Throughout this section, let $d = d_\ell$ be a length metric on X, and $M = M(d)$ be the diameter of X. Set $\ell_m = \min\limits_{x\neq y\in VX}|x-y|$. For every $\delta > 0$, let us choose $L = L(\delta)$ big enough so that $\int_L^\infty e^{-t}dt \leq \frac{1}{4M}\delta$. We will denote by $\tilde{X}$ a fixed universal cover of X and by $\tilde{d}$ the pull-back distance induced by d on $\tilde{X}$.

Let $\mathcal{G}_d\tilde{X}$ denote the set of isometries $f : \mathbb{R} \to \tilde{X}$ such that $f(0)$ is a vertex of $\tilde{X}$. Let $\mathcal{G}_dX$ denote the set of local isometries $f : \mathbb{R} \to X$ such that $f(0)$ is a vertex of X. There is a surjective map

$p : \mathcal{G}_d\tilde{X} \to \mathcal{G}_d X$, defined by $p(f) = \pi \circ f$ where $\pi : \tilde{X} \to X$ is the natural projection. In other words, $\mathcal{G}_d X = \mathcal{G}_d\tilde{X}/\Gamma$ where Γ is the fundamental group of the finite graph X, acting by composition in the target on $\mathcal{G}_d\tilde{X}$.

Proposition 2.4.1. *The space of geodesics $\mathcal{G}_d X$ is a compact space for the compact open topology (which coincides with the quotient topology of the compact open topology on $\mathcal{G}_d\tilde{X}$). Its topology is induced by the metric*

$$D(f,g) = \int_{-\infty}^{\infty} d(f(t), g(t))e^{-|t|}dt.$$

Proof. This is well-known (see for instance [Bo]). Here we give a proof for the sake of completeness. Recall that the compact open topology on $\mathcal{C}_0(\mathbb{R}, \tilde{X})$ is equal to the topology of uniform convergence on compact sets since $\tilde{X}$ is a metric space. Suppose that we are given an arbitrary sequence $(f_n)_{n\in\mathbb{N}}$ of local geodesics in X. We want to show that there exists a subsequence converging to a local geodesic on X.

For any natural number m, consider the sequence $(f_n|_{I_m})_{n\in\mathbb{N}}$ of f_n restricted to the interval $I_m = [-m, m]$. Since f_n are geodesics, the sequence is equicontinuous and bounded. By Arzela-Ascoli theorem, there exists a uniformly converging subsequence. By passing to subsequences if necessary as m increases, we conclude that the sequence f_n converges to a fixed geodesic g uniformly on any interval I_m. Therefore $\mathcal{G}_d X$ is a compact space.

Let us show that the topology is induced by the metric given in the proposition. To show that any open ball of radius ϵ for the metric D is an open set with respect to the compact open topology $\mathcal{T}$, let us show that the complement $C = \{g \in \mathcal{G}_d X : D(f,g) \geq \epsilon\}$ is closed for $\mathcal{T}$. Suppose that the sequence $(g_n)_{n\in\mathbb{N}}$ in C converges to g for $\mathcal{T}$. Take an arbitrary small positive number δ. By the definition of the compact open topology, there exists n big enough such that $d(g_n(t), g(t)) \leq \frac{1}{4}\delta$ for every t in $[-L, L]$. Then

$$D(g_n, g) \leq \int_{-L}^{L} d(g_n(t), g(t))e^{-|t|}dt + 2M\int_{L}^{\infty} e^{-|t|}dt \leq \delta/2 + 2M\frac{\delta}{4M} = \delta.$$

It follows that

$$D(f,g) \geq D(f, g_n) - D(g_n, g) \geq \epsilon - \delta.$$

Since δ is arbitrary, we conclude that g is in the set C, thus C is a closed set.

It remains to show that any open set containing f with respect to the compact open topology contains an open ball containing f with respect to the given metric. Let us recall that the collection of subsets $\{f : f(K) \subset U\}$ for a compact set K and an open set U forms a subbasis of the compact open topology. Let $K_1, \cdots, K_n$ be fixed compact sets in $\mathbb{R}$ and $U_1, \cdots, U_n$ be fixed open sets in X. Let $B = \{f \in \mathcal{G}_d X : f(K_i) \subset U_i, \quad \forall i = 1, \cdots, n\}$ be an element of the basis. We want to show that there exists an $\epsilon > 0$ such that if $D(f,g) < \epsilon$ and $f \in B$, then $g \in B$.

From $D(f,g) < \epsilon$, it follows that $\int_{K_i} d(f(t), g(t))e^{-|t|} \leq \epsilon$. Since f and g are local geodesic lines (parametrized by arc length), the slope of the real function $t \mapsto d(f(t), g(t))$ (outside the countably many t for which $f(t)$ or $g(t)$ is a vertex) is one of 2, 0 and -2. By the triangular inequality, for every t_0, if $d(f(t_0), g(t_0)) > \delta$, then $d(f(t), g(t)) > \frac{\delta}{2}$ for any $t \in [t_0 - \delta/4, t_0 + \delta/4]$, thus $\int_{K_i} d(f(t), g(t))dt > \frac{\delta}{2} \min\{\frac{\delta}{2}, \mathrm{diam} K_i\}$. Therefore by letting $r_0 = \frac{1}{2} \min\{\mathrm{diam}\, K_1, \cdots, \mathrm{diam}\, K_n, \frac{\delta}{2}\}$, we conclude that if $\int_{K_i} d(f(t), g(t))dt < \delta r_0$, then $d(f(t), g(t)) < \delta$ for any t in K_i. For each i, set $m_i = \min\{e^{-|t|} : t \in K_i\}$, so that $\int_{K_i} d(f(t), g(t))e^{-|t|}dt < \epsilon$ implies $d(f(t), g(t)) < \frac{\epsilon}{m_i r_0}$ for any t in K_i.

Now choose $\epsilon_i > 0$ such that the ϵ_i-neighborhood $B(f(K_i), \epsilon_i)$ of $f(K_i)$ is still contained in U_i. Then, by choosing ϵ such that $\epsilon < \epsilon_i m_i r_0$, it follows that $\int d(f(t), g(t))e^{-|t|}dt < \epsilon$ implies $\int_{K_i} d(f(t), g(t))e^{-|t|}dt < \epsilon$ for each i, thus it implies that $d(f(t), g(t)) < \frac{\epsilon}{m_i r_0} < \epsilon_i$ for any t in K_i, therefore $g(t) \in U_i$ for any $t \in K_i$. □

There is a natural flow $\phi = (\phi^s)_{s \in \mathbb{R}}$ on $\mathcal{G}_d X$ defined by the $\mathbb{R}$-action on the domain, i.e., by the rule $\phi^s f(t) = f(t+s)$. This flow is called the *geodesic flow* on $\mathcal{G}_d X$.

Proposition 2.4.2. *The map $\phi^s : f \mapsto \{t \mapsto f(t+s)\}$ is a homeomorphism of $\mathcal{G}_d X$ onto itself. In particular, the time-one map ϕ^1 is a homeomorphism.*

Proof. Let us first show that ϕ^1 is a homeomorphism. The map ϕ^1 is obviously bijective. Let ϵ be a given positive number. Choose δ so that $\delta(e^{-1} + e) < \epsilon$. Suppose that $D(f,g) < \delta$. Then

$$\begin{aligned} D(\phi^1(f), \phi^1(g)) &= \int_{-\infty}^{1} d(f(t), g(t))e^{t-1}dt + \int_{1}^{\infty} d(f(t), g(t))e^{1-t}dt \\ &= e^{-1} \int_{-\infty}^{0} d(f(t), g(t))e^{t}dt + e \int_{0}^{\infty} d(f(t), g(t))e^{-t}dt \\ &+ \int_{0}^{1} d(f(t), g(t))e^{t-1}dt - \int_{0}^{1} d(f(t), g(t))e^{1-t}dt. \end{aligned}$$

Since $e^{t-1} \leq 1$ on $[0,1]$, the value of the last two terms

$$\int_0^1 d(f(t), g(t))e^{t-1}dt - \int_0^1 d(f(t), g(t))e^{1-t}dt = \int_0^1 d(f(t), g(t))(e^{t-1} - \frac{1}{e^{t-1}})dt$$

is at most 0. Thus $D(\phi'(f), \phi'(g)) \leq (e^{-1} + e)\delta < \epsilon$. We just showed that ϕ^1 is continuous (and even $(e + e^{-1})$-Lipschitz). The map $(\phi^1)^{-1}$ maps t to $f(t-1)$, and the proof of its continuity (and being $(e + e^{-1})$-Lipschitz), is analogous to the proof for phi^1. Therefore ϕ^1 is a homeomorphism of $\mathcal{G}_d X$ to $\mathcal{G}_d X$. In exactly the same way, we can show that ϕ^s is a homeomorphism for any s between 0 and 1 since $e^{t-s} \leq 1$ on $[0, 1]$. On the other hand, since ϕ^1 is a homeomorphism, the map ϕ^n is a homeomorphism for any integer n. Thus ϕ^s is a homeomorphism for any real number s. $\square$

Let $h(\phi) = h_{top}(\phi)$ be the topological entropy of the flow ϕ. Let us recall that the number $h(\phi)$ is independent of the choice of metric on $\mathcal{G}_d X$ but depends on d and that it is defined in the two following equivalent ways.

Remark. The topological entropy $h(\phi^1)$ where ϕ^1 is a map coincides with the entropy defined below (see [Man] for instance).

A subset Y of $\mathcal{G}_d X$ is called a *(T,δ)-separated set for ϕ* if for any two different f and g in Y, there exists some t, $0 \leq t \leq T$ such that $D(\phi^t f, \phi^t g) \geq \delta$. Let $N(T,\delta)$ be the maximum cardinality of a (T,δ)-separated set. Then

$$h(\phi) = \sup_{\delta>0} \limsup_{T\to\infty} \frac{\log N(T,\delta)}{T}.$$

A subset Z of $\mathcal{G}_d X$ is called a *(T,δ)-spanning set for ϕ* if for any f in $\mathcal{G}_d X$, there exists g in Z such that $D(\phi^t f, \phi^t g) \leq \delta$ for every t with $0 \leq t \leq T$. Let $M(T,\delta)$ be the minimum cardinality of a (T,δ)-spanning set. Then, $h(\phi) = \sup_{\delta>0} h(\phi,\delta)$ where

$$h(\phi,\delta) = \limsup_{T\to\infty} \frac{\log M(T,\delta)}{T}.$$

In [Man], Manning showed that for a compact Riemannian manifold of non-positive curvature, $h(\phi) = h_{\mathrm{vol}}(d)$. Here is the analogous result, claimed without proof in [Gui].

Theorem 2.4.3. *The volume entropy $h = h_{\mathrm{vol}}(d)$ is equal to the topological entropy $h(\phi)$ of the geodesic flow on $\mathcal{G}_d \tilde{X}$.*

Proof. If X (which has no terminal vertex) is reduced to a cycle, then $h = 0$. Hence we assume that $\pi_1(X)$ is not cyclic. We will call a subset Y of $\tilde{X}$ *δ-separated* if for any x, y in Y, the distance $d(x, y)$ between them is at least δ. For simplicity, let us denote $B(r) = B_{\tilde{X},\tilde{d}}(x_0, r)$.

Let us consider an "annulus" $B(r+\delta/2) - B(r)$. By the definition of $h_{\mathrm{vol}}(d)$ and the remark following it, for any $\epsilon > 0$, there exists r_ϵ such that

$$\exp((h-\epsilon)r) \leq \ell(B(r)) \leq \exp((h+\epsilon)r),$$

for any $r \geq r_\epsilon$. Moreover, there exists a sequence $(r_i)_{i\in\mathbb{N}}$ covering to ∞ such that for every $i \in \mathbb{N}$,

$$\exp((h-\epsilon)r_i) \leq \ell(B(r_i+\delta/2) - B(r_i)) \leq \exp((h+\epsilon)r_i),$$

for otherwise, by a summation argument, the growth rate of $\ell(B(r))$ would be bounded above by $h-\epsilon$.

Let us choose such an r in $(r_i)_{i\in\mathbb{N}}$ and take a maximal 2δ-separated subset Q_r of $B(r+\delta/2)-B(r)$. Then

$$|Q_r| \geq \frac{\ell(B(r+\delta/2) - B(r))}{\sup_{x\in X} \ell(B(x,\delta))} \geq c_\delta \exp((h-\epsilon)r).$$

For any q in Q_r, let f_q be a geodesic line from x_0 to q such that $f_q(0) = x_0$. We want to show that the set of geodesics $\{\pi \circ f_q : q \in Q_r\}$ in $\mathcal{G}_d X$ is a (r,δ)-separated set. For let f_q and $f_{q'}$ be two geodesics with $q \neq q'$. Since $d(f_q(r), q) < \delta/2$ and $d(f_{q'}(r), q') < \delta/2$, we have $d(f_q(r), f_{q'}(r)) > 2\delta - \delta = \delta$. Thus there exists s such that $0 < s < r$ and that $f_q = f_{q'}$ on $[0,s]$ and $f_q \neq f_{q'}$ on $]s, s+l_{\min}]$. Note also that $\pi(f_q) \neq \pi(f_{q'})$ on $]s, s+l_m]$ since they coincide from time 0 to s. Therefore,

$$D(\phi^s(\pi(f_q)), \phi^s(\pi(f_{q'}))) > \int_0^{l_{\min}/4} 2te^{-t}dt = 2(1 - \frac{l_{\min}/4+1}{e^{l_{\min}/4}}) > \delta,$$

if δ is small enough. Thus

$$h(\phi) \geq \limsup_{n\to\infty} \frac{\log |Q_{r_n}|}{r_n} \geq h - \epsilon.$$

Since ϵ is arbitrary, the topological entropy $h(\phi)$ is greater or equal to the volume entropy h.

Now for the inequality $h(\phi) \leq h$, let us choose a bounded fundamental domain F of X in $\tilde{X}$ which contains the vertex x_0 and let α be the diameter of F. Let Q_r be a maximal δ-separated set in an "annulus", this time, $B(F,r) - B(F, r-\alpha) = \{x \in \tilde{X} : r - \alpha \leq \tilde{d}(x,F) = \inf_{y\in F} \tilde{d}(x,y) \leq r\}$. Let E be a maximal δ-separated set in F. With $L = L(\delta)$ as in the beginning of Section 2.4, let $S = S(E,L) = \{x \in \tilde{X} : \exists y \in E, d(x,y) = L\}$. Since the set E is finite, so is S.

Claim. The set $\{\pi \circ f_{p,o,q} : p \in S,\ o \in E,\ q \in Q_r\}$ where $f_{p,o,q}$ is a fixed geodesic line through p, q such that $f_{p,o,q}(0)$ is of distance at most δ from o is an $(r-L, 4\delta)$-spanning set in $\mathcal{G}_d X$.

Proof of claim. For any element of $\mathcal{G}_dX$, we can find a geodesic line $g \in \mathcal{G}_d\tilde{X}$ which represent the given element of $\mathcal{G}_dX$ such that $g(0)$ is in the fundamental domain F. Since E is a maximal δ-separated set, there exists $o \in E$ such that $d(g(0), o) < \delta$. Since Q_r is a maximal δ-separated set, there exists $q \in Q_r$ such that $d(g(r), q) < \delta$. Choose a point p in S such that $d(g(-L), p) < \delta$. Then the geodesic from p to q passes through a δ-neighborhood of a point $o \in E$, and if L and r are large enough, we can normalize so that $d(f_{p,o,q}(0), o) = \delta$ (there are only two possible choices). Then for every s in $[0, r-L]$,

$$\begin{aligned}
D(\phi^s\pi \circ g, \phi^s\pi \circ f_{p,o,q}) &\leq \int_{-\infty}^{\infty} d(\pi \circ g(t), \pi \circ f_{p,o,q}(t))e^{-|t-s|}dt \\
&\leq \int_{\infty}^{-L+s} Me^{|t-s|}dt + \int_{-L+s}^{L+s} 3\delta e^{-|t-s|}dt + \int_{L+s}^{\infty} Me^{-|t-s|}dt \\
&\leq \delta/2 + 3\delta \int_{-L}^{L} e^{-|t|}dt + \delta/2 \\
&\leq \delta + 6\delta(1 - e^{-L}) \leq 7\delta.
\end{aligned}$$

The second inequality above holds since $L + s \leq r$ and $d(g(t), f_{p,o,q}(t)) \leq 3\delta$, for every t in $[-L, r]$, as $d(f_{p,o,q}(0), g(0)) \leq 2\delta$. □

By the above claim, we have a $(r-L, 7\delta)$-spanning set in $\mathcal{G}_dX$ of cardinality at most $2|S|\cdot|E|\cdot|Q_r|$. Thus $h(\phi, \delta) \leq \limsup\limits_{r\to\infty} \frac{\log(|S||E||Q_r|)}{r} \leq \limsup\limits_{r\to\infty} \frac{\log(|S||E||B_{\tilde{X},\tilde{d}}(x_0, r+a)|)}{r} = h_{\text{vol}}(d)$. The second inequality holds simply because the Q_r is a subset of the ball $B_{\tilde{X},\tilde{d}}(x_0, r+a)$ and the last equality holds by definition (and since $|S||E|$ are constants once δ is fixed). Since δ is arbitrary, we conclude that $h(\phi) \leq h_{\text{vol}}(d)$. □

2.4.1 Entropy associated to the first return map of geodesic flows

Consider the space $\mathcal{G}_dX^{(0)} = \{f \in \mathcal{G}_dX : f(0) \in VX\}$ of geodesics whose value at time 0 is a vertex of X with the induced topology. The *first return map* of geodesic flow is the map $R_d : \mathcal{G}_dX^{(0)} \to \mathcal{G}_dX^{(0)}$ defined as follows. If $\tau : \mathcal{G}_dX^{(0)} \to \mathbb{R}$ is the *first return time* of the geodesic flow in $\mathcal{G}_d(X)^{(0)}$, i.e. $\tau(f) = \inf\limits_{t>0} \{t > 0 : \phi^t f(0) \in VX\}$, then $R_d(f)$ coincides with $\phi^{\tau(f)}(f)$. (Note that the map ϕ on the right side depends on the metric d.) In other words, $\mathcal{G}_dX^{(0)} = \bigcup\limits_{x\in VX} \bigcup\limits_{e\in EX, i(e)=x} B_e$, where $B_e = \{f \in \mathcal{G}_dX : f([0, \tau(f)]) = e\}$ is an open and closed subset of $\mathcal{G}_dX^{(0)}$ (but not open in $\mathcal{G}_dX$). (Recall that $i(e)$ is the initial vertex of the edge e.) On each B_e, the map R_d coincides with $\phi^{l(e)}$.

Proposition 2.4.4. *The space $\mathcal{G}_dX^{(0)}$ is compact and R_d is a continuous map on $\mathcal{G}_dX^{(0)}$.*

Proof. Recall that $\mathcal{G}_d X$ is a compact space with the quotient topology of the topology of uniform convergence on compact sets. Then if a sequence of geodesics $(f_n \in \mathcal{G}_d X^{(0)})_{n\in\mathbb{N}}$ converges to a geodesic $f \in \mathcal{G}_d X$, the sequence $(f_n(0))_{n\in\mathbb{N}}$ converges to $f(0) \in X$. Since $f_n(0) \in VX$ for every n and $VX \subset X$ is discrete (thus closed), it follows that $f(0) \in VX$. Hence $\mathcal{G}_d X^{(0)}$ is a closed subset of a compact space, thus it is compact.

As $\mathcal{G}_d X^{(0)}$ is the union of the open sets B_e for e in EX, and R_d coincides with the continuous map $\phi^{l(e)}$ on B_e, the map R_d is continuous. □

Let $h_{top}(R_d)$ be the topological entropy of the continuous map R_d. We will see in the next section that it is a combinatorial object and it depends only on the graph structure of X.

2.4.2 Symbolic coding for the first return map of the geodesic flow

Let F be a finite set, equipped with the discrete topology. Consider the space $\Sigma_F = F^{\mathbb{Z}}$ with the product topology. For $n_1 < n_2 < \cdots < n_k$ and $a_1, \cdots, a_k \in F$ we call

$$C^{n_1,\cdots,n_k}_{a_1,\cdots,a_k} = \{\omega \in \Sigma_F : \omega_{n_j} = a_j, \;\; \text{for } j = 1, \cdots, k\}$$

a cylinder and k the rank of that cylinder. Cylinders form a base for the product topology of Σ_F. The topology is given by any metric

$$d_\lambda(\omega, \omega') = \lambda^{\max\{n\in\mathbb{N}:\, \omega_k = \omega'_k, \;\; |k|\leq n\}}$$

with $\lambda \in (0, 1)$. Then any symmetric cylinder $C^{-n,\cdots,n}_{a_{-n},\cdots,a_n}$ of rank $2n+1$ is a λ^n-ball. Let σ be the shift on $\Sigma : \sigma(\omega)_n = \omega_{n+1}$. Then (Σ_F, σ) is called a *symbolic dynamical system.*

Now let X be a finite oriented graph. Consider the space of two-sided sequences of edges $EX^{\mathbb{Z}}$ where EX is the set of oriented edges of X. We can define the product topology, and a shift σ on it as in the previous paragraph. Let (Σ_{EX}, σ) be the symbolic dynamical system defined in this way. Let $A = (A_{ef})_{e,f\in EX}$ be a matrix with entries $a_{ef} = 1$ if the terminal vertex of e coincides with the initial vertex of f and $\overline{f} \neq e$, and $A_{ef} = 0$ otherwise. Let

$$\Sigma_A = \{\omega \in \Sigma_{EX} : A_{\omega_n \omega_{n+1}} = 1, \forall n \in \mathbb{Z}\}.$$

The set Σ_A is obviously σ-invariant. The restriction $\sigma|_{\Sigma_A} = \sigma_A$ is call the *subshift of finite type associated to A.*

Remark. For any geodesic flow on the unit tangent bundle of a compact C^∞-Riemannian manifold of negative curvature, or more generally for Anosov flow, Bowen and Ratner constructed a suspension flow of a subshift of finite type "more or less" representing the given flow. In [CP], Coornaert and Papadopoulos studied symbolic coding for the geodesic flow associated to word hyperbolic groups. Since $(\mathcal{G}_d X^{(0)}, R_d)$ is purely combinatorial, i.e., it does not depend, up to isomorphism of continuous dynamical systems , on the length distance d on X, its coding is obtained in a similar way (see [CP] p. 488-489 where the case when X is the Cayley graph of a free group is described).

Proposition 2.4.5. *The dynamical system $(\mathcal{G}_d X^{(0)}, R_d)$ is topologically conjugate to the two-sided subshift of finite type (Σ_A, σ_A).*

Proof. We want to show that there exists a homeomorphism $q : \mathcal{G}_d X^{(0)} \to \Sigma_A$ satisfying $q \circ R_d = \sigma_A \circ q$.

Since every geodesic line f is entirely determined by the sequence

$$x = x_f = (\cdots, x_{-1}, x_0, x_1, \cdots) \in \Sigma_{\mathrm{N}}$$

of its consecutive edges such that $i(x_0) = f(0)$, the map q is naturally defined by sending f to the sequence x_f where $i(x_n) = (R_d)^n f(0)$ is the n-th edge of the geodesic f along the positive direction. The sequence x_f is clearly an element of Σ_A. It is also clear that $T \circ q(f)_n = q \circ R_d(f)_n$.

Let us show that q is continuous. Recall that $\mathcal{G}_d X^{(0)}$ is equipped with the restriction of the topology of uniform convergence on compact sets where as Σ_A is equipped with the product topology of $EX^{\mathbb{Z}}$ with discrete topology on EX, in other words, the topology of pointwise convergence. Suppose that the sequence $(f_n)_{n \in \mathbb{N}}$ of local geodesics converges to a local geodesic f. Then $f_n(0)$ converges to $f(0)$. Since $f_n(0) \in VX$ and VX is discrete, $f_n(0) = f(0)$ for large enough n. Similarly for any k in $\mathbb{Z}$, $(R_d)^k f_n(0)$ converges to $(R_d)^k f(0)$, thus the k-th edge of f_n coincides with that of f for large enough n. Therefore for large enough n, $q(f_n)_k = q(f)_k$, thus $q(f_n)$ converges pointwise to $q(f)$.

The inverse map q^{-1} sends x to a local geodesic line f whose n-th edge is x_n, for every $n \in \mathbb{Z}$. If x_m converges to x pointwise, then again by the discreteness of EX, $x_m = (x_n)$ for any large enough n, thus $q^{-1}((x_n)_m) = q^{-1}((x_n))$. Therefore q^{-1} is also continuous. □

By the above proposition, the topological entropy of $(\mathcal{G}_d X^{(0)}, R_d)$ is equal to that of (Σ_A, σ_A). Let us recall that the topological entropy of a subshift of finite type is the spectral radius of A (See

[HK], pp. 120-121 for instance). In fact, we can also find the entropy maximizing measure as follows. For the given irreducible matrix A (see section 3.1), let $q = (q_1, \cdots, q_N)$ and $v = (v_1, \cdots, v_N)$ be positive eigenvectors of A and A^T with positive eigenvalue λ equal to the spectral radius of A, respectively, normalized so that $\sum_{i=1}^N q_i v_i = 1$. These vectors exist and are unique up to scalars by Perron-Frobenius theorem. Let $\Pi = (\pi_{ij})$ be the matrix given by $\pi_{ij} = \frac{a_{ij} v_i}{\lambda v_j}$. It is easy to see that Π is a stochastic matrix. The Markov measure μ_Π of Π on Σ_A (which is σ_A invariant) is called the Parry measure and it is the unique measure of maximal entropy. (See [HK] pp. 174-177.)

2.4.3 Suspension flow on finite subshift and the geodesic flow

Now let us fix a length metric d_ℓ on the graph X. Let $r : \Sigma_A \to \mathbb{R}^+$ be the height function defined by $r(x) = \ell(x_0)$. We define the suspension flow (Σ_A^r, σ_A^r) on Σ_A as follows :

$$\Sigma_A^r = \{(x,t) \in \Sigma_A \times \mathbb{R} : 0 \leq t \leq r(x)\}/(x, r(x)) \sim (\sigma(x), 0),$$

$$\sigma_A^r(x,t) = (\sigma_A(x), t).$$

Proposition 2.4.6. *The geodesic flow $(\mathcal{G}_d X, \phi_t)$ is topologically conjugate to the suspension flow (Σ_A^r, σ_t^r) of the subshift of finite type (Σ_A, σ).*

Proof. A geodesic line f in $\mathcal{G}_d X$ determines a unique sequence $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$ of its consecutive edges, such that $f(0)$ belongs to x_0, and does not belong to x_1 (but $f(0)$ can belong to both x_{-1} and x_0 if it is the origin of x_0). Let us define a map $q : \mathcal{G}_d X \to \Sigma_A^r$ by sending f to (x, s) where s is the distance between the initial vertex $i(x_0)$ of the edge at time 0 and the point $f(0)$. The inverse map q^{-1} sends (x, s) to a local geodesic line f whose n-th edge is x_n and such that $f(0)$ is of distance s from $i(x_0)$. The maps q and q^{-1} are clearly extensions of the functions q and q^{-1}, respectively, defined in the proof of Proposition 2.4.5. The proof continuity of q and q^{-1} is similar to the proof of Proposition 2.4.5. □

By the above proposition, $h_{vol}^A(d) = h_{top}(\sigma_A^r)$. Now $h_{top}(\sigma_A^r) = max_{\bar{\mu}} h_{\bar{\mu}}(\sigma_A^r)$ where $h_{\bar{\mu}}(\sigma_A^r)$ is the measure theoretic entropy of (Σ_A^r, σ_A^r) and the maximum is over all σ_A^r-invariant probability measures $\bar{\mu}$ on Σ_A^r.

There is a one-to-one correspondence between the σ-invariant probability measures on Σ_A and σ_A^r-invariant probability measures on Σ_A^r. Let μ be a σ-invariant probability measure, and $h_\mu(\sigma)$ be the measure-theoretic (metric) entropy of σ with respect to μ. If we denote by $\bar{\mu} = \mu \times dt$ the probability measure on Σ^r_A corresponding to μ, then

$$h_{\bar{\mu}}(\sigma^r_A) = \frac{h_\mu(\sigma_A)}{\int r d\mu} \tag{2.6}$$

(see [Ab]).

Remark. The characterization of the volume entropy described in this section gives an alternative approach to find the minimal volume entropy, namely, by finding the minimal value of $h_\mu(\sigma_A)/\int r d\mu$ where μ varies over all σ-invariant probability measures on Σ_A.

2.5 Another characterization of the volume entropy

In this section, we give another characterization of the volume entropy, using Equation 2.6 in the last section. As in the last section, μ denotes a σ-invariant probability measure, $\overline{\mu}$ the $(\sigma^r)_A$-invariant probability measure, and $h_{\overline{\mu}}$ the measure-theoretic entropy of σ^r_A with respect to $\overline{\mu}$.

Now let us calculate the metric entropy of the suspension flow. Let $\mathcal{P}_m$ be the set of admissible sequences of length m in Σ_A. For simplicity, let us denote any admissible sequence $a_0a_1\cdots a_{m-1}$ in $\mathcal{P}_m$ by $\underline{a}$. For any $\underline{a} \in \mathcal{P}_m$, let $C^{k,\cdots,m+k-1}_{\underline{a}}$ be the set of bi-infinite sequences x whose entries x_{k+i} are a_i, for $i = 0,\cdots,m-1$. Let $\mathcal{B}^k_m$ be the collection of cylinders $C^{k,\cdots,m+k-1}_{\underline{a}}$ where $\underline{a}$ run over all the elements in $\mathcal{P}_m$. Denote $\mathcal{B}_m = \mathcal{B}^0_m$ and $C^{0,\cdots,m-1}_{\underline{a}} = C_{\underline{a}}$ for simplicity. For every $\underline{a} \in \mathcal{P}_m$, let $\ell_{\underline{a}} = \ell_{a_0} + \ell_{a_1} \cdots + \ell_{a_{m-1}}$, and let $\mu_{\underline{a}} = \mu(B_{\underline{a}})$. Let $S_m r(x) = r(x) + r(\sigma x) + r(\sigma^2 x) + \cdots + r(\sigma^{m-1}x) = \ell(x_0) + \cdots + \ell(x_{m-1})$. If we let $H_m = -\frac{1}{m}\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}} \log \mu_{\underline{a}}$, it is well known that $h_\mu(\sigma_A) = \lim_{m\to\infty} H_m$. For any σ_A-invariant probability measure μ, it is easy to see that $\int r d\mu = 1/m \int S_m r d\mu$. Therefore by the last equality in Section 2.4.3,

$$\begin{aligned} h_{\bar{\mu}}(\sigma^r_A) &= \lim_{m\to\infty} \frac{H_m}{(1/m)\int S_m r d\mu} = \lim_{m\to\infty} \frac{-\frac{1}{m}\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}} \log \mu_{\underline{a}}}{\frac{1}{m}\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}} l_{\underline{a}}} \\ &= \lim_{m\to\infty} \frac{-\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}} \log \mu_{\underline{a}}}{\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}} l_{\underline{a}}} \end{aligned}$$

Lemma 2.5.1. *Let $n \geq 1$, and let $0 < \mu_i < 1$ and $\alpha = \sum_{i=1}^n \mu_i \in (0,1)$. Let $\ell_i > 0$, for $i = 1,\cdots,n$, be given positive number which are not all equal. Then the function*

$$f(\mu_1,\cdots,\mu_n) = -\frac{\sum \mu_i \log \mu_i}{\sum \mu_i \ell_i}$$

is maximized when $\mu_i = \exp(-c\ell_i)$, where c is the unique positive constant which satisfies

$$\sum_{i=1}^{n} exp(-c\ell_i) = \alpha.$$

Proof. Let us use calculus of variation. The maximum is attained when $\nabla f = \lambda \nabla g$ for some constant λ, where $g : (\mu_1, \cdots, \mu_n) \mapsto \sum_{i=1}^{n} \mu_i$. Since $\frac{\partial g}{\partial \mu_j} = 1$, we should find μ_i's so that $\frac{\partial f}{\partial \mu_j}$ does not depend on j.

$$\frac{\partial f}{\partial \mu_j} = -\frac{(1+\log \mu_j)(\sum \mu_i \ell_i) - (\sum \mu_i \log \mu_i)\ell_j}{(\sum \mu_i \ell_i)^2} = \lambda \tag{2.7}$$

It follows that $(\log \mu_j - \log \mu_i)(\sum \mu_i l_i) = (l_j - l_i)(\sum \mu_i \log \mu_i)$, for every i, j. Let $\log \mu_j = c_1 + c_2 \ell_j$. Then

$$\frac{\partial f}{\partial \mu_j} = -\frac{(1+c_1+c_2\ell_j)(\sum \mu_i \ell_i) - (\sum \mu_i (c_1 + c_2 \ell_j))\ell_j}{(\sum (\mu_i \ell_i))^2} = -\frac{(\sum \mu_i \ell_i) + c_1(\sum \mu_i \ell_i - \alpha \ell_j)}{(\sum (\mu_i \ell_i))^2} \tag{2.8}$$

Thus $c_1 = 0$, unless all the ℓ_i are of the same length. Therefore the extremal value is attained when $\mu_j = \exp(-c\ell_j)$ where c satisfies $\sum_{i=1}^{n} \exp(-c\ell_i) = \alpha$. The value of f at this point is c and since $\frac{\partial f}{\partial \mu_j} = -\frac{1}{\sum \mu_i \ell_i} < 0$ for all j at this point, it is a maximal value. On the boundary, say when $\mu_1 = 0$ for example, by the same calculation, the value of f is c' where $\sum_{i=2}^{n} exp(-c'\ell_i) = 1$. Since $c' < c$, the value of f on the boundary is strictly smaller than c. Thus the global maximum of f is attained at $\mu_i = \exp(c\ell_i)$ with maximum value c. □

By Lemma 2.5.1, we have the inequality

$$\frac{-\sum\limits_{\underline{a} \in \mathcal{P}_m} \mu_{\underline{a}} \log \mu_{\underline{a}}}{\sum\limits_{\underline{a} \in \mathcal{P}_m} \mu_{\underline{a}} l_{\underline{a}}} \leq c_m,$$

where c_m is the only positive number such that $\sum\limits_{\underline{a} \in \mathcal{P}_m} e^{-c_m \ell_{\underline{a}}} = 1$.

Lemma 2.5.2. $\lim\limits_{m \to \infty} c_m$ *exists.*

Proof. Let $n < m$. Since any path of length m is a concatenation of a path of length n and a path of length $m-n$, we have $\mathcal{B}_m^0 \subset \mathcal{B}_n^0 \cap \mathcal{B}_{m-n}^n := \{C_{a_0, \cdots, a_{n-1}}^{0, \cdots, n-1} \cap C_{a_n, \cdots, a_{m-1}}^{n, \cdots, m} : a_0 \cdots a_{n-1}$ and $a_n \cdots a_{m-1}$ are admissible paths$\}$. It follows that

$$\sum_{\underline{b} \in \mathcal{P}_n} e^{-c_m \ell_{\underline{b}}} \sum_{\underline{c} \in \mathcal{P}_{m-n}} e^{-c_m \ell_{\underline{c}}} \geq \sum_{\underline{a} \in \mathcal{P}_m} e^{-c_m \ell_{\underline{a}}} = 1.$$

Thus for every $n < m$, either $c_m \leq c_n$ or $c_m \leq c_{m-n}$, and clearly $c_m \leq c_{m/d}$ for any d which divides m. Therefore $c_m \leq \inf\{\sup c_{n_1}, \cdots, c_{n_r} : m \in \mathbb{N}n_1 + \cdots + \mathbb{N}n_r\}$.

Now let $c = \liminf_{n\to\infty} c_n$ and let $\epsilon > 0$. By the above observation, it suffices to find two integers p and q relatively prime such that $c_p < c+\epsilon$ and $c_q < c+\epsilon$. (Then any big enough integer m can be written as a linear combination of p and q, thus $c_m < c+\epsilon$, thus $\lim c_n \leq c$.)

Suppose that such p and q do not exist. Let $d = \min\{\gcd(p,q) : c_p < c+\epsilon, c_q < c+\epsilon\} > 1$. Fix $D \in d\mathbb{N}$ such that $c_D < c+\epsilon$. By the definition of d, $c_{Dt+1} \geq c+\epsilon$ for any integer t.

$$\begin{aligned}1 &\leq \sum_{\underline{a}\in\mathcal{P}_{Dt+1}} \exp(-c_{Dt+1}l_{\underline{a}}) \leq (\sum_{\underline{b}\in\mathcal{P}_D} \exp(-c_{Dt+1}l_{\underline{b}}))^t(\sum_{\underline{c}\in\mathcal{P}_1} \exp(-c_{Dt+1}l_{\underline{c}}))\\ &\leq (\sum_{\underline{b}\in\mathcal{P}_D} \exp(-(c+\epsilon)l_{\underline{b}}))^t(\sum_{\underline{c}\in\mathcal{P}_1} \exp(-(c+\epsilon)l_{\underline{c}}))\end{aligned}$$

Therefore

$$\frac{1}{\sum_{\underline{c}\in\mathcal{P}_1} \exp(-(c+\epsilon)\ell_{\underline{c}})} \leq (\sum_{\underline{b}\in\mathcal{P}_D} \exp(-(c+\epsilon)l_{\underline{b}}))^t.$$

Since $c_D < c+\epsilon$, it follows that $(\sum_{\underline{b}\in\mathcal{P}_D} e^{-(c+\epsilon)\ell_{\underline{b}}}) < 1$, which leads to a contradiction since the right hand side in the above equation tends to 0 as t tends to infinity, where as the left hand side of the equation is a positive constant. Therefore we conclude that $d = 1$ and the infimum limit $c = \liminf_{m\to\infty} c_m$ is in fact the limit of c_m as $m \to \infty$. □

Therefore

$$h_{\bar{\mu}}(\sigma_A^r) = \lim_{m\to\infty} \frac{-\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}} \log \mu_{\underline{a}}}{\sum_{\underline{a}\in\mathcal{P}_m} \mu_{\underline{a}}(l_{\underline{a}})} \leq c$$

We just showed that for any σ-invariant probability measure μ, we have $h_\mu \leq c$. Since the volume entropy $h_{\text{vol}}(d)$ is the supremum of the metric entropy $h_{\bar{\mu}}(\sigma_A^r)$ over all σ-invariant probability measure $\bar{\mu}$ on the space Σ_A^r, we conclude that $h_{\text{vol}}(d) \leq c$.

Proposition 2.5.3. $h_{\text{vol}} = c$.

Proof. First notice that $c = \lim_{m\to\infty} c_m \leq c_m$ for any m since c_{m_i} is decreasing for any geometric sequence m_i. It follows that for any integer m,

$$\sum_{i,j\in EX} A(c)_{i,j}^{m-1} \geq \sum_{i,j} A(c_m)_{i,j}^{m-1} = \sum_{\underline{a}\in\mathcal{P}_m} e^{-c_m\ell_{\underline{a}}} = 1. \tag{2.9}$$

To show the inequality $c \leq h_{\text{vol}}$, let us use the criterion from Theorem 2.1.2 : the volume entropy is the unique positive number h such that the matrix $A(h)$ defined by $a(h)_{ij} := a_{ij}exp(-h\ell_j)$ has the largest eigenvalue 1. If $c > h_{\text{vol}}$, then the entries of $A(c)$ are strictly smaller than $A(h_{\text{vol}})$, thus all the absolute values of eigenvalues of $A(c)$ are strictly less than 1. Thus $A(c)^n$ tends to the zero matrix as $n \longrightarrow \infty$, which is a contradiction to the equation 2.9.

Proposition 2.5.4. *If the graph X is regular, then* $\lim c_m(\ell_0) \leq \lim c_m(\ell)$ *for any* $\ell \neq \ell_0$.

Proof. We want to show that $c_m(\ell_0) < c_m(\ell)$ for any m, for any $\ell \neq \ell_0$.

By strict convexity of the function e^{-x}, we have that

$$\frac{1}{|\mathcal{P}_m|}\sum_{\underline{a}\in\mathcal{P}_m} e^{-c_m l(\underline{a})} > e^{-c_m \frac{\sum \ell(\underline{a})}{|\mathcal{P}_m|}}.$$

Note that

$$\{\underline{a} \in \mathcal{P}_m : \underline{a}_j = e\} = k^{m-1}$$

since once we fix an alphabet at position j, we have k choices for each position $i \neq j$, where the graph is $k+1$-regular. Since we have $|EX|$ choices for the alphabet at position j, we have

$$|\mathcal{P}_m| = |EX|k^{m-1}.$$

Therefore

$$\begin{aligned}\sum_{\underline{a}\in\mathcal{P}_m} \ell(\underline{a}) &= \sum_{j=1}^{m}\sum_{e}\{\underline{a} \in \mathcal{P}_m : \underline{a}_j = e\}\ell(e)\\ &= \sum_{j=1}^{m}\sum_{e} k^{m-1}\ell(e) = mk^{m-1}\sum_{e}\ell(e)\\ &= |\mathcal{P}_m| m \frac{\sum \ell(e)}{|EX|} = |\mathcal{P}_m|\ell_0(\underline{a})\end{aligned}$$

(the last equality comes from the definition of ℓ_0). We just showed that $\frac{\sum_{\underline{a}\in\mathcal{P}_m} \ell(\underline{a})}{|\mathcal{P}_m|} = \ell_0(\underline{a})$, thus we conclude that

$$1 = \sum_{\underline{a}\in\mathcal{P}_m} e^{-c_m(\ell)\ell_{\underline{a}}} > |\mathcal{P}_m| e^{-c_m(\ell)\ell(\underline{a})} = \sum_{\underline{a}\in\mathcal{P}_m} e^{-c_m(\ell)\ell_0\underline{a}} = 1,$$

i.e, $c_m(l_0) < c_m(l)$ for any m, therefore $c(l_0) \leq c(l)$. □

The following theorem is a corollary of the above proposition.

Theorem 2.5.5. *If the graph X is regular, then $h^d_{vol} \geq h^{d_0}_{vol}$, i.e., the volume entropy is minimized when the length of the edges are all equal.*

Chapitre 3

Overlattices in automorphism groups of trees

3.1 Overlattices and coverings of graphs of groups

In this section, we briefly recall some background on group actions on trees and the theory of graphs of groups, and we explain the correspondence between overlattices and coverings of graphs of groups. We refer the reader to [Se], [Ba] and [BL] for details on the standard material, gathered in section 3.1.1.

Throughout the paper, we denote by T a locally finite tree, i.e., a tree having finite valence at each vertex. We denote by $Aut(T)$ the group of automorphisms without inversions of the tree T. A subgroup Γ of $Aut(T)$ is *discrete* if the stabilizer Γ_x is finite for some, thus for every, vertex x of T. The *covolume* of Γ is defined by

$$Vol\ (\Gamma\backslash\backslash T) = \sum_{x \in \Gamma\backslash VT} \frac{1}{|\Gamma_x|}.$$

A discrete subgroup is a *lattice* if its covolume is finite. In this case, $Aut(T)$ is unimodular, and the covolume is equal (up to a constant depending only on T) to the volume of $\Gamma\backslash Aut(T)$ induced by the Haar measure on the locally compact group $Aut(T)$ [BL]. A lattice Γ is called *cocompact* if the quotient graph $\Gamma\backslash T$ is finite. An *overlattice* of Γ is a lattice of $Aut(T)$ containing Γ with finite index.

3.1.1 Cocompact lattices and finite graphs of finite groups

By a graph of groups $(X, G_\bullet)$, we mean a connected graph X, groups G_x and $G_e = G_{\overline{e}}$ assigned to each vertex x in VX and each edge e in EX, together with injections $G_e \to G_x$ for each edge e

In [P], Pyber showed that the number of isomorphism classes of groups of order n with a given Sylow set, namely the set of Sylow p_i-subgroups defined up to conjugacy, is at most $n^{75\mu+16}$. Together with the result of Sims ([Si]), namely $f(p^k) \leq p^{\frac{2}{27}k^3+\frac{1}{2}k^{\frac{8}{3}}}$, we get the following upper bound for $f(n)$:

$$\begin{aligned} f(n) &\leq \prod_{i=1}^{t} p_i^{\frac{2}{27}k_i^3+\frac{1}{2}k_i^{8/3}} n^{75\mu+16} \\ &\leq n^{\frac{2}{27}\mu^2+\frac{1}{2}\mu^{5/3}+75\mu+16} \end{aligned}$$

Let $g(n) = \frac{2}{27}\mu^2(n) + \frac{1}{2}\mu^{5/3}(n) + 75\mu(n) + 16$ so that $f(n) \leq n^{g(n)}$.

On the other hand, Lucchini and Guralnick showed that if every Sylow subgroup of G can be generated by d elements, then $d(G) \leq d+1$ ([Luc], [Gur]). Combining with the basic fact that $d(H) \leq n$ for any group H of order p^n ([Si]), we deduce that

$$d(G) \leq \mu + 1.$$

Using these results, we obtain the following upper-bound for $u(n)$.

Theorem 3.2.1. *Let Γ be a cocompact lattice of $Aut(T)$. Then there are some positive constants C_0 and C_1 depending only on Γ, such that*

$$\forall n > 1, \qquad u_\Gamma(n) \leq C_0 n^{C_1 log^2(n)}.$$

Lemma 3.2.2. *Any covering $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ is strongly isomorphic to a covering $\phi'_\bullet = (\phi', \phi'_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ where each γ'_x for $x \in VX \cup EX$ is a word, in $h_y \in G_y$'s ($y \in VY'$) and the edges $e \in EY'$, of length at most $12K$, where K is the diameter of X.*

Proof. Fix $x_0 \in X$. Associated to $\phi_\bullet$ is a lattice $\Gamma' \subset Aut((\widetilde{X, G_\bullet, x_0}))$ containing $\pi_1(X, G_\bullet, x_0)$. From $(X, G_\bullet, x_0)$ we construct (R, S, g_e) such that the quotient of $(\widetilde{X, G_\bullet, x_0})$ by $\pi_1(X, G_\bullet, x_0)$ is exactly $(X, G_\bullet)$. Namely, first fix a maximal tree τ in X. We may choose R to be the set of paths $e_1 \cdots e_n$ from x_0 in τ, S to be the set of paths $e_1 \cdots e_n e_{n+1}$ such that $e_1 \cdots e_n$ is a path in τ and $g_e = e'_1 \cdots e'_l {e_{n+1}}^{-1} \cdots {e_1}^{-1}$ where $e'_1 \cdots e'_l$ is a path in τ from x_0 to $t(e)$, and where e is the edge connecting $e_1 \cdots e_n$ to $e_1 \cdots e_{n+1}$. In particular, g_e is a product of at most twice the diameter of X number of generators of $\Pi(X, G_\bullet)$. Now we choose R', S' subsets of R, S in such a way that the restriction of the projection $(\widetilde{X, G_\bullet, x_0}) \to \Gamma' \backslash (\widetilde{X, G_\bullet, x_0})$ on R' is bijective for vertices (resp. the restriction of S' is bijective on edges). We also choose g'_e in a similar fashion as above, hence g'_e is also a product of at most twice the diameter of X number of generators of $\Pi(X, G_\bullet)$. From

$\pi_1(X, G_\bullet, x_0) = \pi[x_0, x_0]$ acts on $\widetilde{(X, G_\bullet, x_0)}$ by the natural left action. The graph $\widetilde{(X, G_\bullet, x_0)}$ is a tree and moreover, for any other universal cover (T, Γ) of $(X, G_\bullet)$, there is an isomorphism ψ between Γ and $\pi_1(X, G_\bullet, x_0)$ and a ψ-equivariant graph isomorphism between T and $\widetilde{(X, G_\bullet, x_0)}$, see for example [Se].

A graph of groups is called *faithful (or effective)* if there is no edge subgroup family $(N_e)_{e \in EX}$ satisfying the following conditions :

i) for each e and e' in EX such that $o(e) = o(e')$, the images of N_e and $N_{e'}$ coincide : $\alpha_e(N_e) = \alpha_{e'}(N_{e'})$. Let us denote it by $N_{o(e)}$.

ii) For each x in VX, N_x is a nontrivial normal subgroup in G_x.

It is shown in [Ba] that the graph of groups $(X, G_\bullet)$ is faithful if and only if its fundamental group Γ is a subgroup of $Aut(T)$ for its universal cover T, i.e., if and only if the map $\Gamma \longrightarrow Aut(T)$ is injective. The fundamental group of a faithful finite graph of finite groups is a cocompact lattice in the automorphism group of its universal covering tree and conversely, a quotient graph of groups of a cocompact lattice in the automorphism group of a locally finite tree is a faithful finite graph of finite groups.

In [Ba], Bass defines a covering of graphs of groups in such a way that the induced map between the corresponding fundamental groups is a group monomorphism.

Definition 3.1.1. *Let $(X, G_\bullet)$ and $(Y, H_\bullet)$ be two graphs of groups. We call a morphism of graphs of groups, which we denote by $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$, the following data*

(i) a graph morphism $\phi : X \to Y$,

(ii) group homomorphisms $\phi_x : G_x \to H_{\phi(x)}$ and $\phi_e : G_e \to H_{\phi(e)}$, for every vertex x and every edge e of X,

(iii) families of elements $(\gamma_x)_{x \in VX} \in \pi_1(Y, H_\bullet, \phi(x))$ and $(\gamma_e)_{e \in EX} \in \Pi(Y, H_\bullet)$

such that for every edge e of X with origin x, we have $\gamma_x^{-1}\gamma_e \in H_{\phi(x)}$ and the following diagram commutes.

$$\begin{tikzcd}
G_e \arrow[r, "\alpha_e"] \arrow[d, "\phi_e"] & G_x \arrow[d, "\phi_x"] \\
H_{\phi(e)} \arrow[r, "ad(\gamma_x^{-1}\gamma_e) \circ \alpha_{\phi(e)}"] & H_{\phi(x)}
\end{tikzcd}$$

The *induced homomorphism of path groups* $\Phi = \Phi_{\phi_\bullet} : \Pi(X, G_\bullet) \to \Pi(Y, H_\bullet)$, is defined as follows on generators (see [Ba]) : $\Phi(g) = \gamma_x \phi_x(g) \gamma_x^{-1}$ for $g \in G_x$ and $x \in VX$, $\Phi(e) = \gamma_e \phi(e) \gamma_{\bar{e}}^{-1}$ for $e \in EX$. The induced homomorphism on path groups restricts to a homomorphism $\pi_1(X, G_\bullet, x_0) \to \pi_1(Y, H_\bullet, \phi(x_0))$, which we will denote again by Φ.

The induced homomorphism $\Phi = \Phi_{\phi_\bullet} : \pi_1(X, G_\bullet, x_0) \to \pi_1(Y, H_\bullet, \phi(x_0))$ gives a Φ_{x_o}-equivariant graph isomorphism $\tilde{\phi} : (X, \widetilde{G_\bullet, x_o}) \to (Y, \widetilde{H_\bullet, \phi}(x_0))$ defined by

$$[g] \in \pi[x_0, x]/G_x \mapsto [\Phi(g)\gamma_x] \in \pi[\phi(x_0), \phi(x)]/H_{\phi(x)}.$$

A morphism $\phi_\bullet = (\phi, \phi_x, \gamma_x)_{x \in VX \cup EX}$ of graphs of groups is an *isomorphism of graphs of groups* if ϕ is a graph isomorphism and ϕ_x are all group isomorphisms. In this case, $\phi_\bullet^{-1} = (\phi^{-1}, \phi'_y, \gamma'_y)$ where $\phi'_y = \phi_{\phi^{-1}(y)}$ and $\gamma'_y = \Phi^{-1}(\gamma_{\phi^{-1}(y)})^{-1}$ for $y \in VY \cup EY$.

Definition 3.1.2. *A morphism of graphs of groups $\phi_\bullet$ is furthermore called a covering if*

(a) the maps ϕ_e and ϕ_x are injective for all x and e,

(b) for every edge f of Y with origin $\phi(x)$,where x is in VX, the well-defined map

$$\begin{aligned} \Phi_{x/f} : \coprod_{e \in \phi^{-1}(f), o(e) = x} G_x/\alpha_e(G_e) &\longrightarrow H_{\phi(x)}/\alpha_f(H_f) \\ [g]_e &\longmapsto [\phi_x(g)\gamma_x^{-1}\gamma_e]_f \end{aligned}$$

is bijective.

By the condition (b) in Definition 3.1.2, we have $\sum_{e \in \phi^{-1}(f), o(e) = x} \frac{|G_x|}{|G_e|} = \frac{|H_{\phi(x)}|}{|H_f|}$ for every edge f of Y with origin $\phi(x)$. Summing over all vertices x such that $\phi(x) = y$, it follows that the value of

$$n := \sum_{x \in \phi^{-1}(y)} \frac{|H_y|}{|G_x|} = \sum_{e \in \phi^{-1}(f)} \frac{|H_f|}{|G_e|}$$

does not depend on vertices and edges, since the graph Y is connected. Note that n is an integer since $\phi_x(G_x)$ is a subgroup of H_y for each x such that $\phi(x) = y$. A covering graph of groups with the above n is said to be *n-sheeted.*

Note also that by the condition (b), a covering of graphs of groups induces a covering of the corresponding edge-indexed graphs. Recall that a *covering* $\phi : (X, i) \to (Y, i)$ *of edge-indexed graphs* is a graph morphism ϕ such that $\sum_{e \in \phi^{-1}(e'), o(e) = x} i(e) = i(e')$, for every x and for every e' of origine $\phi(x)$.

Theorem 3.1.3 ([Ba], Prop. 2.7). *The morphism $\phi_\bullet$ is a covering if and only if $\Phi : \pi_1(X, G_\bullet, x_0) \to \pi_1(Y, H_\bullet, \phi(x_0))$ is injective and $\tilde{\phi} : (X, \widetilde{G_\bullet, x_0}) \to (Y, \widetilde{H_\bullet, \phi}(x_0))$ is an isomorphism.*

3.1.2 Counting overlattices

Let Γ be a cocompact lattice in $Aut(T)$. Set

$$U(n) = U_\Gamma(n) = \{\Gamma' : \Gamma \subset \Gamma' \subset Aut(T),\ [\Gamma' : \Gamma] = n\}$$

and let $u(n) = u_\Gamma(n) = |U(n)|$ be the number of overlattices of Γ of index n. It is shown in [BK] that $u(n)$ is finite. We are interested in the asymptotic behavior of $u(n)$. For that purpose, we will show in this section that there is a bijection between overlattices of Γ and isomorphisms classes of coverings of graphs of groups by the quotient graph of groups of Γ, in the following sense.

Definition 3.1.4. *Let $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet = (\psi, \psi_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ be two coverings of graphs of groups. An isomorphism between them is an isomorphism of graphs of groups $\theta_\bullet = (\theta, \theta_y, \rho_y) : (Y, H_\bullet) \to (Y', H'_\bullet)$ such that $\theta \circ \phi = \psi$ as a map of graphs and the corresponding induced diagram of isomorphisms between universal covers*

$$\begin{tikzcd}
(\widetilde{X, G_\bullet, x_0}) \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"] & (\widetilde{Y, H_\bullet, \phi(x_0)}) \arrow[d, "\tilde{\theta}"] \\
& (\widetilde{Y', H', \psi(x_0)})
\end{tikzcd}$$

commutes.

It will also be useful to consider a more restricted notion of isomorphism of coverings. (As for now, we do not know whether the notions of isomorphism and strong isomorphism are equivalent. We use the following definition of strong isomorphism to prove the bijection between the set of overlattices and the isomorphism classes of coverings of complexes of groups.)

Definition 3.1.5. *Let $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet = (\psi, \psi_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ be two coverings of graphs of groups. A strong isomorphism between them consists of a pair $\{\theta_\bullet = (\theta, \theta_y, \rho_y) : (Y, H_\bullet) \to (Y', H'_\bullet), (\zeta_x)_{x \in VX \cup EX}\}$ where $\theta_\bullet$ is an isomorphism of graphs of groups $(Y, H_\bullet) \to (Y', H'_\bullet)$ and $(\zeta_x) \in H'_{\psi(x)}$ are such that*

a) $\theta \circ \phi = \psi$ as a map of graphs,

b) For any $x \in VX \cup EX$, we have $\psi_x = ad(\zeta_x^{-1})\theta_{\phi(x)} \circ \phi_x$ as maps $G_x \to H'_{\psi(x)}$,

c) $\gamma'_x = \Theta(\gamma_x)\rho_{\phi(x)}\zeta_x$ for any $x \in VX \cup EX$.

Lemma 3.1.6. *Any two strongly isomorphic coverings $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet = (\psi, \psi_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ are isomorphic.*

Proof. We have a triangle of morphisms of path groups

$$\begin{tikzcd} \Pi(X,G_\bullet) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \Pi(Y,H_\bullet) \arrow[d, "\Theta"] \\ & \Pi(Y',H'_\bullet) \end{tikzcd}$$

We claim that this triangle commutes. It is enough to check it on generators : let $x \in VX$ and $s \in G_x$. We have $\Phi(s) = \gamma_x \phi_x(s) {\gamma_x}^{-1}$, $\Psi(s) = \gamma'_x \psi_x(s) {\gamma'_x}^{-1}$ and on the other hand

$$\begin{aligned} \Theta \circ \Phi(s) =& \Theta(\gamma_x)\Theta(\phi_x(s))\Theta(\gamma_x)^{-1} \\ =& \Theta(\gamma_x)\rho_{\phi(x)}\theta_{\phi(x)}(\phi_x(s))\rho^{-1}_{\phi(x)}\Theta(\gamma_x)^{-1} \\ =& \Theta(\gamma_x)\rho_{\phi(x)}\zeta_x\psi_x(s)\zeta_x^{-1}\rho^{-1}_{\phi(x)}\Theta(\gamma_x)^{-1} \end{aligned} \tag{3.1}$$

(using property (b) of strong isomorphism of coverings), and this is equal to

$$= \gamma'_x \psi_x(s) {\gamma'_x}^{-1} = \Psi(s)$$

by property (c) and the definition of Ψ. Similarly, for $e \in EX$,

$$\begin{aligned} \Theta \circ \Phi(e) =& \Theta(\gamma_e)\Theta(\phi(e))\Theta(\gamma_{\bar{e}})^{-1} \\ =& \Theta(\gamma_e)\rho_{\phi(e)}\theta(\phi(e))\rho^{-1}_{\phi(\bar{e})}\Theta(\gamma_{\bar{e}})^{-1} \\ =& \Theta(\gamma_e)\rho_{\phi(e)}\psi(e)\rho^{-1}_{\phi(\bar{e})}\Theta(\gamma_e)^{-1} = \gamma'_e\psi(e){\gamma'_{\bar{e}}}^{-1}. \end{aligned} \tag{3.2}$$

The last equality comes from the fact that since $\zeta_e \in H'_{\psi(e)}$, by definition of the fundamental group,

$$\psi(e) = \zeta_e \psi(e) \zeta_{\bar{e}}^{-1}.$$

Thus we have a commuting triangle of morphisms of fundamental groups

$$\begin{tikzcd} \pi_1(X,G_\bullet,x_0) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \pi_1(Y,H,\phi(x_0)) \arrow[d, "\Theta"] \\ & \pi_1(Y',H',\psi(x_0)) \end{tikzcd}$$

(where Θ is an isomorphism), and a triangle of *isomorphisms* of trees, which is equivariant with respect to the above triangle of groups :

$$\begin{tikzcd} (\widetilde{X,G_\bullet,x_0}) \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & (Y,\widetilde{H_\bullet,\phi(x_0)}) \arrow[d, "\tilde{\theta}"] \\ & (Y',\widetilde{H',\psi(x_0)}). \end{tikzcd}$$

We claim that this triangle is also commutative. Indeed, by definition, if $g \in \pi[x_0, x]/G_x \subset (\widetilde{X, G_\bullet, x_0})$ then

$$\begin{aligned}\tilde{\theta}(\tilde{\phi}(g)) &= \tilde{\theta}(\Phi(g)\gamma_x) = \Theta(\Phi(g))\Theta(\gamma_x)\rho_{\phi(x)} \\ &= \Psi(g)\gamma'_x\zeta_x^{-1} = \Psi(g)\gamma'_x = \tilde{\psi}(g)\end{aligned}$$

where we used relation (c) together with the fact that $\zeta_x \in H'_{\psi(x)}$. Observe that

$$\Psi(g)\gamma'_x \in \pi[\psi(x_0), \psi(x)]/H'_{\psi(x)}.$$

The Lemma is proved. □

For a given overlattice Γ' of Γ, we can construct a covering $m^{\Gamma'}$ of graphs of groups as follows. Let $Y = \Gamma' \backslash T$ and $p' : T \to Y$ be the canonical projection.

Define subtrees R' and S' of R and S, respectively, in the following way. For each vertex y of Y, choose one vertex from each set $\{{p'}^{-1}(y)\} \cap VR$ and call it $\tilde{y}$. Let R' be the subgraph of R with vertices $\{\tilde{y} : y \in Y\}$. Since R is a tree, we can choose vertices $\tilde{y}$ so that R' is connected. Let S' be the maximal subtree of S containing R' such that $p'|_{S'}$ is injective on the edges. For $e \in EY$, choose elements $g'_e \in \Gamma'$ such that $g'_e o(\tilde{e}) = \widetilde{o(e)}$. The graph of groups $(Y, H_\bullet)$ is defined with respect to R', S' and g''s, as $(X, G_\bullet)$ is defined in section 1.1.

Now the covering of graphs of groups, which will be denoted by $m = m^{\Gamma'} : (X, G_\bullet) \to (Y, H_\bullet)$, is defined as follows. For the graph morphism $m : X \to Y$, take the natural projection π. For the group morphisms $m_x : G_x \to H_{m(x)}$, take an element σ_x in Γ' which sends $\tilde{x}$ to $\widetilde{m(x)}$. We can choose $\sigma_x = 1$ if $\tilde{x} \in VR' \cup ES'$. Note that $m(x)$ is a vertex of Y, thus $\widetilde{m(x)} \in R'$ whereas x is a vertex of X, thus $\tilde{x} \in R$. Let $m_x = ad(\sigma_x) \circ \iota$ be the injection followed by the conjugation $(g \mapsto \sigma_x g \sigma_x^{-1})$. Since G_x stabilizes $\tilde{x} \in VT \cup ET$, the group $\sigma_x G_x \sigma_x^{-1}$ stabilizes $\widetilde{p(x)} \in VT \cup ET$, thus it is a subgroup of $H_{p(x)} = \Gamma'_{\widetilde{p(x)}}$, for $x \in VX \cup EX$. For the elements γ_x, γ_e in (iii) of Definition 1.1, take $\gamma_x = \sigma_x^{-1}$ and $\gamma_e = g_e \sigma_e^{-1} g'^{-1}_{m(e)}$. It follows that

$$\begin{aligned}ad(\gamma_x^{-1}\gamma_e) \circ \alpha_{m(e)} \circ m_e &= ad(\gamma_x^{-1}\gamma_e) \circ ad(g'_{m(e)}) \circ ad(\sigma_e) \\ &= ad(\sigma_x g_e \sigma_e^{-1} g'^{-1}_{m(e)}) \circ ad(g'_{m(e)}) \circ ad(\sigma_e) \\ &= ad(\sigma_x g_e) = ad(\sigma_x) \circ ad(g_e) = m_x \circ \alpha_e.\end{aligned}$$

Since γ_x's are the elements of Γ', the map $m^{\Gamma'}$ is a morphism of graphs of groups. The maps m_x are clearly injective, thus it remains to show that the map $\Phi_{x/f}$ (in Definition 3.1.2 (b)) is bijective. Suppose that for $e, e' \in EX$ and $g, g' \in G_x$, we have $[\phi_x(g)\gamma_x^{-1}\gamma_e]_f = [\phi_x(g')\gamma_x^{-1}\gamma_{e'}]_f$ in $H_{\phi(x)}/\alpha_f H_f$. In other words,

$$\gamma_e^{-1}\gamma_x\phi_x(g^{-1}g')\gamma_x^{-1}\gamma_{e'} \in \alpha_f(H_f)$$

$$g_{m(e)}\sigma_e g_e^{-1}\sigma_x^{-1}\sigma_x g^{-1}g'\sigma^{-1}\sigma g_{e'}\sigma_{e'}^{-1}{g'_{m(e)}}^{-1} \in ad(g'_f)(H_f)$$

$$\sigma_e g_e^{-1}g^{-1}g' g_{e'}\sigma_{e'}^{-1} \in H_f = Stab_{\Gamma'}(\tilde{f})$$

Since σ_e sends $\tilde{e}$ to $\tilde{f}$ and $\sigma_{e'}$ sends $\tilde{e'}$ to $\tilde{f}$, the element $g_e^{-1}g^{-1}g'g_{e'}$ of Γ should send $\tilde{e'}$ to $\tilde{e}$. We conclude that $e = e'$ since no element of Γ sends $\tilde{e}$ to $\tilde{e'}$ where $e' \neq e$ in $X \simeq \Gamma\backslash T$. We conclude that $e = e'$ and $g^{-1}g' \in G_e$, i.e. $[g]_e = [g']_{e'}$. Therefore $m^{\Gamma'}$ is indeed a covering of graphs of groups.

Proposition 3.1.7. *Let Γ be a cocompact lattice of $Aut(T)$ and $(X, G_\bullet)$ be its quotient graph of groups. The map $\Gamma' \mapsto m^{\Gamma'}$ induces a bijection m between the set of overlattices of Γ of index n and the set of isomorphism classes of the n-sheeted coverings of faithful graphs of groups by $(X, G_\bullet)$.*

The following lemma shows that the map $m : \Gamma' \mapsto m^{\Gamma'}$ is well-defined.

Lemma 3.1.8. *Let Γ be a lattice in T, and let $\Gamma' \supset \Gamma$ be an overlattice. Fix (R, S, g_e) giving rise to a graph of groups structure $(X, G_\bullet)$ on $\Gamma\backslash T$ (as in section 1.1.). Let (R', S', g'_e) (resp. (R'', S'', g''_e)) be a data giving rise to a graph of groups structure $(Y, H_\bullet)$ (resp. $(Y', H'_\bullet)$) on $\Gamma'\backslash T$, and let $(\sigma'_x)_{x\in VX\cup EX}$ (resp. $(\sigma''_x)_{x\in VX\cup EX}$) be a data giving rise to a covering $\phi_\bullet = (\phi, \phi_x, \gamma'_x) : (X, G_\bullet) \to (Y, H_\bullet)$ (resp. $\psi_\bullet = (\psi, \psi_x, \gamma''_x) : (X, G_\bullet) \to (Y', H'_\bullet)$). Then the two coverings $\phi_\bullet$ and $\psi_\bullet$ are strongly isomorphic.*

Proof. Recall that by definition, we have $\sigma'_x : \tilde{x} \mapsto \widetilde{\phi(x)}$ and $\sigma''_x : \tilde{x} \mapsto \widetilde{\psi(x)}$, where σ'_x and σ''_x are in Γ'. Recall also that $\gamma'_x = {\sigma'_x}^{-1}, \gamma''_x = {\sigma''_x}^{-1}$ for $x \in VX$ and $\gamma'_e = g_e{\sigma'_e}^{-1}{g'_{\phi(e)}}^{-1}$, $\gamma''_e = g_e{\sigma''_e}^{-1}{g''_{\psi(e)}}^{-1}$ for $e \in EX$. Now we want to construct a strong isomorphism $\{\theta_\bullet : (Y, H_\bullet) \to (Y', H'_\bullet), \zeta_x\}$ of coverings of graphs of groups. First notice that there is a canonical bijection $\theta : Y \simeq \Gamma'\backslash T \simeq Y'$. It lifts to a bijection $\tilde{\theta} : R' \to R''$ and it extends to a unique bijection $\tilde{\theta} : S' \to S''$. Let us choose arbitrary elements $\xi_y \in \Gamma'$ for $y \in VY \cup EY$ such that $\xi_y(\tilde{y}) = \widetilde{\theta(y)}$ and define maps

$$\theta_y : H_y = \Gamma'_{\tilde{y}} \to \Gamma'_{\widetilde{\theta(y)}} = H'_{\theta(y)}, h \mapsto \xi_y h \xi_y^{-1}.$$

We have a morphism of graphs of groups $\theta_\bullet = (\theta, \theta_y, \rho_y) : (Y, H_\bullet) \to (Y', H'_\bullet)$ by setting $\rho_y = \xi_y^{-1}$ for $y \in VY$ and $\rho_e = {g'_e}^{-1}{\xi_e}^{-1}g''_{\theta(e)}$ for $e \in EY$. It is clear by construction that this is an isomorphism of graphs of groups (all maps are isomorphisms of groups). Note that there is a commutative diagram of isomorphisms

$$\begin{tikzcd}
\pi_1(Y, H_\bullet, y_0) \arrow[r, "i_Y"] \arrow[d, "\Theta"] & \Gamma' \arrow[d, "Id"] \\
\pi_1(Y', H'_\bullet, \theta(y_0)) \arrow[r, "i_{Y'}"] & \Gamma'
\end{tikzcd}$$

where we have denoted by i_Y, i'_Y the isomorphisms $\pi_1(Y, H_\bullet, y_0) \simeq \Gamma'$, and $\pi_1(Y', H'_\bullet, \theta(y_0)) \simeq \Gamma'$, respectively.

Finally, put $\zeta_x = \xi_{\phi(x)} \sigma'_x \sigma''^{-1}_x$. For any vertex x, there holds

$$\begin{aligned} Ad(\zeta_x^{-1}) \theta_{\phi(x)} \phi_x &= ad((\sigma''_x)(\sigma'_x)^{-1} \xi^{-1}_{\phi(x)}) \circ ad(\xi_{\phi(x)}) \circ ad(\sigma'_x) \\ &= ad((\sigma''_x)) = \psi_x \end{aligned}$$

as desired. A similar computation holds for $\psi_e : G_e \to H'_{\psi(e)}$ when $e \in EX$. This proves condition (b) in the definition of strong isomorphism of coverings. Condition (c) follows from the very definition of $\zeta_x, \sigma_y, \gamma'_x$ and γ''_x. □

Now let us define the inverse map $\phi_\bullet \mapsto \Gamma_\phi$ of m as follows. Set $\Gamma_Y := \pi_1(Y, H_\bullet, \phi(x_0)) \subset Aut((\widetilde{Y, H_\bullet, \phi(x_0)}))$. We define an embedding $i_\phi : \Gamma_Y \to Aut((\widetilde{X, G_\bullet, x_0}))$ as follows :

$$i_\phi(u) \cdot v = \tilde{\phi}^{-1}(u \cdot \tilde{\phi}(v)) \qquad \textit{for } u \in \Gamma_Y \textit{ and } v \in V(\widetilde{X, G_\bullet, x_0}) \cup E(\widetilde{X, G_\bullet, x_0}).$$

Let us denote by $\Gamma_\phi \subset Aut((\widetilde{X, G_\bullet, x_0}))$ the image of i_ϕ. The following lemma shows that this map is well-defined.

Lemma 3.1.9. *If $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ and $\psi_\bullet : (X, G_\bullet) \to (Y', H'_\bullet)$ are isomorphic coverings of graphs of groups, then the corresponding subgroups $\Gamma_\phi \subset Aut((\widetilde{X, G_\bullet, x_0}))$ and $\Gamma_\psi \subset Aut((\widetilde{X, G_\bullet, x_0}))$ coincide.*

Proof. By definition of isomorphic coverings, we have a triangle of *isomorphisms* of trees

$$\begin{tikzcd}
(\widetilde{X, G_\bullet, x_0}) \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & (\widetilde{Y, H_\bullet, \phi(x_0)}) \arrow[d, "\tilde{\theta}"] \\
& (\widetilde{Y', H', \psi(x_0)})
\end{tikzcd}$$

which is equivariant with respect to the action of the corresponding fundamental groups. Define $\Gamma_Y \subset Aut((\widetilde{Y, H_\bullet, \phi(x_0)}))$, an embedding $i_\phi : \Gamma_Y \to Aut((\widetilde{X, G_\bullet, x_0}))$ and put $\Gamma_\phi = Im(i_\phi) \subset$

$Aut((\widetilde{X, G_\bullet, x_0}))$ as above, and define Γ_ψ in the same fashion. We claim that $\Gamma_\phi = \Gamma_\psi$. Indeed, if $u \in \Gamma_Y$ then $\Theta(u) \in \Gamma_{Y'}$ and for $v \in (\widetilde{X, G_\bullet, x_0})$ we have

$$\begin{aligned} i_\phi(u) \cdot v &= \tilde{\phi}^{-1}(u \cdot \tilde{\phi}(v)) = \tilde{\phi}^{-1}(\tilde{\theta}^{-1}(\Theta(u) \cdot \tilde{\theta}(\tilde{\phi}(v))) \\ &= \tilde{\psi}^{-1}(\Theta(u) \cdot \tilde{\psi}(v)) = i_\psi(\Theta(u)) \cdot v. \end{aligned}$$

We deduce that $\Gamma_\phi \subset \Gamma_\psi$. Replacing $\theta_\bullet$ by its inverse and exchanging the roles of $\psi_\bullet$ and $\phi_\bullet$ we obtain the reverse inclusion $\Gamma_\psi \subset \Gamma_\phi$. Thus $\Gamma_\psi = \Gamma_\phi$ as desired. $\square$

Proof of Proposition 3.1.7. It remains to show that the map $\phi_\bullet \mapsto \Gamma_\phi$ is the inverse map of m. To see this, let $\Gamma' \supset \Gamma$ be an overlattice of Γ. The quotient graph of groups $\Gamma\backslash\backslash T = (X, G_\bullet)$ is formed relative to some datum (R, S, g_x) ; let us similarly choose datum (R', S', g'_x) inducing a quotient graph of groups $(Y, H_\bullet) = \Gamma'\backslash\backslash T$. Recall that by [Se], §5.4, there are, for any $x_0 \in VX$ and $y_0 \in VY$, canonical isomorphisms $\Gamma \simeq \pi_1(X, G_\bullet, x_0)$, $T \simeq (\widetilde{X, G_\bullet, x_0})$ and $\Gamma' \simeq \pi_1(Y, H_\bullet, y_0)$, $T \simeq (\widetilde{Y, H_\bullet, y_0})$. Choosing furthermore some elements σ_x's as in the proof of Lemma 3.1.8 we get a covering (see [Ba], Section 4.2)

$$m^{\Gamma'} : (X, G_\bullet) \to (Y, H_\bullet).$$

From [Ba], Proposition 4.2, the following diagrams commute :

$$\begin{tikzcd} T \arrow[r, "id"] \arrow[d] & T \arrow[d] \\ (\widetilde{X, G_\bullet, x_0}) \arrow[r, "\widetilde{m^{\Gamma'}}"] & (\widetilde{Y, H_\bullet, y_0}) \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & \Gamma' \arrow[d] \\ \pi_1(X, G_\bullet, x_0) \arrow[r, "M^{\Gamma'}"] & \pi_1(Y, H_\bullet, y_0) \end{tikzcd}$$

where we denote $M^{\Gamma'}$ the morphism of path groups induced by the covering $m^{\Gamma'}$.

In particular, the pullback of $\pi_1(Y, H_\bullet, y_0)$ via the composition of isomorphisms $T \simeq (\widetilde{X, G_\bullet, x_0}) \xrightarrow{\widetilde{m^{\Gamma'}}} (\widetilde{Y, H_\bullet, y_0})$ is equal to Γ'. This shows that $\phi_\bullet \mapsto \Gamma_\phi$ is a left inverse of $\Gamma' \mapsto m^{\Gamma'}$.

To prove the other direction, let $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ be a covering of $(X, G_\bullet)$ and set $\Gamma' = \Gamma_\phi \subset Aut((\widetilde{X, G_\bullet, x_0}))$. Now let $(Y', H'_\bullet)$ be the quotient graph of groups associated as in Section 3.1.1 to the action of Γ' on $(\widetilde{X, G_\bullet, x_0})$, relative to some choices, and let $\psi_\bullet : (X, G_\bullet) \to (Y', H')$ be a covering constructed as in Section 3.1.2. By construction there is an isomorphism $\tilde{\psi} : (\widetilde{X, G_\bullet, x_0}) \xrightarrow{\sim} (\widetilde{Y', H'_\bullet, \psi(x_0)})$, equivariant with respect to an embedding $\Psi : \pi_1(X, G_\bullet, x_0) \hookrightarrow \pi_1(Y', H'_\bullet, \psi(x_0))$, and by the first part of the proof of Proposition 3.1.7., we have $\Gamma' = i_\psi(\pi_1(Y', H'_\bullet, \psi(x_0)))$. Thus, composing $\tilde{\psi}^{-1}$ with $\tilde{\phi}$ and Ψ^{-1} with Φ yields an isomorphism of trees $\tilde{\theta} : (\widetilde{Y', H'_\bullet, \psi(x_0)}) \xrightarrow{\sim} (\widetilde{Y, H_\bullet, \phi(x_0)})$ which is equivariant with respect to an isomorphism $\Theta : \pi_1(Y', H'_\bullet, \psi(x_0)) \xrightarrow{\sim} \pi_1(Y, H_\bullet, \phi(x_0))$. At this point, we use the following Lemma :

Lemma 3.1.10 ([Ba], Prop. 4.4, Cor. 4.5.). *Let $(Z, K_\bullet)$ and $(W, J_\bullet)$ be two graphs of groups. For any isomorphism of trees $\tilde{\sigma} : (\widetilde{Z, K_\bullet, z_0}) \xrightarrow{\sim} (\widetilde{W, J_\bullet, w_0})$ which is equivariant with respect to an isomorphism of fundamental groups $\Sigma : \pi_1(Z, K_\bullet, z_0) \xrightarrow{\sim} \pi_1(W, J_\bullet, w_0)$ there exists a (unique) isomorphism of graphs of groups $\omega_\bullet(Z, K_\bullet) \to (W, J_\bullet)$ such that $\tilde{\Sigma} = \tilde{\omega}$ and $\Sigma = \Omega$.*

Using the above Lemma, we conclude that there exists an isomorphism $\theta_\bullet : (Y', H'_\bullet) \to (Y, H_\bullet)$ making the diagram

$$\begin{tikzcd}
(\widetilde{X, G_\bullet, x_0}) \arrow[r, "\tilde{\phi}"] \arrow[dr, "\tilde{\psi}"'] & (\widetilde{Y, H_\bullet, \phi(x_0)}) \arrow[d, "\tilde{\theta}"] \\
& ((\widetilde{Y', H', \psi(x_0)})
\end{tikzcd}$$

commute. Hence the coverings $\phi_\bullet$ and $\psi_\bullet$ are indeed isomorphic as desired.

Finally, we check that the above bijection sends an overlattice of index n to an n-sheeted covering. Let Γ' be an overlattice of Γ of index n. We claim that m^Γ is an n'-sheeted covering with $n = n'$. Indeed, we have

$$n = [\Gamma' : \Gamma] = \frac{vol(\Gamma \backslash\backslash T)}{vol(\Gamma' \backslash\backslash T)} = \frac{\sum_{x \in VX} \frac{1}{|G_x|}}{\sum_{y \in VY} \frac{1}{|H_y|}} = \frac{\sum_{y \in VY} \sum_{x \in \phi^{-1}(y)} \frac{1}{|G_x|}}{\sum_{y \in VY} \frac{1}{|H_y|}} = \frac{\sum_{y \in VY} \frac{n'}{|H_y|}}{\sum_{y \in VY} \frac{1}{|H_y|}} = n'.$$

Note that the first equality comes from the fact that T is a left Γ'-set (and Γ-set) with finite stabilizers (see [BL], page 16). □

It follows from Proposition 1.7 that to find $u(n)$, it suffices to count the number of isomorphism classes of coverings of faithful graphs of groups by $(X, G_\bullet)$.

3.2 Main results

3.2.1 Upper bound

Let G be a group of order n and let $n = \prod_{i=1}^{t} p_i^{k_i}$ be the prime decomposition of n. Let $\mu = \mu(n)$ be the maximum of k_i. We denote by $d(G)$ the minimal cardinality of a generating set of G and by $f(n)$ the number of isomorphism classes of groups of order n.

In [P], Pyber showed that the number of isomorphism classes of groups of order n with a given Sylow set, namely the set of Sylow p_i-subgroups defined up to conjugacy, is at most $n^{75\mu+16}$. Together with the result of Sims ([Si]), namely $f(p^k) \leq p^{\frac{2}{27}k^3+\frac{1}{2}k^{\frac{8}{3}}}$, we get the following upper bound for $f(n)$:

$$\begin{aligned} f(n) &\leq \prod_{i=1}^{t} p_i^{\frac{2}{27}k_i^3+\frac{1}{2}k_i^{8/3}} n^{75\mu+16} \\ &\leq n^{\frac{2}{27}\mu^2+\frac{1}{2}\mu^{5/3}+75\mu+16} \end{aligned}$$

Let $g(n) = \frac{2}{27}\mu^2(n) + \frac{1}{2}\mu^{5/3}(n) + 75\mu(n) + 16$ so that $f(n) \leq n^{g(n)}$.

On the other hand, Lucchini and Guralnick showed that if every Sylow subgroup of G can be generated by d elements, then $d(G) \leq d+1$ ([Luc], [Gur]). Combining with the basic fact that $d(H) \leq n$ for any group H of order p^n ([Si]), we deduce that

$$d(G) \leq \mu + 1.$$

Using these results, we obtain the following upper-bound for $u(n)$.

Theorem 3.2.1. *Let Γ be a cocompact lattice of $Aut(T)$. Then there are some positive constants C_0 and C_1 depending only on Γ, such that*

$$\forall n > 1, \qquad u_\Gamma(n) \leq C_0 n^{C_1 log^2(n)}.$$

Lemma 3.2.2. *Any covering $\phi_\bullet = (\phi, \phi_x, \gamma_x) : (X, G_\bullet) \to (Y, H_\bullet)$ is strongly isomorphic to a covering $\phi'_\bullet = (\phi', \phi'_x, \gamma'_x) : (X, G_\bullet) \to (Y', H'_\bullet)$ where each γ'_x for $x \in VX \cup EX$ is a word, in $h_y \in G_y$'s ($y \in VY'$) and the edges $e \in EY'$, of length at most $12K$, where K is the diameter of X.*

Proof. Fix $x_0 \in X$. Associated to $\phi_\bullet$ is a lattice $\Gamma' \subset Aut(\widetilde{(X, G_\bullet, x_0)})$ containing $\pi_1(X, G_\bullet, x_0)$. From $(X, G_\bullet, x_0)$ we construct (R, S, g_e) such that the quotient of $\widetilde{(X, G_\bullet, x_0)}$ by $\pi_1(X, G_\bullet, x_0)$ is exactly $(X, G_\bullet)$. Namely, first fix a maximal tree τ in X. We may choose R to be the set of paths $e_1 \cdots e_n$ from x_0 in τ, S to be the set of paths $e_1 \cdots e_n e_{n+1}$ such that $e_1 \cdots e_n$ is a path in τ and $g_e = e'_1 \cdots e'_l {e_{n+1}}^{-1} \cdots {e_1}^{-1}$ where $e'_1 \cdots e'_l$ is a path in τ from x_0 to $t(e)$, and where e is the edge connecting $e_1 \cdots e_n$ to $e_1 \cdots e_{n+1}$. In particular, g_e is a product of at most twice the diameter of X number of generators of $\Pi(X, G_\bullet)$. Now we choose R', S' subsets of R, S in such a way that the restriction of the projection $\widetilde{(X, G_\bullet, x_0)} \to \Gamma' \backslash \widetilde{(X, G_\bullet, x_0)}$ on R' is bijective for vertices (resp. the restriction of S' is bijective on edges). We also choose g'_e in a similar fashion as above, hence g'_e is also a product of at most twice the diameter of X number of generators of $\Pi(X, G_\bullet)$. From this data, we construct a graph of groups $(Y', H'_\bullet)$ as usual, and we have a canonical injection $\Gamma' \subset \Pi(Y', H'_\bullet)$. For $x \in X$ there exists a unique lift $\tilde{x} \in R$ and a unique $\tilde{x'} \in R'$ in the Γ'-orbit of $\tilde{x}$. Choose $\theta_x \in \Gamma' \subset \Pi(X, G_\bullet)$ such that $\theta_x(\tilde{x}) = \tilde{x'}$ and such that θ_x is a product of at most $l(\tilde{x}, \tilde{x_0}) + l(\tilde{x_0}, \tilde{x'})$ generators of $\Pi(X, G_\bullet)$: here $l(a, b)$ is the distance in the tree R between a and b. This is possible since we may first choose a path in the path group $\Pi(X, G_\bullet)$ from $\tilde{x}$ to $\tilde{x_0}$ of length $\leq l(\tilde{x}, \tilde{x_0})$ and then a path from $\tilde{x_0}$ to $\tilde{x'}$ of length $\leq l(\tilde{x_0}, \tilde{x'})$. Observe that since we chose $R' \subset R$, we have $l(\tilde{y}, \tilde{w}) \leq K$ for any vertices $w, y \in VX$. We do the same thing for edges in S, to define $\theta_e \in \Pi_1(X, G_\bullet)$ such that $\theta_e(\tilde{e}) = \tilde{e'}$ and θ_e is a product of at most $2K$ generators of $\Pi(X, G_\bullet)$. Then we can construct from θ_x and θ_e's a covering $\phi'_\bullet : (X, G_\bullet) \to (Y', H'_\bullet)$, with $\gamma'_x = \theta_x^{-1}$ and $\gamma'_e = g_e \theta_e^{-1} g_{e'}^{-1}$, which are both products of at most $6K$ generators of $\Pi(X, G_\bullet)$. Observe that a word of length l in generators of $\Pi(X, G_\bullet)$ belonging to Γ' is also expressible as a word of length l in generators of $\Pi(Y', H'_\bullet)$. Finally, by the proposition on bijection of isomorphism classes of coverings and overlattices, $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ is isomorphic to $\phi'_\bullet$. □

Proof of Theorem 3.2.1 Let us fix a quotient graph of groups $(X, G_\bullet)$ of Γ as in section 1.1. There exist only finitely many coverings of edge-indexed graphs by the edge-indexed graphs underlying $(X, G_\bullet)$, thus it is enough to show the assertion for the number of overlattices with a fixed edge-indexed graph. Thus we want to count isomorphism classes of n-sheeted coverings of graphs of groups $\phi_\bullet : (X, G_\bullet) \to (Y, H_\bullet)$ such that Y is a fixed quotient graph (with fixed indices) of X and $\phi : X \to Y$ the natural projection.

If two coverings are isomorphic, then the corresponding groups H_y are isomorphic. Thus we count the number of isomorphism classes of H_y, and we consider fixed H_y's. They are of order nc_y.

If two such coverings are isomorphic, then the corresponding graphs of groups $(Y, H_\bullet)$ are isomorphic. Up to isomorphism of graphs of groups, to prescribe the edge groups H_f and the monomorphisms $\alpha_f : H_f \to H_{o(f)}$, it suffices to consider, for each y, a subgroup H_f^* of $H_{o(f)}$, whose index is $c_{o(f)}/c_f$, and an isomorphism $\varphi : H_f^* \to H_{\bar{f}}^*$.

We thus count the number of subgroups of index $c_{o(f)}/c_f$ of a group of order $nc_{o(f)}$, for each f, and the number of isomorphisms between two groups of order nc_f for each f.

Let $c_x = |G_x|$ for any x in $VX \cup EX$ and let $c_y = (\sum_{x \in \phi^{-1}(y)} c_x^{-1})^{-1}$. By the definition of n-sheeted covering, the cardinality $|H_y|$ satisfies $|H_y| = nc_y$, for any y in $VY \cup EY$.

Now we claim that for any group H of order n, there are at most $(m!)^{\mu(n)+1}$ subgroups of index

m. For to any transitive H-action on the set $\{1,\cdots,m\}$, we can associate a subgroup of H with index m, namely the stabilizer of 1. This map $\{\rho : H \to S_m\} \longrightarrow \{H' \subset H | [H : H'] = m\}$ is surjective since for any subgroup H' of H with index m, the action of H on the cosets H/H' gives (among many) an action on $\{1,\cdots,m\}$, where we let 1 stand for the trivial coset H'. Again by the theorem of Lucchini and Guralnick ([Luc] [Gur]), there are at most $(m!)^{\mu(n)+1}$ transitive H-action on the set $\{1,\cdots,m\}$, as claimed.

There are at most $\prod_{y\in VY}(c_y n)^{g(c_y n)}$ isomorphism classes of H_y's. By the above claim, the number of subgroups $\alpha_f(H_f)$ of H_y is at most $((c_y/c_f)!)^{\mu(c_y n)+1}$. There are at most $\prod_{f\in EY}(c_f n)^{\mu(c_f n)+1}$ isomorphisms $\varphi : \alpha_f H_f \to \alpha_{\bar{f}} H_f$ and at most $\prod_{x\in VX}(c_{\phi(x)}n)^{\mu(c_x)+1}$ injections $\phi_x : G_x \to H_{\phi(x)}$. By Lemma 3.2.2, there are at most $(\sum_{y\in VY}|H_y|)^{12K}$ choices for each γ_x or γ_e, where $K =$ *diameter of* X. Hence

$$\#\{(\gamma_x,\gamma_e)\} \le \prod_{x\in VX}\max_{y\in VY}(c_y n)^{12K}\prod_{e\in EX}\max_{y\in VY}(c_y n)^{12K}$$

which is bounded by $(Mn)^{(12K)(|VX|+|EX|)}$, where $M = \max\limits_{y\in VY\cup EX} c_y$.

Note that by the condition of injectivity and the commutativity of the diagram,

$$\begin{tikzcd} G_e \arrow[r, "\alpha_e"] \arrow[d, "\phi_e"] & G_x \arrow[d, "\phi_x"] \\ H_{\phi(e)} \arrow[r, "ad(\gamma_x^{-1}\gamma_e)\circ\alpha_{\phi(e)}"] & H_{\phi(x)} \end{tikzcd}$$

the group morphism $\phi_e : G_e \to H_\phi(e)$ is completely determined by the morphism $\phi_x : G_x \to H_x$.

Let $M = \max\limits_{y\in VY\cup EY} c_y$, $\mu = \mu(Mn)$. Let $c_0 = |VY|$, $c_1 = |EY|$, $c_2 = \max\limits_{\{f\in EY\}}\{(\frac{c_{o(f)}}{c_f})!\}$, let $c_3 = \sum\limits_{x\in VX}\mu(c_x)+1$. Combining all the estimates above, we get the following upper bound for $u(n)$,

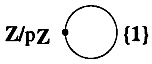
$$\begin{aligned} u_\Gamma(n) &\le \prod_{y\in VY}(c_y n)^{g(c_y n)}\prod_{x\in VX}(c_{\phi(x)}n)^{\mu(c_x)+1}\prod_{f\in EY}(c_f n)^{\mu(c_f n)+1} \\ &\quad \prod_{f\in EY}((c_{o(e)}/c_e)!)^{\mu_{c_{o(e)}n}+1}\cdot((Mn)^{12K(|VX|+|EX|)}) \\ &\le \prod_{y\in VY}(Mn)^{g(Mn)}\prod_{x\in VX}(Mn)^{\mu(c_x)+1}\prod_{f\in EY}(Mn)^{\mu(Mn)+1} \\ &\quad \prod_{f\in EY}(c_2)^{\mu_{Mn}+1}\cdot((Mn)^{12K(|VX|+|EX|)}) \end{aligned}$$

$$\begin{aligned}
u_\Gamma(n) \leq &(Mn)^{c_0 g(Mn)+c_3+c_1(\mu(Mn)+1)}(c_2)^{c_1(\mu(Mn)+1)}(Mn^{12K(|VX|+|EX|)})\\
\leq &(Mn)^{\frac{2}{27}c_0\mu^2+\frac{c_0}{2}\mu^{5/3}+(75c_0+2c_1)\mu+(16c_0+2c_1+c_3)}c_2^{c_1(\mu+1)}((Mn)^{12K(|VX|+|EX|)})\\
\leq &(C_0 n)^{C_1\mu^2} \leq (C_0 n)^{C_1'(\log n)^2}
\end{aligned}$$

where $C_0 = max\{M, c_2\}$, $C_1 = c_0\left(\frac{2}{27} + \frac{1}{2} + 75 + 16 + 2\right) + 6c_1 + c_3 + 12K(|VX| + |EX|)$ and $C_1' = \frac{C_1}{(\log 2)^2}$. □

3.2.2 Study in the case of a loop

Let p be a prime number. From now on, we assume that T is a $2p$-regular tree and that Γ is a cocompact lattice in $Aut(T)$ with a quotient graph of groups given by



The aim of this section is to give, in this situation, a smaller upper bound on $u_\Gamma(n)$ than the previous one, as well as a lower bound.

Theorem 3.2.3. *Let $n = p_0^{k_0}p_1^{k_1}\cdots p_t^{k_t}$ be the prime decomposition of n with $p_0 = p$. We fix $t, p_1, \dots, p_t, k_1, \dots, k_t$'s and let k_0 tend to infinity. Then there exist positive constants c_0, c_1 such that $\limsup_{k_0\to\infty}\frac{u_\Gamma(n)}{n^{c_1 \log n}} \leq c_0$. For $n = p_0^{k_0}$ $(k_0 \geq 3)$, we also have $n^{\frac{k_0}{50}-4} \leq u_\Gamma(n) \leq n^{\frac{k_0}{2}+1}$.*

In the following lemma, we denote by $[g, h]$ the commutator $ghg^{-1}h^{-1}$ in G.

Lemma 3.2.4. *Let $A = (a_{s,t})_{1\leq s,t\leq k-1}$ be a lower triangular matrix with coefficients in $0, \cdots, p-1$ and $G = G(A)$ be a group defined by the generators $\bar{g}_0, \bar{g}_1, \cdots, \bar{g}_k$ and the following relators*

$$\begin{array}{ll}
\bar{g}_i^p = 1, & i = 0, 1, \cdots, k\\
[\bar{g}_i, \bar{g}_{i+1}] = 1, & i = 0, 1, \cdots, k-1\\
[\bar{g}_i, \bar{g}_{i+2}] = \bar{g}_{i+1}^{a_{1,1}}, & i = 0, 1, \cdots, k-2\\
[\bar{g}_i, \bar{g}_{i+3}] = \bar{g}_{i+1}^{a_{2,1}}\bar{g}_{i+2}^{a_{2,2}}, & i = 0, 1, \cdots, k-3\\
\vdots & \\
[\bar{g}_i, \bar{g}_{i+k}] = \bar{g}_{i+1}^{a_{k-1,1}}\bar{g}_{i+2}^{a_{k-1,2}}\cdots\bar{g}_{i+k-1}^{a_{k-1,k-1}}, & i = 0.
\end{array} \qquad (**)$$

Then any element of G can be written as $\bar{g}_0^{i_0}\cdots\bar{g}_k^{i_k}$ where $0 \leq i_j < p$, and G is a group of order at most p^{k+1}.

Proof. We proceed by induction on $k \geq 1$. It is clear for the case $k = 1$ since the generators $\bar{g}_0$ and $\bar{g}_1$ commute. Now suppose that the assertion is true for all $k \leq m - 1$. For $k = m$, consider the subgroup G_1 generated by $\bar{g}_0, \bar{g}_1, \cdots, \bar{g}_{m-1}$. It is a quotient of $G(A')$ with $A' = (a_{s,t})_{1 \leq s,t \leq k-2}$, by induction hypothesis, any element of G_1 can be written as $\bar{g}_0^{i_0} \cdots \bar{g}_{m-1}^{i_{m-1}}$ where $0 \leq i_j < p$. Now we only need to consider the elements of $G - G_1$. By an easy induction, it suffices to consider the elements $\bar{g}_m \bar{g}_i = [\bar{g}_i, \bar{g}_m]^{-1} \bar{g}_i \bar{g}_m$ for $i = 0, 1, \cdots, m - 1$. Since $[\bar{g}_i, \bar{g}_m] \in [G, G] \subset G_1$, the element $[\bar{g}_i, \bar{g}_m]^{-1} \bar{g}_i$ is an element in G_1, thus can be expressed as $\bar{g}_0^{i_0} \bar{g}_1^{i_1} \cdots \bar{g}_{m-1}^{i_{m-1}}$ for some i_j in $\{0, \cdots, p-1\}$. Therefore we get $\bar{g}_m \bar{g}_i = \bar{g}_0^{i_0} \cdots \bar{g}_{m-1}^{i_{m-1}} \bar{g}_m$. It follows that G has order at most p^{k+1}. □

Lemma 3.2.5. *Let G be a group of order $p^{k+1} (k \geq 1)$, G_1 and G_2 be two isomorphic subgroups of index p in G and φ be an isomorphism from G_1 to G_2. Suppose that G_1 contains no subgroup N which is normal in G and φ-invariant. Then*

(a) there exist elements g_i in G for $i = 0, \cdots, k$, such that $\varphi(g_i) = g_{i+1}$ for $i = 0, \cdots, k - 1$, $G = \langle g_0, \cdots, g_k \rangle$ and $G_1 = \langle g_0, \cdots, g_{k-1} \rangle$,

(b) There exists a lower triangular matrix A with coefficients in $0, \cdots, p - 1$ such that the map $\psi : G(A) \to G$ defined by $\bar{g}_i \mapsto g_i$ is well-defined and is an isomorphism.

Proof. We proceed by induction on $k \geq 1$, using the fact that any maximal proper subgroup of a p-group is normal, see for instance [Su].

We first consider the case $k = 1$, that is when G has order p^2. Since G_1 and G_2 are maximal, they are normal. Thus they are not equal by the normality assumption and $G = \langle G_1, G_2 \rangle$. Let g_0 be an element in $G_1 - G_2$ and set $g_1 = \varphi(g_0)$. Then clearly $G_1 = \langle g_0 \rangle, G_2 = \langle g_1 \rangle$ and $G = \langle g_0, g_1 \rangle$. Moreover, since $|G| = p^2$, G is abelian [Su]. Thus $[g_0, g_1] = 1$ and $G = \{g_0^{i_0} g_1^{i_1} : 0 \leq i_0, i_1 \leq p - 1\}$, which shows that ψ in (b) is well-defined and surjective. Since $G(A)$ has cardinality at most p^2 by Lemma 3.2.4, and G has cardinality p^2, the map ψ is an isomorphism.

Now suppose that the assertion is true for all $k < m$. For $k = m$, consider G, G_1, G_2 and φ as in the statement of the lemma. As above, G_1 and G_2 are normal, distinct and $G = \langle G_1, G_2 \rangle$. Since $[G_2 : G_2 \cap G_1] \leq [\langle G_1, G_2 \rangle : G_1] = p$, we have $[G_2 : G_1 \cap G_2] = p$ and similarly $[G_1 : G_1 \cap G_2] = p$. Therefore $G_1 \cap G_2$ is maximal, thus normal in G_1 and G_2. Since G is generated by G_1 and G_2, the subgroup $G_1 \cap G_2$ is normal in G. By the assumption, $\varphi(G_1 \cap G_2) \neq G_1 \cap G_2$.

Claim. If a subgroup N of $G_1 \cap G_2$ is normal in G_2 and φ-invariant, then N is normal in G.

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & & \\
G_1 \arrow[rr, "\varphi"] \arrow[dr, dash] & & G_2 \arrow[dl, dash] \arrow[dr, dash] & \\
 & G_1 \cap G_2 \arrow[rr, "\varphi"] \arrow[dr, dash] & & \varphi(G_1 \cap G_2) \arrow[dl, dash] \\
 & & N &
\end{tikzcd}$$

Proof. Consider gNg^{-1}, for any $g \in G_1$. As $\varphi(gNg^{-1}) = \varphi(g)\varphi(N)\varphi(g)^{-1} = \varphi(g)N\varphi(g)^{-1} = N$ (since $\varphi(g) \in G_2$) and φ is an isomorphism, we deduce that $gNg^{-1} = \varphi^{-1}(N) = N$. Therefore $G_1 \subseteq N_G(N)$ and similarly $G_2 \subseteq N_G(N)$. Thus G, as a group generated by G_1 and G_2, is also contained in $N_G(N)$. Hence N is normal in G. $\square$

By the above claim, we can use the induction hypothesis on $G' = G_2$, $G'_1 = G_1 \cap G_2$, $G'_2 = \varphi(G_1 \cap G_2)$ and $\varphi' = \varphi|_{G_1 \cap G_2}$. It follows that there is an element g_1 in G_2 such that $G_2 = \langle g_1, \cdots, g_m \rangle$, $G_1 \cap G_2 = \langle g_1, \cdots, g_{m-1} \rangle$ and $\varphi(g_i) = g_{i+1}$ for $i = 1, \cdots, m-1$.

Let $g_0 = \varphi^{-1}(g_1)$. If $g_0 \in G_1 \cap G_2$, then $g_1 \in \varphi(G_1 \cap G_2)$, which contradicts $G_1 \cap G_2 \neq \varphi(G_1 \cap G_2)$. Thus g_0 is an element of $G_1 - G_2$. Since G_2 is maximal in G, the group G is generated by G_2 and g_0, i.e., $G = \langle g_0, g_1, \cdots, g_m \rangle$. It is clear that $G_1 = \langle g_0, \cdots, g_{m-1} \rangle$ as $G_1 \cap G_2$ is maximal in G_1 and $g_0 \in G_1 - (G_1 \cap G_2)$.

To prove the assertion (b), note that $[g_0, g_i] = \varphi^{-1}([g_1, g_{i+1}]) = \varphi^{-1}(g_2^{a_{i,1}} \cdots g_i^{a_{i,i}}) = g_1^{a_{i,1}} \cdots g_{i-1}^{a_{i,i}}$ for $1 \leq i \leq m-1$ and ${g_i}^p = 1$, for all $i \geq 1$ by induction hypothesis. Thus we only need to consider $[g_0, g_m]$ and ${g_0}^p$. The element g_0 clearly has order p since φ is an isomorphism and $g_1 = \varphi(g_0)$ has order p. It is easy to see that if two subgroups H and K are normal subgroups of a group G, then so is the commutator subgroup $[H, K]$ and we have $[H, K] \subset H \cap K$. Since $g_0 \in G_1$ and $g_m \in G_2$, it follows that $[g_0, g_m] \in G_1 \cap G_2 = \langle g_1, g_2, \cdots, g_{m-1} \rangle$, which proves that ψ is a well-defined homomorphism. By the previous paragraph, it is surjective. Since G and $G(A)$ are of cardinality p^{k+1} and at most p^{k+1} respectively, the map ψ is an isomorphism. $\square$

Lemma 3.2.6. *Let $\mathcal{A}$ be the set of $(k-1) \times (k-1)$ lower triangular matrices with coefficients in $\{0, 1, \cdots, p-1\}$. There are at least $p^{(\frac{k}{10}-2)(\frac{k}{5}-2)}$ elements A in $\mathcal{A}$ such that the group $G(A)$ is of order p^{k+1}.*

Proof. The idea is to consider only the nilpotent groups of degree 2 (with large center) among the groups $G(A)$'s of $\mathcal{A}$. Let us consider the subset of $\mathcal{A}$ which consists of matrices $A = (a_{i,j})$ such that $a_{j-1,t} = 0$ for all t if $j < [\frac{9}{10}k]$ and $t = 1, \cdots, [\frac{j}{2} - \frac{k}{10}] - 1, [\frac{j}{2} + \frac{k}{10}] + 1, \cdots, j-1$ if $j \geq [\frac{9k}{10}]$. Then the group G is a group described in Lemma 3.2.4 with the following additional relations :

$$[\bar{g}_i, \bar{g}_j] = 1, \text{ if } |j-i| < \left[\frac{9}{10}k\right]$$
$$[\bar{g}_i, \bar{g}_{i+j}] = \bar{g}_{i+[\frac{j}{2}-\frac{k}{10}]}^{a_1^j} \cdots \bar{g}_{i+[\frac{j}{2}+\frac{k}{10}]}^{a^j_{\frac{2k}{10}}}, \text{ if } j \geq \left[\frac{9k}{10}\right]$$

where $[a]$ denotes the largest integer smaller than or equal to a.

Now we claim that the group G is isomorphic to a semidirect product

$$G = (\mathbb{Z}/p\mathbb{Z})^{k-[\frac{9k}{10}]} \ltimes (\mathbb{Z}/p\mathbb{Z})^{[\frac{9k}{10}]+1},$$

defined as follows. Let $g_0, \cdots, g_{k-[\frac{9k}{10}]}$ and $g_{k-[\frac{9k}{10}]+1}, \cdots, g_k$ be generators of the group $(\mathbb{Z}/p\mathbb{Z})^{k-[\frac{9k}{10}]}$ and $(\mathbb{Z}/p\mathbb{Z})^{[\frac{9k}{10}]+1}$, respectively.

For every integer i such that $0 \leq i \leq k - [\frac{9k}{10}] - 1$, we define a linear automorphism σ_i of $(\mathbb{Z}/p\mathbb{Z})^{[\frac{9k}{10}]+1}$, by

$$\sigma_i(g_u) = g_u \text{ if } u < i + \left[\frac{9}{10}k\right], \ i \leq k - \left[\frac{9}{10}k\right]$$
$$\sigma_i(g_{i+j}) = g_{i+[\frac{j}{2}-\frac{k}{10}]}^{a_1^j} \cdots g_{i+[\frac{j}{2}+\frac{k}{10}]}^{a^j_{[\frac{2k}{10}]}} g_{i+j}, \ j > \left[\frac{9k}{10}\right].$$

Then $\sigma_i(g_u)$ and $\sigma_i(g_{i+j})$ are all of order p since g_u are of order p and $\sigma_i(g_{i+j})$ are in the center. The automorphisms σ_i mutually commute i.e. $\sigma_i\sigma_j(g_u) = \sigma_j\sigma_i(g_u)$ since $\sigma_i(g_j)$ are all in the center. Note also that σ_i are of order p, because every g_u's appears in $\sigma_i(g_{i+j})$ are are all in the center. Therefore there is an action by automorphisms of $(\mathbb{Z}/p\mathbb{Z})^{k-[\frac{9k}{10}]}$ on $(\mathbb{Z}/p\mathbb{Z})^{[\frac{9k}{10}]+1}$.

Now for every matrix described in the beginning of the proof, it is easy to see that the group $G(A)$ is isomorphic to the semidirect product described just above : the relations we get from the definition of a semidirect product $g_i g_j g_i^{-1} = \sigma_i(g_j)$ (where $0 \geq i < k - [\frac{9k}{10}] - 1$ and $k - [\frac{9k}{10}] - 1 \geq j \geq k$) are precisely the relations on commutators in the definition of $G(A)$. Hence the group $G(A)$ is clearly a group of order p^{k+1}. Note also that the subgroup G' generated by $g_0, \cdots, g_{k-1}$ may also be described as a semidirect product, hence is a group of order p^k. There are

$$\sum_{j=[\frac{9k}{10}]}^{k-1} \left(\left[\frac{j}{2} + \frac{k}{10}\right] - \left[\frac{j}{2} - \frac{k}{10}\right]\right) \geq \left(\frac{k}{10} - 2\right)\left(\frac{k}{5} - 2\right)$$

components a_{ij} of A which are not zero, thus there are at least $p^{(\frac{k}{10}-2)(\frac{k}{5}-2)}$ elements of $\mathcal{A}$ which are of order p^{k+1}. □

Proof of Theorem 3.2.3 By Proposition 3.1.7, the number u_n is the number of isomorphism classes of n-sheeted coverings of faithful graphs of groups $\phi_\bullet : \Gamma\backslash\backslash T \to (X, G_\bullet)$. As already seen, we may assume that $X = \Gamma\backslash T$. The following commutative diagram summarizes the data defining $\phi_\bullet$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[d] \\ G_e \arrow[r, shift left, "\alpha_e"] \arrow[r, shift right, "\alpha_{\bar{e}}"'] & G_x \end{tikzcd}$$

Let's first consider the case when $n = p^k$. Let $G = G_x$, $G_1 = \alpha_e(G_e)$ and $G_2 = \alpha_{\bar{e}}(G_e)$. The cardinality of G is p^{k+1} since it is the index n times the cardinality of $\mathbb{Z}/p\mathbb{Z}$.

By the condition of faithfulness, G_1 and G_2 are distinct as they are normal subgroups of G. Hence if we let $\varphi = \alpha_{\bar{e}} \circ \alpha_e^{-1} : G_1 \to G_2$, then φ is an isomorphism and there is no subgroup of G_1 which is normal in G and φ-invariant. Thus we can use Lemma 3.2.5 to find an element g_0 in G_x such that $G_x = \langle g_0, g_1, \cdots g_k \rangle$ where $\varphi(g_j) = g_{j+1}$. Moreover, the group G is isomorphic to $G(A)$ for some matrix A, which is determined by A. (note that A also determines G_e and the maps α_e and $\alpha_{\bar{e}}$.) Thus we have at most $p^{\sum_{j=0}^{k-1} j}$ choices for G_x, G_e, α_e and $\alpha_{\bar{e}}$, which is exactly the number of choices of $(a_{st})_{1 \le t \le s \le k-1}$. Once we have fixed G_x, G_e, α_e and $\alpha_{\bar{e}}$, an injection i from $\mathbb{Z}/p\mathbb{Z}$ into G_x is determined by the image of a generator in the domain, which implies that we have at most $|G_x| = p^{k+1}$ choices for i. Therefore we have an upper bound $u_\Gamma(n) \le p^{\sum_{j=1}^{k-1} j} p^{k+1} = p^{\frac{(k-1)k}{2}+k+1} = p^{\frac{k^2+k+2}{2}}$.

Now let us construct mutually non-isomorphic faithful coverings of graphs of groups to deduce a lower bound of $u(n)$. Let $\mathcal{A}'$ be a subset of $\mathcal{A}$ consisting of lower triangular matrices $A = (a_{st})$ $1 \le t \le s \le k-1$ such that furthermore $a_{k-1,j} = 0$ for $j = 0, \cdots, k-1$. Note that there are at least p to the

$$\sum_{j=[\frac{9k}{10}]}^{k-2} \left(\left[\frac{j}{2} + \frac{k}{10} \right] - \left[\frac{j}{2} - \frac{k}{10} \right] \right) \ge \left(\frac{k}{10} - 3 \right) \left(\frac{k}{5} - 2 \right)$$

elements of $\mathcal{A}'$. Let us fix a matrix A in $\mathcal{A}'$ and let $G_x = G(A)$ be the group described in Lemma 3.2.6.

Let us define the covering graph of groups $\phi_\bullet(A) = (Id_{\Gamma\backslash T}, \phi_\bullet, \gamma_\bullet) : \Gamma\backslash\backslash T \to G_\bullet$ and we show that there are many elements A's of $\mathcal{A}'$ which give rise to mutually non-isomorphic coverings of graphs of groups. Let G_e be the subgroup of G_x generated by $\bar{g}_0, \cdots, \bar{g}_{k-1}$. Let the injection α_e be the inclusion map and the other inclusion $\alpha_{\bar{e}}$ be defined by $\alpha_{\bar{e}}(\bar{g}_i) = \bar{g}_{i+1} \in G_x$, which is indeed a monomorphism by the definition of $G(A)$ (for every A in $\mathcal{A}$. Therefore, the group morphism $\varphi = \varphi_A$ defined by $\varphi(\bar{g}_i) = \bar{g}_{i+1}$ is an isomorphism from $\alpha_e(G_e)$ onto $\alpha_{\bar{e}}(G_e)$. For any nontrivial element

$h = \bar{g}_{j_0}^{i_{j_0}} \cdots \bar{g}_{j_t}^{i_{j_t}}$ in G_e (with nonzero i_{j_t}), $\varphi^{k-j_t}(h) = \bar{g}_{j_0+k-j_t}^{i_{j_0}} \cdots \bar{g}_k^{i_{j_t}} \notin G_e$. Therefore the data $G_\bullet$ defines a faithful graph of groups, as there is no φ-invariant subgroup of G_e.

Let g be a generator of $\mathbb{Z}/p\mathbb{Z}$ and set $\phi_x(g) = \bar{g}_0\bar{g}_k$. This defines a group monomorphism $\phi_x : \mathbb{Z}/p\mathbb{Z} \to G_x$ since the order of $g_0 g_k$ is p by assumption. Let $\gamma_x = \gamma_e = \gamma_{\bar{e}} = 1$. Then map

$$\Phi_{x/e} : \mathbb{Z}/p\mathbb{Z}/\alpha_e(1) \to G_x/\alpha_e(G_e) \simeq \langle g_k \rangle$$

$$[g]_e \mapsto [g_0 g_k]_{\bar{e}}$$

$$\Phi_{x/\bar{e}} : \mathbb{Z}/p\mathbb{Z}/\alpha_{\bar{e}}(1) \to G_x/\alpha_{\bar{e}}(G_{\bar{e}}) \simeq \langle g_0 \rangle$$

$$[g]_{\bar{e}} \mapsto [g_0 g_k]_{\bar{e}}$$

clearly satisfies the condition (b) of Definition 3.1.2 for a covering of graphs of groups since $g_0 g_k$ is nontrivial and has order p in $G_x/\alpha_e(G_e) \simeq \langle g_0 \rangle$ and has order p in $G_x/\alpha_{\bar{e}}(G_{\bar{e}}) \simeq \langle g_k \rangle$. Thus we have constructed a covering of graphs of groups. Now suppose that the coverings of graphs of groups $\phi_\bullet(A)$ and $\phi_\bullet(A') : \Gamma\backslash\backslash T \to (X, G'_\bullet)$ are isomorphic. Now we want to show that there are sufficiently many A's which give rise to mutually non-isomorphic coverings.

Lemma 3.2.7. *For a given A, there are at most p^{2k+4} number of A' such that $\phi_\bullet(A)$ and $\phi_\bullet(A')$ are isomorphic.*

Proof. Let A and A' be in $\mathcal{A}'$. Let $\phi_\bullet(A)$ and $\phi_\bullet(A')$ be the corresponding coverings, respectively. Let $(G_x, G_e = G_{\bar{e}}, \alpha_e, \alpha_{\bar{e}})$ and $(H_x, H_e = H_{\bar{e}}, \alpha_e, \alpha_{\bar{e}})$ be the vertex group, the edge group, and the injections of the target graphs of groups of $\phi_\bullet(A)$ and $\phi_\bullet(A')$, respectively.

Let us denote by $g_0, \cdots, g_n$ and $h_0, \cdots, h_n$ the generating set of G_x and H_x defined by the matrices A and A', respectively.

Now suppose that $\theta_\bullet = (Id_{\Gamma\backslash T}, \theta_\bullet, \rho_\bullet) : (X, G_x) \to (X, H_x)$ is an isomorphism of coverings. Since the map on the universal covers commute, and the elements $\gamma_x, \gamma_e, \gamma_{\bar{e}}$ are all trivial for $\phi_\bullet(A)$ and $\phi_\bullet(A')$, the map $\widetilde{\theta}$ on the universal covers of $(X, G_\bullet)$ and $(X, H_\bullet)$ maps $[g_0 g_k]$ to $[h_0 h_k]$. In other words, $\rho_x \theta_x(g_0 g_k)\rho_x^{-1}(h_0 h_k)^{-1} \in H_x$. Since $\theta_x(g_0 g_k)$ and $(h_0 h_k)^{-1}$ are elements in $H_{\theta(x)}$, we conclude that $\rho_x \in H_{\theta x}$ and $\rho_e \in H_{\theta(x)}$. By substituting ρ_x by 1 and ρ_e by $\rho_x^{-1}\rho_e$ (with the same maps θ_x and θ_e), we may assume that $\rho_x = 1$.

Therefore, by the definition of morphisms, there are elements $\rho_e, \rho_{\bar{e}} \in H_x$ such that the following diagrams commute.

$$\begin{tikzcd} G_e \arrow[r, "\alpha_e"] \arrow[d, "\theta_e"] & G_x \arrow[d, "\theta_x"] \\ H_{\theta(e)} \arrow[r, "Ad(\rho_e)\circ\alpha_{\theta(e)}"] & H_{\theta(x)} \end{tikzcd} \qquad \begin{tikzcd} G_{\bar{e}} \arrow[r, "\alpha_{\bar{e}}"] \arrow[d, "\theta_{\bar{e}}"] & G_x \arrow[d, "\theta_x"] \\ H_{\theta(\bar{e})} \arrow[r, "Ad(\rho_{\bar{e}})\circ\alpha_{\theta(e)}"] & H_{\theta(x)} \end{tikzcd}$$

Let us define the isomorphisms σ and β so that the diagrams

$$\begin{tikzcd} G_e \arrow[r, "id"] \arrow[d, "\alpha_e"] & G_{\bar{e}} \arrow[d, "\alpha_{\bar{e}}"] \\ \alpha_e(G_e) \arrow[r, "\sigma"] & \alpha_{\bar{e}}(G_{\bar{e}}) \end{tikzcd} \qquad \begin{tikzcd} H_e \arrow[r, "id"] \arrow[d, "\alpha_{\bar{e}}"] & H_{\bar{e}} \arrow[d, "\alpha_{\bar{e}}"] \\ \alpha_e(G_e) \arrow[r, "\beta"] & \alpha_{\bar{e}}(H_{\bar{e}}) \end{tikzcd}$$

commute.

Combining these diagrams, we obtain the following commutative diagram

$$\begin{tikzcd} \alpha_e(G_e) \arrow[r, "\sigma"] \arrow[d, "\alpha_e\theta_e\alpha_e^{-1}"] & \alpha_{\bar{e}}(G_{\bar{e}}) \arrow[d, "\alpha_{\bar{e}}\theta_{\bar{e}}\alpha_{\bar{e}}^{-1}"] \\ \alpha_e(H_e) \arrow[r, "\beta"] & \alpha_{\bar{e}}(H_{\bar{e}}). \end{tikzcd}$$

By definition of isomorphism of graphs of groups, we have $\theta_e(G_e) = H_e$. Since the groups $\alpha_e(H_e)$ is stable under the adjoint action by any element of H_x (recall that the commutators are in the center and the center is a subgroup of $\alpha_e(G_e)$ and $\alpha_{\bar{e}}(G_{\bar{e}})$), we have $Ad(\rho_e)(\alpha_e(H_e)) = \alpha_e(H_e)$. Thus, by definition of morphisms, we have

$$\theta_x(\alpha_e(G_e)) = Ad(\rho_e)(\alpha_e(\theta_e(G_e))) = \alpha_e(H_e),$$

and similarly

$$\theta_x(\alpha_{\bar{e}}(G_{\bar{e}})) = \alpha_{\bar{e}}(H_{\bar{e}}).$$

Now let us define a group isomorphism ψ so that the following diagram commutes.

$$\begin{tikzcd} \alpha_e(G_e) \arrow[r, "\sigma"] \arrow[d, "\theta_x"] & \alpha_{\bar{e}}(G_{\bar{e}})\alpha_{\bar{e}}^{-1} \arrow[d, "\theta_x"] \\ \alpha_e(H_e) \arrow[r, "\psi"] & \alpha_{\bar{e}}(H_{\bar{e}}). \end{tikzcd}$$

We claim that ψ is completely determined by ρ_e and $\rho_{\bar{e}}$. Indeed, ψ maps $\theta_x(\alpha_e(g)) = \rho_e^{-1}(\alpha_e(\theta_e(g)))\rho_e$ to $\theta_x(\sigma(\alpha_e(g))) = \theta_x(\alpha_{\bar{e}}(g)) = \rho_{\bar{e}}^{-1}(\alpha_{\bar{e}}(\theta_{\bar{e}}(g)))\rho_{\bar{e}}$ for any $g \in G_e$. In other words, ψ maps $\rho_e^{-1}u\rho_e$ to $\rho_{\bar{e}}^{-1}\beta(u)\rho_{\bar{e}}$ for every $u \in \alpha_e(H_e)$, i.e. $\psi = Ad(\rho_e\rho_{\bar{e}}^{-1})\beta$.

Next, we will show that, under θ_x, the subgroups $G_{[i,j]} = \langle g_i, \cdots, g_j \rangle \subset G_x$ are necessarily mapped to certain subgroups $\widetilde{H}_{[i,j]}$, uniquely characterized by $\alpha_e(H_e)$, $\alpha_{\bar{e}}(H_{\bar{e}})$ and the map ψ. Let us define the subgroups $\widetilde{H}_{[i,j]}$ by descending induction on $|i-j|$. We have

$$\widetilde{H}_{[0,n-1]} = H_{[0,n-1]} = \alpha_e(H_e) = \psi(G_{[0,n-1]}) = \theta_x(G_{[0,n-1]})$$

and

$$\widetilde{H}_{[1,n]} = H_{[1,n]} = \alpha_{\bar{e}}(H_{\bar{e}}) = \psi(G_{[1,n]}) = \theta_x(G_{[1,n]}).$$

Assume that $\widetilde{H}_{[i,j]}$ is defined for all pairs (i,j) such that $|i-j| > l$ and that $\theta_x(G_{[i,j]}) = \widetilde{H}_{[i,j]}$ for all such (i,j).

If $n - l > i > 0$, then we set

$$\widetilde{H}_{[i,i+l]} = \widetilde{H}_{[i-1,i+l]} \cap \widetilde{H}_{[i,i+l+1]}.$$

Note that $\widetilde{H}_{[i,i+l]} = \theta_x(G_{[i-1,i+l]}) \cap \theta_x(G_{[i,i+l]}) = \theta_x(G_{[i,i+l]})$ so that $|\widetilde{H}_{[i,i+l]}| = p^l$.

If $i = 0$, then we put

$$\widetilde{H}_{[0,l]} = \psi^{-1}(\widetilde{H}_{[1,l+1]}).$$

Note that $\theta_x(G_{[0,l]}) = \theta_x(\sigma^{-1}(G_{[i,i+l]})) = \psi^{-1}\theta_x(G_{[1,l+1]}) = \psi^{-1}\widetilde{H}_{[1,l+1]} = \widetilde{H}_{[0,l]}$.

If $i = n - l$, then we set

$$\widetilde{H}_{[n-l,n]} = \psi\left(\widetilde{H}_{[n-l-1,n-1]}\right).$$

Again, note that $\theta_x(G_{[n-l,n]}) = \widetilde{H}_{[n-l,n]}$.

At the end, we have thus defined groups $\widetilde{H}_{[i,i]}$, purely in terms of $\rho_e, \rho_{\bar{e}}$ and β such that we necessarily have $\theta_x(G_{[i,i]}) = \widetilde{H}_{[i,j]}$. Let $\widetilde{h}_0$ be a generator of $\widetilde{H}_{[0,0]}$, and set $\widetilde{h}_i = \psi^i \widetilde{h}_0$, so that

$$\widetilde{H}_{[i,i]} = \{\widetilde{h}_i^\gamma : \gamma \in \mathbb{Z}/p\mathbb{Z}\}.$$

Hence, θ_x necessarily sends g_i to $\widetilde{h}_i^{\gamma_i}$ for some $\gamma_i \in (\mathbb{Z}/p\mathbb{Z})^*$. In fact, since $\sigma(g_i) = g_{i+1}$ and $\psi(\widetilde{h}_i) = \widetilde{h}_{i+1}$ we even have $\gamma_i = \gamma_0, \forall i$.

Thus if $(\theta_\bullet, \rho_e, \rho_{\bar{e}}) : (X, G) \to (X, H)$ is an isomorphism of coverings, there exists $\gamma \in \mathbb{Z}/p\mathbb{Z}$ such that

$$\begin{aligned} \theta_x : G_x &\to \widetilde{H}_x \\ g_i &\mapsto \widetilde{h}_i^\gamma \quad \forall i. \end{aligned}$$

But then

$$[g_i, g_j] = \theta_x^{-1}([\widetilde{h_i}, \widetilde{h_j}])$$

is completely determined by $\rho_e, \rho_{\bar{e}}, \gamma$ and β.

In other words, for a given A', there are at most $p^{2(k+1)}p(p-1) \leq p^{2k+4}$ possible A such that $\phi(A)$ and $\phi(A')$ are isomorphic coverings. $\square$

Combining Lemma 3.2.6 and Lemma 3.2.7 there are at least $p^{(\frac{k}{10}-3)(\frac{k}{5}-1)-2k-4}$ thus at least $p^{k(k/50-37/10)}$ non-isomorphic coverings $\phi(A)$'s.

Now let's consider the general case. Recall that $|G_e| = \prod_{i=0}^{t} p_i^{k_i}$ and $|G_x| = \prod_{i=0}^{t} p_i^{k_i} p = p_0^{k_0+1} \prod_{i=1}^{t} p_i^{k_i}$, thus the order of the Sylow p_i-subgroup of G_e and that of G_x are the same for all $i \neq 0$. Let $G_e^{(p_i)}$ be *a* Sylow p_i-subgroup of G_e. For $i \neq 1$, let $G_x^{(p_i)} = \alpha_e(G_e^{(p_i)})$. Choose one p-Sylow subgroup $G_x^{(p)}$ of G_x containing $\alpha_e(G_e^{(p)})$.

We are now going to show that the faithfulness condition is inherited to the Sylow p-subgroups $G_e^{(p)}$, $G_x^{(p)}$ of G_e and G_x, from which we can use the upper bound given in the first part of the proof. Conjugating $\alpha_{\bar{e}}$ by an element of G_x, if necessary, we may assume that $\alpha_{\bar{e}}(G_e^{(p)}) \subset G_x^{(p)}$, thus we have the following diagram :

$$\begin{tikzcd} G_e \arrow[r, shift left, "\alpha_e"] \arrow[r, shift right, "\alpha_{\bar{e}}"'] & G_x \\ G_e^{(p)} \arrow[u, hook] \arrow[r, shift left, "\alpha_e"] \arrow[r, shift right, "\alpha_{\bar{e}}"'] & G_x^{(p)} \arrow[u, hook] \end{tikzcd}$$

Suppose that $N \lhd G_e^{(p)}$ and $\mathcal{N} = \alpha_e(N) = \alpha_{\bar{e}}(N) \lhd G_x^{(p)}$. Let $\overline{N} = \langle gNg^{-1} : g \in G_e \rangle$ (respectively $\overline{\mathcal{N}} = \langle g\mathcal{N}g^{-1} : g \in G_x \rangle$) be the smallest normal subgroup of G_e (respectively G_x) containing N (respectively $\mathcal{N}$).

Note that α_i $(i = e, \overline{e})$ induces a bijection between left cosets

$$\begin{aligned} G_e/G_e^{(p)} &\longrightarrow G_x/G_x^{(p)} \\ gG_e^{(p)} &\longmapsto \alpha_i(g)G_x^{(p)}. \end{aligned}$$

For since α_i is injective, if $gG_e^{(p)}$ is mapped to $G_x^{(p)}$, then g is in $\alpha_i(G_e) \cap G_x^{(p)}$, which is a p-group in $\alpha_i(G_e)$ containing $\alpha_i(G_e^{(p)})$. Since $\alpha_i(G_e^{(p)})$ is a Sylow p-subgroup, $\alpha_i(G_e^{(p)}) \cap G_x^{(p)}$ is equal to $\alpha_i(G_e)$. Thus $\alpha_i(g^{-1}h)$ is contained in $G_x^{(p)}$ if and only if $g^{-1}h \in G_e^{(p)}$ and the map is injective. It is surjective since the source and the target have the same cardinality. Thus any element g in G_x can be written as $\alpha_i(g')h_i$ for some $g' \in G_e$, $h_i \in G_x^{(p)}$ and we have $g\mathcal{N}g^{-1} = \alpha_i(g')h_i\mathcal{N}h_i^{-1}\alpha_i(g'^{-1}) = \alpha_i(g')\mathcal{N}\alpha_i(g'^{-1})$ $(i = 1, 2)$. Therefore $\alpha_1(\overline{\mathcal{N}}) = \alpha_2(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$ and it is normal in G_x. As a consequence, $G_e^{(p)}$ and $G_x^{(p)}$ satisfy the condition of faithfulness, i.e., there is no subgroup N of $G_e^{(p)}$ such that $\alpha_e(N) = \alpha_{\bar{e}}(N)$ is normal in $G_x^{(p)}$. By the first part of the proof, this implies that the number of choices for $G_x^{(p)}, G_e^{(p)}, \alpha_e|_{G_e^{(p)}}$ and $\alpha_{\bar{e}}|_{G_e^{(p)}}$ is at most $p^{\frac{k_0^2+k_0+2}{2}}$.

Since all the other $G_e^{(p_i)}$ and $G_x^{(p_i)}$ have fixed cardinality, we have a constant total number of choices for them and the injections $\alpha_e|_{G_e^{p_i}}$, say c_0. Recall that once all the $G_x^{(p_i)}$'s and $G_e^{(p_i)}$'s are chosen, the number of G_x with a given fixed Sylow system is at most $(pn)^{75\mu(pn)+16}$([P]). Recall also that the injections α_e are determined by its restriction to Sylow subgroups of G_e since they generate the group. Finally we have the following upper bound.

$$u_n \leq c_0 p^{\frac{(k_0+1)^2+(k_0+1)+2}{2}} (pn)^{75\mu(pn)+16} \leq c_0 (c_1)^{\frac{k_0^2+5k_0+8}{2}} (pn)^{75\mu+16}$$

where $c_1 = p$ and $\mu = \mu(pn)$. □

Remark. It follows from the proof above that each prime factor of $|G_x|$ is less than or equal to p, thus in the case $p = 2$, $u(n) = 0$ if n is not a power of 2.

Chapitre 4

Overlattices in automorphism groups of buildings

4.1 Introduction

The theory of graphs of groups describes group actions on trees; see [Se], [Ba] and [BL]. The theory of complexes of groups, due to Gersten–Stallings [GS], Corson [Co] and Haefliger [H2], [BH], generalizes Bass–Serre theory to higher dimensions. Our first aim is to develop some basic tools for complexes of groups, analogous to those in [Ba], and which do not appear in [BH].

We recall Haefliger's theory of complex of groups in Section 4.2.4. Briefly, the action of a group G on a simply connected polyhedral complex K induces a complex of groups $G(\mathcal{Y})$ over the quotient $\mathcal{Y} = K\backslash G$. The fundamental group $\pi_1(G(\mathcal{Y}))$ of $G(\mathcal{Y})$ then acts on the simply connected universal cover $\widetilde{G(\mathcal{Y})}$ of $G(\mathcal{Y})$, with $\pi_1(G(\mathcal{Y}))$ isomorphic to G, and $\widetilde{G(\mathcal{Y})}$ equivariantly isometric to K. An arbitrary complex of groups $G(\mathcal{Y})$ is developable if it is induced by a group action in this way. A key difference between Bass–Serre theory and complexes of groups is that complexes of groups need not be developable. However, if a complex of groups has nonpositive curvature (see Section 4.2.4), it is developable.

In [Ba], Bass developed a "covering theory for graphs of groups". To translate in the framework of complexes of groups the general theory of coverings of étale groupoids, Haefliger developed a covering theory for complexes of groups ([H2]), similar to the covering theory of Bass. It seems interesting to compare two theories and to complete the analogy between them. We hope that the following additional results will be useful.

Our first main result describes the functoriality of coverings, and corresponds to Proposition 2.7, [Ba].

Remark. Haefliger, by a personal correspondence, explained to us that Theorem 4.1.1 and Proposition 4.1.2 are also consequences of a functorial 1-1 correspondence between the coverings of an etale groupoid and the coverings of its classifying space. He furthermore indicated that if there is a covering from $G(\mathcal{Y})$ to $G(\mathcal{Y}')$ then $G(\mathcal{Y})$ is developable if and only if $G(\mathcal{Y}')$ is developable, an assertion stronger than Proposition 4.1.2. Here we present our results as they are proved without refering to the theory of etale groupoid.

Theorem 4.1.1. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of developable complexes of groups. Then λ induces a monomorphism of fundamental groups*

$$\Lambda : \pi_1(G(\mathcal{Y})) \to \pi_1(G'(\mathcal{Y}'))$$

and a Λ-equivariant isometry of universal covers

$$L : \widetilde{G(\mathcal{Y})} \to \widetilde{G'(\mathcal{Y}')}.$$

We prove Theorem 4.1.1 in Section 4.3.2, using material from Section 4.3.1. The proof that L is an isometry is more difficult for complexes of groups than for graphs of groups, because, unlike trees, a local isometry of polyhedral complexes is not in general an isometry.

In Section 4.3.3, we characterize the group

$$N = \ker\left(\pi_1(G(\mathcal{Y})) \to \mathrm{Aut}(\widetilde{G(\mathcal{Y})})\right)$$

where $G(\mathcal{Y})$ is developable. This corresponds to Proposition 1.23, [Ba]. If N is trivial, then the complex of groups $G(\mathcal{Y})$ is said to be *faithful*, and we may identify the fundamental group $\pi_1(G(\mathcal{Y}))$ with a subgroup of $\mathrm{Aut}(\widetilde{G(\mathcal{Y})})$.

We also develop other, more technical, results. For example, the material in Section 4.3.4 corresponds to Section 4, [Ba]. As described in Proposition 2.1 of [Th], Haefliger's definition of morphism, when restricted to complexes of groups over 1-dimensional spaces, is not the same as a morphism of graphs of groups. Also, the universal covers of graphs of groups and of complexes of groups are defined with respect to different choices. Hence, our proofs differ in many details from those of [Ba].

An additional consideration for complexes of groups, which has no analogue in Bass–Serre theory, is the relationship between coverings and developability. In Section 4.3.5, we show :

Proposition 4.1.2. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups.*

1. *If $G'(\mathcal{Y}')$ is developable, then $G(\mathcal{Y})$ is developable.*

2. *If $G(\mathcal{Y})$ has nonpositive curvature (hence is developable), then $G'(\mathcal{Y}')$ has nonpositive curvature, hence $G'(\mathcal{Y}')$ is developable.*

We then apply this covering theory to the study of lattices in locally compact groups (see Section 4.2.1 for the basic definitions concerning lattices). Let K be a locally finite polyhedral complex, so that $\mathrm{Aut}(K)$ is a locally compact topological group (see Section 4.2.2). Let Γ be a cocompact lattice in $\mathrm{Aut}(K)$. An *overlattice* of Γ of index n is a lattice $\Gamma' \leq \mathrm{Aut}(K)$ containing Γ with $[\Gamma' : \Gamma] = n$. Let $u_\Gamma(n)$ be the number of overlattices of Γ of index n. By arguments similar to those for tree lattices (Theorem 6.5, [BK]), $u_\Gamma(n)$ is finite.

The asymptotics of $u_\Gamma(n)$ in the case K is a tree are treated in Chapter 3. In higher dimensions, suppose K is the building associated to a higher-rank algebraic group G, such as $PSL_3(\mathbb{Q}_p)$. Then G has finite index in $\mathrm{Aut}(K)$. It follows that for any Γ, $u_\Gamma(n) = 0$ for large enough n, since the covolumes of lattices in G are bounded away from 0 (Borel–Prasad [BP]). In contrast, if K is a right-angled hyperbolic building, such as Bourdon's building $I_{p,q}$ (see [Bo2]), then Thomas [Th] showed that $\mathrm{Aut}(K)$ admits infinite ascending towers of cocompact lattices. Hence there is a Γ such that $u_\Gamma(n) > 0$ for arbitrarily large n.

In order to further study the growth rate of $u_\Gamma(n)$, in Section 4.4 we specify the relationship between coverings, and subgroups of $\mathrm{Aut}(K)$ containing Γ. We define isomorphism of coverings (see Definition 4.4.1) so that the following bijection holds :

Theorem 4.1.3. *Let K be a simply connected polyhedral complex, and let Γ be a subgroup of $\mathrm{Aut}(K)$ (acting without inversions) which induces a complex of groups $G(\mathcal{Y})$. Then there is a bijection between the set of subgroups of $\mathrm{Aut}(K)$ (acting without inversions) which contain Γ, and the set of isomorphism classes of coverings of faithful, developable complexes of groups by $G(\mathcal{Y})$.*

The main ingredients in the proof of Theorem 4.1.3 are Theorem 4.1.1 above, and the results of Section 4.3.4.

We then apply Theorem 4.1.3 above to obtain upper and lower bounds for $u_\Gamma(n)$. As a corollary to Theorem 4.1.3, we show that there is a bijection between n-sheeted coverings, and overlattices of index n. Then in Section 4.5.1, we prove the following upper bound, for very general K :

Theorem 4.1.4. *Let K be a simply connected, locally finite polyhedral complex and $\Gamma \leq \mathrm{Aut}(K)$ a cocompact lattice. Then there are some positive constants C_0 and C_1, depending only on Γ, such* *that*

$$\forall n > 1, \qquad u_\Gamma(n) \le (C_0 n)^{C_1 \log^2(n)}$$

This result is asymptotically the same as the upper bound for tree lattices in [L1], and the proof uses the same deep results of finite group theory. However, the definition of covering of complexes of groups makes this bound easier to obtain (thus giving an alternative proof of the result for trees).

The lower bound below, proved in Section 4.5.2, is for certain right-angled hyperbolic buildings.

Theorem 4.1.5. *Let q be prime and let X be a Bourdon building $I_{p,2q}$. Then there is a cocompact lattice Γ in $\mathrm{Aut}(X)$, and constants C_0 and C_1, such that for any $N > 0$, there exists $n > N$ with*

$$u_\Gamma(n) \ge (C_0 n)^{C_1 \log n}$$

In fact, we construct Γ, and prove this lower bound for more general right-angled buildings. The proof relies on Proposition 4.1.2 above, and applies the Functor Theorem of [Th] to a construction for tree lattices in [L1].

Theorems 4.1.4 and 4.1.5, together with the examples given above for buildings, are presently the only known behaviors for overlattice counting functions in higher dimensions.

4.2 Background

We begin with the basic theory of lattices, in Section 4.2.1. Since the quotient of a simplicial complex by a simplicial group action is not in general a simplicial complex, it is natural to define complexes of groups over polyhedral complexes instead. In Section 4.2.2, we discuss polyhedral complexes and the topology of their automorphism groups. Small categories without loops, or scwols, are algebraic objects that substitute for polyhedral complexes. These are described in Section 4.2.3. We conclude this background material by, in Section 4.2.4, summarizing Haefliger's theory of complexes of groups, as presented in Chapter III.$\mathcal{C}$ of [BH].

4.2.1 Lattices

Let G be a locally compact topological group with left-invariant Haar measure μ. A discrete subgroup Γ of G is a *lattice* if its covolume $\mu(\Gamma\backslash G)$ is finite. A lattice is called *cocompact* or *uniform* if $\Gamma\backslash G$ is compact.

Let $\mathcal{S}$ be a left G-set such that, for each $s \in \mathcal{S}$, the stabilizer G_s is compact and open. For any discrete subgroup Γ of G, the stabilizers Γ_s are finite groups, and we define the *$\mathcal{S}$-covolume* of Γ as

$$\mathrm{Vol}(\Gamma\backslash\backslash\mathcal{S}) = \sum_{s\in\Gamma\backslash\mathcal{S}} \frac{1}{|\Gamma_s|} \leq \infty$$

It is shown in [BL], Chapter 1, that if $G\backslash\mathcal{S}$ is finite and G admits a lattice, then there is a normalization of the Haar measure μ, depending only on $\mathcal{S}$, such that for every discrete subgroup Γ of G,

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash\mathcal{S})$$

It is clear that for two lattices $\Gamma \subset \Gamma'$ of G, the index $[\Gamma' : \Gamma]$ is equal to the ratio of the covolumes $\mu(\Gamma\backslash G) : \mu(\Gamma'\backslash G)$.

4.2.2 Polyhedral complexes

Let M^n_κ be the complete, simply connected, Riemannian n-manifold of constant sectional curvature $\kappa \in \mathbb{R}$, where $n \geq 2$, and $M^1_\kappa = \mathbb{R}$ for $\kappa \leq 0$ and $M^1_\kappa = S_{1/\kappa}$, a sphere of radius $1/\kappa$ for $k > 0$.

Definition 4.2.1 (polyhedral complex). *An M_κ-polyhedral complex K is a CW-complex such that :*

1. *each open cell of dimension n is endowed with an isometry to the interior of a compact convex polyhedron in M^n_κ ; and*
2. *for each cell σ of K, the restriction of the attaching map to each open codimension one face of σ is an isometry onto an open cell of K.*

Theorem 4.2.2 (Bridson, [BH]). *An M_κ-polyhedral complex with finitely many isometry classes of cells endowed with the canonical length metric is a complete geodesic metric space.*

Let K be a locally finite, connected polyhedral complex with finitely many isometry classes of cells, and let $\mathrm{Aut}(K)$ be the group of cellular isometries, or automorphisms, of K. Then $\mathrm{Aut}(K)$ is naturally a locally compact group, with a neighborhood basis of the identity consisting of automorphisms fixing larger and larger balls. With respect to this topology, a subgroup Γ of $\mathrm{Aut}(K)$ is discrete if and only if for each cell σ of K, the stabilizer Γ_σ is finite. A subgroup Γ of $\mathrm{Aut}(K)$ is said to act *without inversions* if whenever $g \in \Gamma$ preserves a cell of K, then g fixes that cell pointwise.

4.2.3 Small categories without loops

In Chapter III.$\mathcal{C}$ of [BH], complexes of groups are presented using the language of scwols, or small categories without loops. As we explain in this section, to any polyhedral complex K one may associate a scwol $\mathcal{X}$, which has a geometric realization $|\mathcal{X}|$ isometric to the barycentric subdivision of K. Morphisms of scwols correspond to polyhedral maps, and group actions on scwols correspond to actions without inversions on polyhedral complexes.

Definition 4.2.3 (scwol). *A small category without loops (scwol) $\mathcal{X}$ is a disjoint union of a set $V(\mathcal{X})$, the vertex set, and a set $E(\mathcal{X})$, the edge set, endowed with maps*

$$i : E(\mathcal{X}) \to V(\mathcal{X}) \quad \text{and} \quad t : E(\mathcal{X}) \to V(\mathcal{X})$$

and, if $E^{(2)}(\mathcal{X})$ denotes the set of pairs (a, b) of edges where $i(a) = t(b)$, with a map

$$E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$$
$$(a, b) \mapsto ab$$

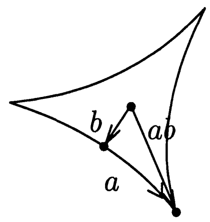
such that :

1. *if $(a, b) \in E^{(2)}(\mathcal{X})$, then $i(ab) = i(b)$ and $t(ab) = t(a)$;*
2. *if $a, b, c \in E(\mathcal{X})$ such that $i(a) = t(b)$ and $i(b) = t(c)$, then $(ab)c = a(bc)$; and*
3. *if $a \in E(\mathcal{X})$, then $i(a) \neq t(a)$.*

For $a \in E(\mathcal{X})$, the vertices $i(a)$ and $t(a)$ are called the *initial vertex* and *terminal vertex* of a respectively. If $(a, b) \in E^{(2)}(\mathcal{X})$ we say a and b are *composable*, and that ab is the *composition* of a and b. We will sometimes write $\alpha \in \mathcal{X}$ for $\alpha \in V(\mathcal{X}) \cup E(\mathcal{X})$. If $\alpha \in V(\mathcal{X})$ then $i(\alpha) = t(\alpha) = \alpha$.

The motivating example of a scwol is the scwol $\mathcal{X}$ associated to a polyhedral complex K. The set of vertices $V(\mathcal{X})$ is the set of cells of K (or the set of barycenters of the cells of K). The set of edges $E(\mathcal{X})$ is the set of 1-simplices of the barycentric subdivision of K, that is, each element of $E(\mathcal{X})$ corresponds to a pair of cells $T \subsetneq S$, with initial vertex S and terminal vertex T. The composition of the edge a corresponding to $T \subsetneq S$ and the edge b corresponding to $S \subsetneq U$ is the edge ab corresponding to $T \subsetneq U$.

Conversely, given a scwol $\mathcal{X}$, we may construct a polyhedral complex, called the *geometric realization*. For an integer $k \geq 0$, let $E^{(k)}(\mathcal{X})$ be the set of sequences $(a_1, a_2, \ldots, a_k)$ of composable edges, that is, $(a_j, a_{j+1}) \in E^{(2)}(\mathcal{X})$ if $k > 1$, $E^{(1)}(\mathcal{X}) = E(\mathcal{X})$, and $E^{(0)}(\mathcal{X}) = V(\mathcal{X})$. The geometric



realization $|\mathcal{X}|$ of $\mathcal{X}$ is defined as a polyhedral complex whose cells of dimension k are standard k-simplices indexed by the elements of $E^{(k)}(\mathcal{X})$, with the obvious attaching maps. For the details of this construction, see [BH], pp. 522–523. If $\mathcal{X}$ is the scwol associated to an M_κ-polyhedral complex K, then $|\mathcal{X}|$ may be realized as an M_κ-polyhedral complex isometric to the barycentric subdivision of K.

For a scwol $\mathcal{X}$, let $E^{\pm}(\mathcal{X})$ be the set of *oriented edges*, that is, the set of symbols a^+ and a^-, where $a \in E(\mathcal{X})$. For $e = a^+$, we define $i(e) = t(a)$, $t(e) = i(a)$ and $e^{-1} = a^-$. For $e = a^-$, we define $i(e) = i(a)$, $t(e) = t(a)$ and $e^{-1} = a^+$.

An *edge path* in $\mathcal{X}$ joining the vertex σ to the vertex τ is a sequence $(e_1, e_2, \ldots, e_n)$ of elements of $E^{\pm}(\mathcal{X})$ such that $i(e_1) = \sigma$, $i(e_{j+1}) = t(e_j)$ for $1 \leq j \leq n-1$ and $t(e_n) = \tau$.

A scwol $\mathcal{X}$ is *connected* if for any two vertices $\sigma, \tau \in V(\mathcal{X})$, there is an edge path joining σ to τ. Equivalently, $\mathcal{X}$ is connected if and only if the geometric realization $|\mathcal{X}|$ is connected. A scwol is *simply connected* if and only if its geometric realization is simply connected as a topological space.

Definition 4.2.4 (morphism of scwols). *Let $\mathcal{X}$ and $\mathcal{X}'$ be two scwols. A morphism $l : \mathcal{X} \to \mathcal{X}'$ is a map that sends $V(\mathcal{X})$ to $V(\mathcal{X}')$ and $E(\mathcal{X})$ to $E(\mathcal{X}')$, such that*

1. *for each $a \in E(\mathcal{X})$, we have $i(l(a)) = l(i(a))$ and $t(l(a)) = l(t(a))$; and*
2. *for each $(a, b) \in E^{(2)}(\mathcal{X})$, we have $l(ab) = l(a)l(b)$.*

   A nondegenerate morphism of scwols is a morphism of scwols such that in addition to (1) *and* (2),
3. *for each vertex $\sigma \in V(\mathcal{X})$, the restriction of l to the set of edges with initial vertex σ is a bijection onto the set of edges of $\mathcal{X}'$ with initial vertex $l(\sigma)$.*

An *automorphism* of a scwol $\mathcal{X}$ is a morphism $l : \mathcal{X} \to \mathcal{X}$ which has an inverse. Note that Condition (3) in Definition 4.2.4 is automatic for automorphisms.

Definition 4.2.5 (covering of scwols). *Let $\mathcal{X}$ be a (nonempty) scwol and let $\mathcal{X}'$ be a connected scwol. A nondegenerate morphism of scwols $l : \mathcal{X} \to \mathcal{X}'$ is called a covering if, for every vertex σ of $\mathcal{X}$, the restriction of l to the set of edges with terminal vertex σ is a bijection onto the set of edges of $\mathcal{X}'$ with terminal vertex $l(\sigma)$.*

Let $\mathcal{X}$ and $\mathcal{X}'$ be scwols associated to polyhedral complexes K and K' respectively. A nondegenerate polyhedral map (i.e. a map which does not decrese dimension of every maximal cell) $K \to K'$ induces a morphism of scwols $\mathcal{X} \to \mathcal{X}'$, and conversely, a morphism $l : \mathcal{X} \to \mathcal{X}'$ induces a continuous polyhedral map $|l| : |\mathcal{X}| \to |\mathcal{X}'|$ (see [BH], p. 526). The morphism l is nondegenerate if and only if the restriction of $|l|$ to the interior of each cell of K induces a homeomorphism onto the interior of a cell of K', and l is a covering if and only if $|l|$ is a (topological) covering. A morphism $l : \mathcal{X} \to \mathcal{X}$ is an automorphism of $\mathcal{X}$ if and only if $|l| : K \to K$ is an automorphism of K.

Definition 4.2.6 (group actions on scwols). *An action of a group G on a scwol $\mathcal{X}$ is a homomorphism from G to the group of automorphisms of $\mathcal{X}$ such that :*

1. *for all $a \in E(\mathcal{X})$ and $g \in G$, we have $g \cdot i(a) \neq t(a)$; and*
2. *for all $g \in G$ and $a \in E(\mathcal{X})$, if $g \cdot i(a) = i(a)$ then $g \cdot a = a$ (no "inversions").*

The action of a group G on a scwol $\mathcal{X}$ induces a *quotient scwol* $\mathcal{Y} = G\backslash\mathcal{X}$, defined as follows. The vertex set is $V(\mathcal{Y}) = G\backslash V(\mathcal{X})$ and the edge set $E(\mathcal{Y}) = G\backslash E(\mathcal{X})$. For every $a \in E(\mathcal{X})$ we have $i(Ga) = Gi(a)$ and $t(Ga) = Gt(a)$, and if $(a, b) \in E^{(2)}(\mathcal{X})$ then the composition of Ga and Gb is Gab. The natural projection $p : \mathcal{X} \to \mathcal{Y}$ is a nondegenerate morphism of scwols.

Let $\mathcal{X}$ be the scwol associated to a polyhedral complex K, and let Γ be a subgroup of $G = \mathrm{Aut}(K)$. Then Γ acts on $\mathcal{X}$, in the sense of Definition 4.2.6, if and only if Γ acts without inversions on K.

In the case K is locally finite, we define the covolume of a discrete subgroup $\Gamma \leq G$ acting on $\mathcal{X}$ by taking the set of vertices $V(\mathcal{X})$ (which corresponds to the set of cells of K) as the Γ-set $\mathcal{S}$ in Section 4.2.1. From now on, we normalize the Haar measure μ on G so that

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash V(\mathcal{X})) = \sum_{\sigma \in \Gamma\backslash V(\mathcal{X})} \frac{1}{|\Gamma_\sigma|}$$

4.2.4 Complexes of groups

In this section, we recall Haefliger's theory of complexes of groups. We mainly follow the notation and definitions of Chapter III.$\mathcal{C}$ of [BH], although at times, such as in Proposition 4.2.23 below, we indicate choices and define maps more explicitly. Section 4.2.4 defines complexes of groups and their morphisms. Section 4.2.4 then discusses groups associated to complexes of groups, in particular the fundamental group, and Section 4.2.4 discusses scwols associated to complexes of groups, in particular the universal cover. In Section 4.2.4 we describe the role of local developments and nonpositive curvature. All references to [BH] in this section are to Chapter III.$\mathcal{C}$, which the reader should consult for further details.

Objects and morphisms of the category of complexes of groups

Definition 4.2.7 (complex of groups). *Let $\mathcal{Y}$ be a scwol. A complex of groups $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ over $\mathcal{Y}$ is given by the following data :*

1. *for each $\sigma \in V(\mathcal{Y})$, a group G_σ, called the local group at σ ;*
2. *for each $a \in E(\mathcal{Y})$, an injective group homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$; and*
3. *for each pair of composable edges $(a,b) \in E^{(2)}(\mathcal{Y})$, a twisting element $g_{a,b} \in G_{t(a)}$;*

with the following properties :

(i) *$Ad(g_{a,b})\psi_{ab} = \psi_a\psi_b$, where $Ad(g_{a,b})$ denotes conjugation by $g_{a,b}$; and*

(ii) *$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$, for each triple $(a,b,c) \in E^{(3)}(\mathcal{Y})$.*

For example, any group G is a complex of groups over a singleton $\mathcal{Y} = \{*\} = V(\mathcal{Y})$, with $G_* = G$; since $E(\mathcal{Y}) = \phi$, no other data is necessary.

Definition 4.2.8 (morphism of complexes of groups). *Let $G(\mathcal{Y})$ be as in Definition 4.2.7 and let $G'(\mathcal{Y}') = (G'_{\sigma'}, \psi_{a'}, g_{a',b'})$ be another complex of groups over a scwol $\mathcal{Y}'$. Let $l : \mathcal{Y} \to \mathcal{Y}'$ be a morphism of scwols. A morphism $\phi = (\phi_\sigma, \phi(a)) : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ of complexes of groups over l consists of*

1. *a group homomorphism $\phi_\sigma : G_\sigma \to G'_{l(\sigma)}$, called the local map at σ, for each $\sigma \in V(\mathcal{Y})$; and*
2. *an element $\phi(a) \in G'_{t(l(a))}$ for each $a \in E(\mathcal{Y})$;*

such that :

(i) *$Ad(\phi(a))\psi_{l(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a$; and*

(ii) *$\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi_{l(a)}(\phi(b))g_{l(a),l(b)}$, for every $(a,b) \in E^{(2)}(\mathcal{Y})$.*

We define a morphism $\phi = (\phi_\sigma, \phi(a)) : G(\mathcal{Y}) \to G$ from a complexes of group $G(\mathcal{Y})$ to a group G as a data consisting of a group homomorphism $\phi_\sigma : G_\sigma \to G'$ for each $\sigma \in V(\mathcal{Y})$ and an element

$\phi(a) \in G'$ for each $a \in E(\mathcal{Y})$ such that

$$\phi_{t(a)}\psi_a = Ad(\phi(a))\phi_{i(a)}$$

and

$$\phi_{t(a)}(g_{a,b}) = \phi(a)\phi(b).$$

A morphism ϕ is an *isomorphism* if l is an isomorphism of scwols and ϕ_σ is a group isomorphism for every $\sigma \in V(\mathcal{Y})$. A morphism ϕ is *injective on the local groups* if ϕ_σ is injective for every σ in $V(\mathcal{Y})$.

The *composition* $\phi' \circ \phi$ of a morphism $\phi = (\phi_\sigma, \phi(a)) : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ over l and a morphism $\phi' = (\phi'_\sigma, \phi'(a)) : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ over l' is the morphism over $l' \circ l$ defined by the homomorphisms $(\phi' \circ \phi)_\sigma = \phi'_{l(\sigma)} \circ \phi_\sigma$ and the elements $(\phi' \circ \phi)(a) = \phi'_{l(t(a))}(\phi(a))\phi'(l(a))$.

Definition 4.2.9 (homotopy). *Let ϕ and ϕ' be two morphisms from $G(\mathcal{Y})$ to a group G', given respectively by $(\phi_\sigma, \phi(a))$ and $(\phi'_\sigma, \phi'(a))$. A homotopy from ϕ to ϕ' is given by a family of elements $k_\sigma \in G'$, indexed by $\sigma \in V(\mathcal{Y})$, such that*

1. *$\phi'_\sigma = \mathrm{Ad}(k_\sigma)\phi_\sigma$ for all $\sigma \in V(\mathcal{Y})$; and*
2. *$\phi'(a) = k_{t(a)}\phi(a)k^{-1}_{i(a)}$ for all $a \in E(\mathcal{Y})$.*

Let G be a group acting on a scwol $\mathcal{X}$ with quotient $\mathcal{Y} = G\backslash\mathcal{X}$, and let $p : \mathcal{X} \to \mathcal{Y}$ be the natural projection. The *complex of groups $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ associated to the action of G on $\mathcal{X}$* is defined as follows.

For each vertex $\sigma \in V(\mathcal{Y})$, choose a vertex $\overline{\sigma} \in V(\mathcal{X})$ such that $p(\overline{\sigma}) = \sigma$. For each edge $a \in E(\mathcal{Y})$ with $i(a) = \sigma$, there exists a unique edge $\overline{a} \in E(\mathcal{X})$ such that $p(\overline{a}) = a$ and $i(\overline{a}) = \overline{\sigma}$, by the 'no inversion' assumption. Choose $h_a \in G$ such that $h_a \cdot t(\overline{a}) = \overline{t(a)}$. For each $\sigma \in V(\mathcal{Y})$, let G_σ be the stabilizer in G of $\overline{\sigma} \in V(\mathcal{X})$. For each $a \in E(\mathcal{Y})$, let $\psi_a : G_{i(a)} \to G_{t(a)}$ be conjugation by h_a, that is,

$$\psi_a : g \mapsto h_a g h_a^{-1}$$

For every pair of composable edges $(a, b) \in E^{(2)}(\mathcal{Y})$, define $g_{a,b} = h_a h_b h_{ab}^{-1}$. Then $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ is a complex of groups.

When precision is needed, we denote the set of choices of $\overline{\sigma}$ and h_a in this construction by $C_\bullet$, and the complex of groups $G(\mathcal{Y})$ constructed with respect to these choices by $G(\mathcal{Y})_{C_\bullet}$. If $C'_\bullet$ is another choice of $\overline{\sigma}'$, h'_a, then an isomorphism $\phi = (\phi_\sigma, \phi(a))$ from $G(\mathcal{Y})_{C_\bullet}$ to $G(\mathcal{Y})_{C'_\bullet}$ is obtained by choosing elements $k_\sigma \in G$, such that for each $\sigma \in V(\mathcal{Y})$, $k_\sigma \cdot \overline{\sigma} = \overline{\sigma}'$. Then put $\phi_\sigma = \mathrm{Ad}(k_\sigma)|_{G_\sigma}$ and $\phi(a) = k_{t(a)} h_a k_{i(a)}^{-1} h_a'^{-1}$.

When $G(\mathcal{Y})$ is a complex of groups associated to an action of a group G, there is a canonical morphism of a complex of groups $G(\mathcal{Y})$ to a group G, $\phi_1 : G(\mathcal{Y}) \to G$, given by $\phi_1 = (\phi_\sigma, \phi(a))$, with $\phi_\sigma = G_\sigma \to G$ the inclusion, and $\phi(a) = h_a$.

Definition 4.2.10 (developable). *A complex of groups $G(\mathcal{Y})$ is developable if it is isomorphic to a complex of groups associated to the action of a group G on a scwol $\mathcal{X}$ in the above sense, with $\mathcal{Y} = G\backslash\mathcal{X}$.*

Proposition 4.2.11 (Corollary 2.15, [BH]). *A complex of groups $G(\mathcal{Y})$ is developable if and only if there exists a morphism ϕ from $G(\mathcal{Y})$ to some group G which is injective on the local groups.*

We now define coverings.

Definition 4.2.12 (covering of complexes of groups). *Let $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism of complexes of groups over a nondegenerate morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}'$ is connected. The morphism ϕ is a covering (of $G'(\mathcal{Y}')$ by $G(\mathcal{Y})$) if for each vertex $\sigma \in V(\mathcal{Y})$,*

1. *the group homomorphism $\phi_\sigma : G_\sigma \to G'_{l(\sigma)}$ is injective, and*
2. *for every $a' \in E(\mathcal{Y}')$ and $\sigma \in V(\mathcal{Y})$ with $t(a') = \sigma' = l(\sigma)$, the map*

$$\coprod_{\substack{a \in l^{-1}(a') \\ t(a)=\sigma}} G_\sigma/\psi_a(G_{i(a)}) \to G'_{\sigma'}/\psi_{a'}(G'_{i(a')})$$

   induced by

$$g \mapsto \phi_\sigma(g)\phi(a)$$

   is bijective.

From Condition (2) of this definition, it follows that

$$\sum_{\substack{a \in l^{-1}(a') \\ t(a)=\sigma}} \frac{|G_\sigma|}{|G_{i(a)}|} = \frac{|G'_{\sigma'}|}{|G'_{i(a')}|}$$

Since $\mathcal{Y}'$ is connected, the value of

$$n := \sum_{\sigma \in\, l^{-1}(\sigma')} \frac{|G'_{\sigma'}|}{|G_\sigma|} = \sum_{a \in l^{-1}(a')} \frac{|G'_{i(a')}|}{|G_{i(a)}|}$$

is independent of the vertex σ' and the edge a'. A covering of complexes of groups with the above n is said to be *n-sheeted.*

We will often use Definition 4.2.13 below, which defines a morphism of complexes of groups induced by an equivariant morphism of scwols.

Definition 4.2.13 (induced morphism). *Let $\mathcal{X}$ and $\mathcal{X}'$ be simply connected scwols, endowed with actions of groups G and G', and let $\mathcal{Y} = G\backslash\mathcal{X}$ and $\mathcal{Y}' = G'\backslash\mathcal{X}'$ be the quotient scwols. Let $L : \mathcal{X} \to \mathcal{X}'$ be a morphism of scwols which is equivariant with respect to a group morphism $\Lambda : G \to G'$. Let $l : \mathcal{Y} \to \mathcal{Y}'$ be the induced morphism of the quotients.*

For any choices $C_\bullet = (\bar{\sigma}, h_a)$ and $C'_\bullet = (\bar{\sigma'}, h_{a'})$ of data for the actions of G and G' on $\mathcal{X}$ and $\mathcal{X}'$, and for any choice $N_\bullet$ of elements $k_\sigma \in G'$ indexed by $\sigma \in V(\mathcal{Y})$ such that $k_\sigma \cdot L(\overline{\sigma}) = \overline{l(\sigma)}$, there is an associated morphism of complexes of groups

$$\lambda = \lambda_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G'(\mathcal{Y}')_{C'_\bullet}$$

over l, given by

$$\begin{aligned} \lambda_\sigma : G_\sigma &\to G'_{l(\sigma)} \\ g &\mapsto k_\sigma \Lambda(g) k_\sigma^{-1} \end{aligned}$$

and

$$\lambda(a) = k_{t(a)} \Lambda(h_a) k_{i(a)}^{-1} h'^{-1}_{l(a)}$$

(see Section 2.9(4), [BH].)

The fundamental group of a complex of groups

There are two definitions of the fundamental group of a complex of groups, which result in canonically isomorphic groups. Both definitions involve the universal group.

Definition 4.2.14 (universal group). *The universal group $FG(\mathcal{Y})$ of a complex of groups $G(\mathcal{Y})$ over a scwol $\mathcal{Y}$ is the group presented by the generating set*

$$\left(\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma \right) \coprod E^{\pm}(\mathcal{Y})$$

with the following relations :

1. *the relations in the groups G_σ ;*

2. $(a^+)^{-1} = a^-$ *and* $(a^-)^{-1} = a^+$;

3. $a^+b^+ = g_{a,b}(ab)^+$, *for every* $(a,b) \in E^{(2)}(\mathcal{Y})$; *and*

4. $\psi_a(g) = a^+ga^-$, *for every* $g \in G_{i(a)}$.

There is a natural morphism $\iota = (\iota_\sigma, \iota(a)) : G(\mathcal{Y}) \to FG(\mathcal{Y})$, where the injections $\iota_\sigma : G_\sigma \to FG(\mathcal{Y})$ takes G_σ to its image in $FG(\mathcal{Y})$, and $\iota(a) = a^+$.

Proposition 4.2.15 (Proposition 3.9, [BH]). *A complex of groups $G(\mathcal{Y})$ over a connected scwol $\mathcal{Y}$ is developable if and only if $\iota : G(\mathcal{Y}) \to FG(\mathcal{Y})$ is injective on the local groups.*

The first definition of the fundamental group of a complex of groups $G(\mathcal{Y})$ involves the choice of a basepoint $\sigma_0 \in V(\mathcal{Y})$. A *$G(\mathcal{Y})$-path* starting from σ_0 is then a sequence $(g_0, e_1, g_1, e_2, \dots, e_n, g_n)$ where $(e_1, e_2, \dots, e_n)$ is an edge path in $\mathcal{Y}$ starting from σ_0, we have $g_0 \in G_{\sigma_0}$, and $g_j \in G_{t(e_j)}$ for $1 \le j \le n$. A $G(\mathcal{Y})$-path joining σ_0 to σ_0 is called a *$G(\mathcal{Y})$-loop* at σ_0.

To each path $c = (g_0, e_1, g_1, e_2, \dots, e_n, g_n)$, we associate the element $\pi(c)$ of $FG(\mathcal{Y})$ represented by the word $g_0e_1g_1 \cdots e_ng_n$. Suppose now that c and $c' = (g_0', e_1', g_1', \dots, e_n', g_n')$ are two $G(\mathcal{Y})$-loops at σ_0. We say c and c' are *homotopic* if $\pi(c) = \pi(c')$, and denote the homotopy class of c by $[c]$. The *concatenation of c and c'* is the $G(\mathcal{Y})$-loop

$$c * c' = (g_0, e_1, \dots, e_n, g_ng_0', e_1', \dots, e_{n'}', g_{n'}')$$

The operation $[c][c'] = [c * c']$ defines a group structure on the set of homotopy classes of $G(\mathcal{Y})$-loops at σ_0.

Definition 4.2.16 (fundamental group of $G(\mathcal{Y})$ at σ_0). *The fundamental group of $G(\mathcal{Y})$ at σ_0 is the set of homotopy classes of $G(\mathcal{Y})$-loops at σ_0, with the group structure induced by concatenation. It is denoted by $\pi_1(G(\mathcal{Y}), \sigma_0)$.*

Different choices of basepoint $\sigma_0 \in V(\mathcal{Y})$ result in isomorphic fundamental groups (in fact, as subgroups of $FG(\mathcal{Y})$, the induced fundamental groups are conjugate). In fact, as subgroups of $FG(\mathcal{Y})$, they are conjugate.

The second definition of the fundamental group of a complex of groups involves the choice of a maximal tree T in the 1-skeleton of the geometric realization $|\mathcal{Y}|$. By abuse of notation, we will say that T is a maximal tree in $\mathcal{Y}$.

Proposition 4.2.17 (Theorem 3.7, [BH]). *For any maximal tree T in $\mathcal{Y}$, the fundamental group $\pi_1(G(\mathcal{Y}), \sigma_0)$ is isomorphic to the abstract group $\pi_1(G(\mathcal{Y}), T)$, presented by the generating set*

$$\left(\coprod_{\sigma \in V(\mathcal{Y})} G_\sigma\right) \coprod E^{\pm}(\mathcal{Y})$$

with the following relations :

1. *the relations in the groups G_σ ;*
2. *$(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$;*
3. *$a^+b^+ = g_{a,b}(ab)^+$, for every $(a,b) \in E^{(2)}(\mathcal{Y})$;*
4. *$\psi_a(g) = a^+ g a^-$, for every $g \in G_{i(a)}$; and*
5. *$a^+ = 1$ for every edge $a \in T$.*

If $\mathcal{Y}$ is simply connected, then $\pi_1(G(\mathcal{Y}), T)$ is isomorphic to the direct limit of the diagram of groups G_σ and monomorphisms ψ_a. The isomorphism $\pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G(\mathcal{Y}), T)$ is the restriction of the natural projection $FG(\mathcal{Y}) \to \pi_1(G(\mathcal{Y}), T)$. Its inverse κ_T is defined in the proof of Proposition 4.2.23 in Section 4.2.4 below.

Let $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Then ϕ induces a homomorphism $F\phi : FG(\mathcal{Y}) \to FG'(\mathcal{Y}')$, defined by $F\phi(g) = \phi_\sigma(g)$ for $g \in G_\sigma$, and $F\phi(a^+) = \phi(a)l(a)^+$. The restriction of $F\phi$ to $\pi_1(G(\mathcal{Y}), \sigma_0)$ is a natural homomorphism

$$\pi_1(\phi, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G'(\mathcal{Y}'), l(\sigma_0))$$

In the particular case of a morphism $\phi : G(\mathcal{Y}) \to G$, where G is a group, the induced homomorphism

$$\pi_1(\phi, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to G$$

is defined by $g \mapsto \phi_\sigma(g)$ for $g \in G_\sigma$, and $a^+ \mapsto \phi(a)$.

Developments and the universal cover

To any morphism from a complex of groups to a group is associated a scwol, called its development.

Definition 4.2.18 (development). *Let $\phi : G(\mathcal{Y}) \to G$ be a morphism from a complex of groups $G(\mathcal{Y})$ to a group G. The scwol $D(\mathcal{Y}, \phi)$, called the development of $G(\mathcal{Y})$ with respect to ϕ, is defined as follows.*

The set of vertices is

$$V(D(\mathcal{Y},\phi)) = \{([g],\sigma) : \sigma \in V(\mathcal{Y}), [g] \in G/\phi_\sigma(G_\sigma)\}$$

and the set of edges is

$$E(D(\mathcal{Y},\phi)) = \{([g],a) : a \in E(\mathcal{Y}), [g] \in G/\phi_{i(a)}(G_{i(a)})\}$$

The maps to initial and terminal vertices are given by

$$i([g],a) = ([g], i(a))$$

and

$$t([g],a) = ([g\phi(a)^{-1}], t(a))$$

and the composition of edges $([g],a)([h],b) = ([h],ab)$ *is defined where* $(a,b) \in E^{(2)}(\mathcal{Y})$, $g,h \in G$ *and* $g^{-1}h\phi(b)^{-1} \in \phi_{i(a)}(G_{i(a)})$.

The group G *acts naturally on* $D(\mathcal{Y},\phi)$ *: given* $g,h \in G$ *and* $\alpha \in \mathcal{Y}$, *the action is* $h \cdot ([g],\alpha) = ([hg],\alpha)$.

Proposition 4.2.19 (Theorems 2.13, 3.14 and 3.15, [BH]). *Let* $G(\mathcal{Y})$ *be a complex of groups over a connected scwol* $\mathcal{Y}$ *and let* G *be a group.*

1. *Let* $\phi : G(\mathcal{Y}) \to G$ *be a morphism which is injective on the local groups. Then* $G(\mathcal{Y})$ *is the complex of groups (with respect to canonical choices) associated to the action of* G *on the development* $D(\mathcal{Y},\phi)$, *and* $\phi : G(\mathcal{Y}) \to G$ *equals the canonical morphism* $\phi_1 : G(\mathcal{Y}) \to G$ *(*ϕ_1 *is defined just above Proposition 4.2.10).*

2. *Suppose* $G(\mathcal{Y})$ *is a complex of groups associated to the action of* G *on a simply connected scwol* $\mathcal{X}$, *and* $\phi_1 : G(\mathcal{Y}) \to G$ *is the canonical morphism. Then* ϕ_1 *induces a group isomorphism*

$$\pi_1(\phi_1,\sigma_0) : \pi_1(G(\mathcal{Y}),\sigma_0) \xrightarrow{\sim} G$$

(see the paragraph after Proposition 4.2.17), and there is a G*-equivariant isomorphism of scwols*

$$\Phi_1 : D(\mathcal{Y},\phi_1) \xrightarrow{\sim} \mathcal{X}$$

given by, for $g \in G$ *and* $\alpha \in \mathcal{Y}$,

$$([g],\alpha) \mapsto g \cdot \overline{\alpha}.$$

The following result, on the functoriality of developments, is used to prove Theorem 4.1.1, stated in the Introduction.

Proposition 4.2.20 (Theorem 2.18, [BH]). *Let $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ be complexes of groups over scwols $\mathcal{Y}$ and $\mathcal{Y}'$. Let $\phi : G(\mathcal{Y}) \to G$ and $\phi' : G'(\mathcal{Y}') \to G'$ be morphisms to groups G and G' and let $\Lambda : G \to G'$ be a group homomorphism. Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism over $l : \mathcal{Y} \to \mathcal{Y}'$.*

Suppose there is a homotopy from $\Lambda\phi$ to $\phi'\lambda$, given by elements $k_\sigma \in G'$ (see Definition 4.2.9). Then there is a Λ-equivariant morphism of the developments

$$L : D(\mathcal{Y}, \phi) \to D(\mathcal{Y}', \phi')$$

given by, for $g \in G$ and $\alpha \in \mathcal{Y}$,

$$([g], \alpha) \mapsto ([\Lambda(g)k_{i(\alpha)}^{-1}], l(\alpha))$$

Moreover, if ϕ and ϕ' are injective on the local groups, and λ and Λ are isomorphisms, then L is an isomorphism of scwols.

We now define the universal cover.

Definition 4.2.21 (universal cover of a developable complex of groups). *Let $G(\mathcal{Y})$ be a developable complex of groups over a connected scwol $\mathcal{Y}$. Choose a maximal tree T in $\mathcal{Y}$. Let*

$$\iota_T : G(\mathcal{Y}) \to \pi_1(G(\mathcal{Y}), T)$$

be the morphism of complexes of groups mapping the local group G_σ to its image in $\pi_1(G(\mathcal{Y}), T)$, and the edge a to the image of a^+ in $\pi_1(G(\mathcal{Y}), T)$. The development $D(\mathcal{Y}, \iota_T)$ is called a universal cover of $G(\mathcal{Y})$.

Theorem 4.2.22 (Theorem 3.13, [BH]). *The universal cover $D(\mathcal{Y}, \iota_T)$ is connected and simply connected.*

As described in Definition 4.2.18, the fundamental group $\pi_1(G(\mathcal{Y}), T)$ acts canonically on $D(\mathcal{Y}, \iota_T)$.

A group action on a scwol induces the following explicit isomorphisms of groups and scwols.

Proposition 4.2.23. *Let G be a group acting on a simply connected scwol $\mathcal{X}$, and let $G(\mathcal{Y})$ be the induced complex of groups (with respect to some choices $C_\bullet = \{\overline{\sigma}, h_a\}$). Choose a maximal tree T in $\mathcal{Y}$ and a vertex $\sigma_0 \in V(\mathcal{Y})$. For $e \in E^\pm(\mathcal{Y})$, let*

$$h_e = \begin{cases} h_a & \textit{if } e = a^+ \\ h_a^{-1} & \textit{if } e = a^- \end{cases}$$

For $\sigma \in V(\mathcal{Y})$, let $c_\sigma = (e_1, e_2, \ldots, e_n)$ be the unique edge-path contained in T, with no backtracking, which joins σ_0 to σ, and let $h_\sigma = h_{e_1} h_{e_2} \cdots h_{e_n}$.

Then there is a group isomorphism

$$\Lambda_T : \pi_1(G(\mathcal{Y}), T) \to G$$

defined on generators by

$$g \mapsto h_\sigma g h_\sigma^{-1} \text{ for } g \in G_\sigma$$
$$a^+ \mapsto h_{t(a)} h_a h_{i(a)}^{-1}$$

and a Λ_T-equivariant isomorphism of scwols

$$\tilde{L}_T : D(\mathcal{Y}, \iota_T) \to \mathcal{X}$$
$$([g], \alpha) \mapsto \Lambda_T(g) h_{i(\alpha)} \cdot \overline{\alpha}$$

Proof. For $\sigma \in V(\mathcal{Y})$, let $\pi_\sigma = e_1 e_2 \cdots e_n$ be the element of $FG(\mathcal{Y})$ corresponding to the edge-path c_σ. Then by Theorem 3.7, [BH], there is a canonical isomorphism

$$\kappa_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} \pi_1(G(\mathcal{Y}), \sigma_0)$$

defined on generators by

$$g \mapsto \pi_\sigma g \pi_\sigma^{-1} \text{ for } g \in G_\sigma$$
$$a^+ \mapsto \pi_{t(a)} a^+ \pi_{i(a)}^{-1}.$$

By Proposition 4.2.19, the canonical morphism of complexes of groups $\phi_1 : G(\mathcal{Y}) \to G$ induces a group isomorphism $\pi_1(\phi_1, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to G$. Composing κ_T with $\pi_1(\phi_1, \sigma_0)$, we obtain the group isomorphism $\Lambda_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} G$ defined above.

We now have the square

$$\begin{tikzcd} G(\mathcal{Y}) \arrow[r, "\iota_T"] \arrow[d, "\lambda = Id"'] & \pi_1(G(\mathcal{Y}), T) \arrow[d, "\Lambda_T"] \\ G(\mathcal{Y}) \arrow[r, "\phi_1"] & G. \end{tikzcd}$$

This commutes up to a homotopy from $\Lambda_T \iota_T$ to $\phi_1 \lambda$, given by the elements h_σ^{-1} for σ in $V(\mathcal{Y})$. Thus, by Proposition 4.2.20, there is a Λ_T-equivariant morphism of scwols

$$L_T : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}, \phi_1)$$
$$([g], \alpha) \mapsto ([\Lambda_T(g) h_{i(\alpha)}], \alpha)$$

which is an isomorphism since ι_T and ϕ_1 are injective on the local groups, and both λ and Λ_T are isomorphisms. Composing L_T with the G-equivariant isomorphism $\Phi_1 : D(\mathcal{Y}, \phi_1) \to \mathcal{X}$ (see Proposition 4.2.19), we obtain a Λ_T-equivariant isomorphism of scwols

$$\tilde{L}_T : D(\mathcal{Y}, \iota_T) \to \mathcal{X}$$
$$([g], \alpha) \mapsto \Lambda_T(g) h_{i(\alpha)} \cdot \overline{\alpha}$$

as required.

Local developments and nonpositive curvature

Let K be a connected polyhedral complex and let $\mathcal{Y}$ be the scwol associated to K, so that $|\mathcal{Y}|$ is the first barycentric subdivision of K. The *star* $\mathrm{St}(\sigma)$ of a vertex $\sigma \in V(\mathcal{Y})$ is the union of the interiors of the simplices in $|\mathcal{Y}|$ which meet σ. If $G(\mathcal{Y})$ is a complex of groups over $\mathcal{Y}$, then each $\sigma \in V(\mathcal{Y})$ has a *local development*, even if $G(\mathcal{Y})$ is not developable. That is, we may naturally associate to each vertex $\sigma \in V(\mathcal{Y})$ an action of G_σ on some simplicial complex $\mathrm{St}(\tilde{\sigma})$ containing a vertex $\tilde{\sigma}$, such that $\mathrm{St}(\sigma)$ is the quotient of $\mathrm{St}(\tilde{\sigma})$ by the action of G_σ. If $G(\mathcal{Y})$ is developable, then for each $\sigma \in V(\mathcal{Y})$, the local development at σ is isomorphic to the star of each lift $\tilde{\sigma}$ of σ in the universal cover $D(\mathcal{Y}, \iota_T)$.

We denote by $\mathrm{st}(\tilde{\sigma})$ the star of $\tilde{\sigma}$ in $\mathrm{St}(\tilde{\sigma})$ (here we follow the notations of Haefliger in [BH], but $\widetilde{\mathrm{st}\,\sigma}$, $\widetilde{\mathrm{St}(\sigma)}$ would be more natural notations).

Lemma 4.2.24 (Lemma 5.2, [BH]). *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups, over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Then for each $\sigma \in V(\mathcal{Y}')$, Condition* (2) *in the definition of a covering (Definition 4.2.12) is equivalent to the existence of a λ_σ-equivariant bijection* $\mathrm{st}(\tilde{\sigma}) \to \mathrm{st}(\widetilde{l(\sigma)})$.

In the case that $\mathcal{Y}$ is the scwol associated to a polyhedral complex K, each local development $\mathrm{St}(\tilde{\sigma})$ has a metric structure induced by that of K (see p. 562, [BH]). A complex of groups $G(\mathcal{Y})$ has *nonpositive curvature* if for all $\sigma \in V(\mathcal{Y})$, the local development at σ has nonpositive curvature (that is, $\mathrm{St}(\tilde{\sigma})$ is locally $\mathrm{CAT}(\kappa)$ for some $\kappa \leq 0$) in this induced metric. The importance of this condition is given by :

Theorem 4.2.25 (Theorem 4.17, [BH]). *If a complex of groups has nonpositive curvature, then it is developable.*

We will use the following condition to establish nonpositive curvature :

Lemma 4.2.26 (Remark 4.18, [BH]). *Let $\mathcal{Y}$ be the scwol associated to an M_κ-polyhedral complex K, with $\kappa \leq 0$. Then $G(\mathcal{Y})$ has nonpositive curvature if and only if, for each vertex τ of K, the geometric link of $\tilde{\tau}$ in $\mathrm{st}(\tilde{\tau})$, with the induced spherical structure, is CAT(1).*

4.3 Covering theory for complexes of groups

In this section we present results for complexes of groups analogous to those for graphs of groups in [Ba]. We consider the functoriality of morphisms of complexes of groups in Section 4.3.1 and that of coverings in Section 4.3.2. Section 4.3.3 discusses faithfulness of complexes of groups. A key technical result, the Main Lemma, is proved in Section 4.3.4. We describe the relationship between coverings and developability in Section 4.3.5.

4.3.1 Functoriality of morphisms

Proposition 4.3.1 below gives explicit constructions of the maps on fundamental groups and universal covers induced by a morphism of developable complexes of groups.

Proposition 4.3.1. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism of complexes of groups over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ and $\mathcal{Y}'$ are connected. Assume $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ are developable. For any choice of $\sigma_0 \in V(\mathcal{Y})$ and maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, λ induces a homomorphism of fundamental groups*

$$\Lambda_{T,T'} = \Lambda^{\lambda}_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

and a $\Lambda_{T,T'}$-equivariant morphism of universal covers

$$L^{\lambda}_{T,T'} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'}).$$

We also have $\Lambda^{-1}_{T,T'} = (\Lambda_{T,T'})^{-1}$.

Proof. Let $\sigma_0' = l(\sigma_0)$. Recall from the proof of Proposition 4.2.23 that there is a canonical isomorphism

$$\kappa_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} \pi_1(G(\mathcal{Y}), \sigma_0)$$

and from the paragraph below Proposition 4.2.17 that the morphism λ induces a group homomorphism $\pi_1(\lambda, \sigma_0) : \pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G'(\mathcal{Y}'), \sigma_0')$ which is the restriction of the morphism

$F\lambda : FG(\mathcal{Y}) \to FG'(\mathcal{Y}')$. The group homomorphism

$$\Lambda_{T,T'} : \pi_1(G(\mathcal{Y}),T) \to \pi_1(G'(\mathcal{Y}'),T')$$

is defined by the composition $\kappa'^{-1}_{T'} \circ \pi_1(\lambda,\sigma_0) \circ \kappa_T$:

$$\pi_1(G(\mathcal{Y}),T) \xrightarrow{\sim} \pi_1(G(\mathcal{Y}),\sigma_0) \longrightarrow \pi_1(G'(\mathcal{Y}'),\sigma'_0) \xrightarrow{\sim} \pi_1(G'(\mathcal{Y}'),T')$$

We now have a square

$$\begin{tikzcd} G(\mathcal{Y}) \arrow[r, "\iota_T"] \arrow[d, "\lambda"'] & \pi_1(G(\mathcal{Y}),T) \arrow[d, "\Lambda_{T,T'}"'] \\ G'(\mathcal{Y}') \arrow[r, "\iota_{T'}"] & \pi_1(G'(\mathcal{Y}'),T') \end{tikzcd}$$

We claim that that there is a homotopy from $\Lambda_{T,T'} \circ \iota_T$ to $\iota_{T'} \circ \lambda$. For $\sigma \in V(\mathcal{Y})$ let $\pi_\sigma = e_1 e_2 \cdots e_n$ be the element of $FG(\mathcal{Y})$ corresponding to the unique path $(e_1, e_2, \ldots, e_n)$ in T without backtracking from σ_0 to σ, and similarly for $\pi'_{l(\sigma)} \in FG'(\mathcal{Y}')$. Then for $g \in G_\sigma$, we have

$$\begin{aligned}(\Lambda_{T,T'} \circ \iota_T)(g) =& \Lambda_{T,T'}(g)\\ =& \kappa'^{-1}_{T'} \circ \pi_1(\lambda,\sigma_0) \circ \kappa_T(g)\\ =& \kappa'^{-1}_{T'} \circ \pi_1(\lambda,\sigma_0)(\pi_\sigma g \pi_\sigma^{-1})\\ =& \kappa'^{-1}_{T'} \left(F\lambda(\pi_\sigma)\lambda_\sigma(g)(F\lambda(\pi_\sigma))^{-1}\right)\\ =& \kappa'^{-1}_{T'} \left(F\lambda(\pi_\sigma)(\pi'_{l(\sigma)})^{-1}\right) (\iota_{T'} \circ \lambda_\sigma)(g)\, \kappa'^{-1}_{T'} \left(\pi'_{l(\sigma)}(F\lambda(\pi_\sigma))^{-1}\right)\end{aligned}$$

Setting

$$u_\sigma = \kappa'^{-1}_{T'} \left(F\lambda(\pi_\sigma)(\pi'_{l(\sigma)})^{-1}\right) \in \pi_1(G'(\mathcal{Y}'),T')$$

we conclude

$$(\Lambda_{T,T'} \circ \iota_T)(g) = u_\sigma(\iota_{T'} \circ \lambda_\sigma)(g)\, u_\sigma^{-1} = \mathrm{Ad}(u_\sigma)(\iota_{T'} \circ \lambda)(g)$$

Similarly, if $a \in E(\mathcal{Y})$, we compute

$$\begin{aligned}(\Lambda_{T,T'} \circ \iota_T)(a) &= \kappa'^{-1}_{T'} \circ \pi_1(\lambda,\sigma_0) \circ \kappa_T(a^+)\\ &= \kappa'^{-1}_{T'} \circ \pi_1(\lambda,\sigma_0)(\pi_{t(a)} a^+ \pi_{i(a)}^{-1})\\ &= u_{t(a)}\lambda(a) l(a)^+ u_{i(a)}^{-1}\\ &= u_{t(a)}(\iota_{T'} \circ \lambda)(a) u_{i(a)}^{-1}\end{aligned}$$

The last equality comes from the definition of composition of morphisms,

$$(\iota_{T'} \circ \lambda)(a) = (\iota_{T'})_{l(t(a))}(\lambda(a))\iota_{T'}(l(a)) = \lambda(a) l(a)^+$$

Hence the desired homotopy from $\Lambda_{T,T'} \circ \iota_T$ to $\iota_{T'} \circ \lambda$ is given by the elements u_σ^{-1}.

By Proposition 4.2.20, there is thus a $\Lambda_{T,T'}$-equivariant morphism of universal covers

$$L^\lambda_{T,T'} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$$

given by

$$([g], \alpha) \mapsto ([\Lambda_{T,T'}(g)u_{i(\alpha)}], l(\alpha))$$

The last assertion holds by definition.

Corollary 4.3.2 below says that if a diagram of morphisms of developable complexes of groups commutes, then the corresponding diagrams of the induced maps on fundamental groups and universal covers, defined in Proposition 4.3.1 above, also commute.

Corollary 4.3.2. *With the notation of Proposition 4.3.1, let $G''(\mathcal{Y}'')$ be a developable complex of groups over a connected scwol $\mathcal{Y}''$, and assume there is a morphism $\lambda' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$. Choose a maximal tree T'' in $\mathcal{Y}''$. Then the composition*

$$\lambda'' = \lambda' \circ \lambda$$

induces a group homomorphism $\Lambda_{T,T''} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G''(\mathcal{Y}''), T'')$ and a $\Lambda_{T,T''}$-equivariant morphism of universal covers $L^{\lambda''}_{T,T''} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$, such that

$$L^{\lambda''}_{T,T''} = L^{\lambda'}_{T',T''} \circ L^\lambda_{T,T'}$$

and

$$\Lambda_{T,T''} = \Lambda_{T',T''} \circ \Lambda_{T,T'}$$

Proof. The proof follows from the constructions given in Proposition 4.3.1 above, and the definition of composition of morphisms.

4.3.2 Functoriality of coverings

In this section we prove Theorem 4.1.1, stated in the Introduction. The maps $\Lambda_{T,T'}$ and $L^\lambda_{T,T'}$ are those defined in Proposition 4.3.1 above.

Proposition 4.3.3. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ and $\mathcal{Y}'$ are connected. Assume $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ are developable.*

For any choice of $\sigma_0 \in V(\mathcal{Y})$ *and maximal trees* T *and* T' *in* $\mathcal{Y}$ *and* $\mathcal{Y}'$ *respectively, the induced homomorphism of fundamental groups*

$$\Lambda_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

is a monomorphism and

$$L^{\lambda}_{T,T'} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$$

is a $\Lambda_{T,T'}$*-equivariant isomorphism of scwols.*

Proof. We begin with Lemma 4.3.4 below, which shows that $L^{\lambda}_{T,T'}$ is a covering of scwols (see Definition 4.2.5). Corollary 4.3.5 of this lemma shows that $L^{\lambda}_{T,T'}$ is an isomorphism of scwols. We then use this result to show that $\Lambda_{T,T'}$ is injective.

Lemma 4.3.4. *The morphism* $L^{\lambda}_{T,T'}$ *is a covering of scwols.*

Proof. Let $g \in \pi_1(G(\mathcal{Y}), T)$ and $\sigma \in V(\mathcal{Y})$. We first show that $L^{\lambda}_{T,T'}$ is injective on the set of edges with terminal vertex $([g], \sigma)$. Suppose a_1 and a_2 are edges of $\mathcal{Y}$ (with $t(a_1) = t(a_2) = \sigma$), that for some $h_1, h_2 \in \pi_1(G(\mathcal{Y}), T)$

$$t\left([h_1], a_1\right) = ([g], \sigma) = t\left([h_2], a_2\right)$$

and that

$$L^{\lambda}_{T,T'}\left([h_1], a_1\right) = L^{\lambda}_{T,T'}\left([h_2], a_2\right)$$

By definition of $L^{\lambda}_{T,T'}$, we then have $l(a_1) = l(a_2) = a'$ say, with $t(a') = l(t(a_1)) = l(\sigma) = \sigma'$. Also, by definition of the map $t : E(D(\mathcal{Y}, \iota_T)) \to V(D(\mathcal{Y}, \iota_T))$, we have, for some $h \in G_\sigma$,

$$h_1 a_1^- = h_2 a_2^- h^{-1}$$

Now by definition of $L^{\lambda}_{T,T'}$, it follows that the group $G'_{i(a')}$ contains

$$\begin{aligned}
&\left(\Lambda_{T,T'}(h_1) u_{i(a_1)}\right)^{-1} \left(\Lambda_{T,T'}(h_2) u_{i(a_2)}\right) \\
&= u_{i(a_1)}^{-1} \Lambda_{T,T'}\left(a_1^- \, h \, a_2^+\right) u_{i(a_2)} \\
&= u_{i(a_1)}^{-1} u_{i(a_1)} l(a_1)^- \lambda(a_1)^{-1} u_{t(a_1)}^{-1} u_\sigma \lambda_\sigma(h) u_\sigma^{-1} u_{t(a_2)} \lambda(a_2) l(a_2)^+ u_{i(a_2)}^{-1} u_{i(a_2)} \\
&= a'^- \lambda(a_1)^{-1} \lambda_\sigma(h) \lambda(a_2) a'^+
\end{aligned}$$

Thus by the relation $a'^+ k a'^- = \psi_{a'}(k)$, for all $k \in G'_{i(a')}$,

$$\lambda(a_1)^{-1} \, \lambda_\sigma(h) \, \lambda(a_2) \in \psi_{a'}(G'_{i(a')})$$

That is, $\lambda(a_1)$ and $\lambda_\sigma(h)\lambda(a_2)$ belong to the same coset of $\psi_{a'}(G'_{i(a')})$ in $G'_{\sigma'}$. By Condition (2) in the definition of a covering (Definition 4.2.12, this implies $a_1 = a_2 = a$, say, and $h \in \psi_a(G_{i(a)})$. It follows that h_1 and h_2 belong to the same coset of $G_{i(a)}$ in $\pi_1(G(\mathcal{Y}), T)$. Thus $L^\lambda_{T,T'}$ is injective on the set of edges with terminal vertex $([g], \sigma)$.

We now show that $L^\lambda_{T,T'}$ surjects onto the set of edges of $D(\mathcal{Y}', \iota_{T'})$ with terminal vertex $L^\lambda_{T,T'}([g], \sigma)$. Suppose

$$t([h'], a') = L^\lambda_{T,T'}([g], \sigma)$$

where $h' \in \pi_1(G'(\mathcal{Y}'), T')$, $a' \in E(\mathcal{Y}')$. Then $t(a') = \sigma' = l(\sigma)$ and by definition of $L^\lambda_{T,T'}$,

$$h'a'^- = \Lambda_{T,T'}(g) u_\sigma k_{\sigma'} \tag{4.1}$$

for some $k_{\sigma'} \in G'_{\sigma'}$. By Condition 2 in the definition of a covering, there exists an edge $a \in E(\mathcal{Y})$ with $l(a) = a'$ and $t(a) = \sigma$, and an element $k_\sigma \in G_\sigma$, such that $\lambda_\sigma(k_\sigma)\lambda(a)$ and $k_{\sigma'}$ belong to the same coset of $\psi_{a'}(G_{i(a')})$ in $G'_{\sigma'}$. Let $h = gk_\sigma a^+ \in \pi_1(G(\mathcal{Y}), T)$ and note that by Definition 4.2.18,

$$t([h], a) = \left([gk_\sigma a^+ \iota_T(a)^{-1}], t(a)\right) = \left([gk_\sigma a^+ a^-], \sigma\right) = ([gk_\sigma], \sigma) = ([g], \sigma)$$

We claim

$$L^\lambda_{T,T'}([h], a) = ([h'], a')$$

By Equation (4.1) above, the choice of a and k_σ and the relation $\psi_{a'}(k') = a'^+ k' a'^-$ for all $k' \in G'_{i(a')}$, we have

$$\begin{aligned}
\Lambda_{T,T'}(h) u_{i(a)} &= \Lambda_{T,T'}(g) u_\sigma \lambda_\sigma(k_\sigma) u_\sigma^{-1} u_{t(a)} \lambda(a) l(a)^+ u_{i(a)}^{-1} u_{i(a)} \\
&= h'a'^- k_{\sigma'}^{-1} \lambda_\sigma(k_\sigma) \lambda(a) a'^+ \\
&\in h' G'_{i(a')}
\end{aligned}$$

Hence,

$$L^\lambda_{T,T'}([h], a) = \left(\left[\Lambda_{T,T'}(h) u_{i(a)}\right], a'\right) = ([h'], a')$$

We conclude that $L^\lambda_{T,T'}$ is a covering of scwols.

Corollary 4.3.5. *Under the assumptions of Proposition 4.3.3, the morphism $L^\lambda_{T,T'} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$ is an isomorphism of scwols.*

Proof. By Lemma 4.3.4, $L^\lambda_{T,T'}$ is a covering morphism. Since $D(\mathcal{Y}', \iota_{T'})$ is connected, $L^\lambda_{T,T'}$ is surjective, and since $D(\mathcal{Y}, \iota_T)$ is connected and $D(\mathcal{Y}', \iota_{T'})$ is simply connected, $L^\lambda_{T,T'}$ is injective. See Remark 1.9(2), [BH].

We complete the proof of Proposition 4.3.3 by showing that $\Lambda_{T,T'}$ is a monomorphism of groups. Suppose $g \in \pi_1(G(\mathcal{Y}), T)$ and $\Lambda_{T,T'}(g) = 1$. Since $L^{\lambda}_{T,T'}$ is injective and $\Lambda_{T,T'}$-equivariant, g must act trivially on $D(\mathcal{Y}, \iota_T)$. In particular,

$$g \cdot ([1], \sigma_0) = ([g], \sigma_0) = ([1], \sigma_0)$$

so $g \in G_{\sigma_0}$. We then calculate

$$\begin{aligned}\Lambda_{T,T'}(g) &= \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T((\iota_T)_{\sigma_0}(g)) \\ &= \kappa'^{-1}_{T'}(\lambda_{\sigma_0}((\iota_T)_{\sigma_0}(g))) \\ &= 1\end{aligned}$$

Since $\kappa'^{-1}_{T'}$, λ_{σ_0} and $(\iota_T)_{\sigma_0}$ are each injective, this implies $g = 1$. Thus $\Lambda_{T,T'}$ is injective.

Corollary 4.3.6. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups. Suppose for some $\kappa \in \mathbb{R}$ that the scwols $\mathcal{Y}$ and $\mathcal{Y}'$ are associated to M_κ-polyhedral complexes with finitely many isometry classes of cells. If $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ are developable, then the geometric realizations of their respective universal covers are isometric (as polyhedral complexes).*

4.3.3 Faithfulness

Definition 4.3.7 (faithful). *Let $G(\mathcal{Y})$ be a developable complex of groups. We say $G(\mathcal{Y})$ is faithful if the natural homomorphism $\pi_1(G(\mathcal{Y}), T) \to \mathrm{Aut}(D(\mathcal{Y}, \iota_T))$ is a monomorphism, for any choice of maximal tree T in $\mathcal{Y}$.*

If $G(\mathcal{Y})$ is a complex of groups associated to the action of a group G on a scwol $\mathcal{X}$, then $G(\mathcal{Y})$ is faithful.

Proposition 4.3.8 below may be used to give sufficient conditions for faithfulness.

Proposition 4.3.8. *Let $G(\mathcal{Y})$ be a developable complex of groups over a connected scwol $\mathcal{Y}$. Choose a maximal tree T in $\mathcal{Y}$, and identify by ι_T each local group G_σ with its image in $\pi_1(G(\mathcal{Y}), T)$ under the morphism ι_T. Let*

$$N_T = \ker(\pi_1(G(\mathcal{Y}), T) \to \mathrm{Aut}(D(\mathcal{Y}, \iota_T)))$$

Then

1. *N_T is a vertex subgroup, that is $N_T \le G_\sigma$ for each $\sigma \in V(\mathcal{Y})$.*
2. *N_T is $\mathcal{Y}$-invariant, that is $\psi_a(N_T) = N_T$ for each $a \in E(\mathcal{Y})$.*

3. N_T is normal, that is $N_T \trianglelefteq G_\sigma$ for each $\sigma \in V(\mathcal{Y})$.

4. N_T is maximal : if N'_T is another $\mathcal{Y}$-invariant normal vertex subgroup then $N'_T \leq N_T$.

Proof. If $h \in N_T$, then for every $\sigma \in V(\mathcal{Y})$,

$$h \cdot ([1], \sigma) = ([h], \sigma) = ([1], \sigma)$$

thus $h \in G_\sigma$. This proves (1). Since N_T is normal in $\pi_1(G(\mathcal{Y}), T)$ it is normal in each G_σ, proving (3).

To prove (2), let $a \in E(\mathcal{Y})$. In the group $\pi_1(G(\mathcal{Y}), T)$ the following relation holds for each $g \in G_{i(a)}$:

$$\psi_a(g) = a^+ g a^-$$

Since N_T is a subgroup of $G_{i(a)}$ and N_T is normal in $\pi_1(G(\mathcal{Y}), T)$, it follows that

$$\psi_a(N_T) = a^+ N_T a^- = N_T$$

as required.

To prove (4), we have, for all $g \in \pi_1(G(\mathcal{Y}), T)$ and $\alpha \in \mathcal{Y}$,

$$N'_T \cdot ([g], \alpha) = g N'_T g^{-1} \cdot ([g], \alpha) = g \cdot ([1], \alpha) = ([g], \alpha)$$

since N'_T is normal in $\pi_1(G(\mathcal{Y}), T)$ and N'_T is a subgroup of $G_{i(\alpha)}$. Hence N'_T is contained in N_T, as claimed.

4.3.4 Other functoriality results

This section contains results similar to those in Section 4, [Ba].

We first prove the following useful characterization of isomorphisms of complexes of groups. This result corresponds to Corollary 4.6, [Ba].

Proposition 4.3.9. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a morphism of developable complexes of groups over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}$ and $\mathcal{Y}'$ are connected scwols. For any choice of $\sigma_0 \in V(\mathcal{Y})$ and of maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, λ is an isomorphism if and only if both of the maps $L^\lambda_{T,T'}$ and $\Lambda_{T,T'}$ are isomorphisms.*

Proof. If λ is an isomorphism, it is clearly a covering. Proposition 4.3.3 thus implies that $L^\lambda_{T,T'}$ is an isomorphism of scwols and $\Lambda_{T,T'}$ is a monomorphism of groups. Since λ^{-1} is also a covering, $\Lambda^{-1}_{T,T'} = (\Lambda_{T,T'})^{-1}$ is also a monomorphism, hence $\Lambda_{T,T'}$ is an isomorphism.

Conversely, suppose λ is not an isomorphism, thus one of λ and λ^{-1} is not a covering. Without loss of generality, we assume λ is not a covering. Then either

1. there exists $\sigma \in V(\mathcal{Y})$ such that the homomorphism $\lambda_\sigma : G_\sigma \to G'_{l(\sigma)}$ is not injective, or
2. there exists $a' \in E(\mathcal{Y}')$ and $\sigma \in V(\mathcal{Y})$ with $t(a') = \sigma' = l(\sigma)$, such that the map

$$\coprod_{\substack{a \in l^{-1}(a') \\ t(a)=\sigma}} G_\sigma/\psi_a(G_{i(a)}) \to G'_{\sigma'}/\psi_{a'}(G'_{i(a')})$$

induced by

$$g \mapsto \lambda_\sigma(g)\lambda(a)$$

is not bijective.

Condition (1) implies that the map $\Lambda_{T,T'}$ is not a monomorphism at G_σ, thus $\Lambda_{T,T'}$ is not an isomorphism. Condition (2) implies that $L^\lambda_{T,T'}$ is not a local bijection at $\mathrm{St}(\tilde{\sigma})$ (see Remark 5.3, [BH]), thus the map $L^\lambda_{T,T'}$ is not an isomorphism.

The Main Lemma below, which corresponds to Proposition 4.4, [Ba], will be used many times in Section 4.4. The data for the Main Lemma is as follows.

Let $\mathcal{X}$ and $\mathcal{X}'$ be simply connected scwols, acted upon by groups G and G' respectively, with quotient scwols $\mathcal{Y} = G\backslash\mathcal{X}$ and $\mathcal{Y}' = G'\backslash\mathcal{X}'$. Let $G(\mathcal{Y})_{C_\bullet}$ and $G'(\mathcal{Y}')_{C'_\bullet}$ be the quotient complexes of groups associated to the actions of G and G', with respect to choices $C_\bullet = (\overline{\sigma}, h_a)$ and $C'_\bullet = (\overline{\sigma'}, h_{a'})$.

Suppose $L : \mathcal{X} \to \mathcal{X}'$ is a morphism of scwols which is equivariant with respect to some group homomorphism $\Lambda : G \to G'$. Let $l : \mathcal{Y} \to \mathcal{Y}'$ be the induced morphism of quotient scwols. Fix $\sigma_0 \in \mathcal{Y}$ and let $\sigma'_0 = l(\sigma_0)$. Let $N_\bullet = \{k_\sigma\}$ be a set of elements of G' such that $k_\sigma \cdot L(\overline{\sigma}) = \overline{l(\sigma)}$ for all $\sigma \in V(\mathcal{Y})$.

With respect to these choices, there is an induced morphism $\lambda = \lambda_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ (see Definition 4.2.13). For any choice of maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$, respectively, let

$$\Lambda^\lambda_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$$

be the homomorphism of groups induced by λ and let

$$L^\lambda_{T,T'} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$$

be the associated $\Lambda^\lambda_{T,T'}$-equivariant morphism of scwols (see Proposition 4.3.1). By Proposition 4.2.23 we have isomorphisms of scwols

$$\tilde{L}_T : D(\mathcal{Y}, \iota_T) \xrightarrow{\sim} \mathcal{X} \quad \text{and} \quad \tilde{L}_{T'} : D(\mathcal{Y}', \iota_{T'}) \xrightarrow{\sim} \mathcal{X}'$$

which are equivariant with respect to group isomorphisms

$$\Lambda_T : \pi_1(G(\mathcal{Y}), T) \xrightarrow{\sim} G \quad \text{and} \quad \Lambda_{T'} : \pi_1(G'(\mathcal{Y}'), T') \xrightarrow{\sim} G'$$

respectively.

Main Lemma 4.3.10. *Suppose $C_\bullet$ and $C'_\bullet$ are chosen so that $L(\overline{\sigma_0}) = \overline{l(\sigma_0)} = \overline{\sigma'_0}$, and $N_\bullet$ is chosen so that $k_{\sigma_0} = 1$. Then the following diagrams commute :*

1.

$$\begin{tikzcd} \pi_1(G(\mathcal{Y}), T) \arrow[r, "\Lambda^{\lambda}_{T,T'}"] \arrow[d, "\Lambda_T"] & \pi_1(G'(\mathcal{Y}'), T') \arrow[d, "\Lambda_{T'}"] \\ G \arrow[r, "\Lambda"] & G' \end{tikzcd}$$

2.

$$\begin{tikzcd} D(\mathcal{Y}, \iota_T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[d, "\tilde{L}_T"] & D(\mathcal{Y}', \iota_{T'}) \arrow[d, "\tilde{L}_{T'}"] \\ \mathcal{X} \arrow[r, "L"] & \mathcal{X}' \end{tikzcd}$$

Proof. We first show the commutativity of (1), and then use this diagram and equivariance to prove that (2) commutes.

By construction,

$$\Lambda_T = \pi_1(\phi_1, \sigma_0) \circ \kappa_T \quad \text{and} \quad \Lambda_{T'} = \pi_1(\phi'_1, \sigma'_0) \circ \kappa'_{T'}$$

where $\phi_1 : G(\mathcal{Y}) \to G$ and $\phi'_1 : G'(\mathcal{Y}') \to G'$ are the canonical morphisms. Also, $\Lambda^{\lambda}_{T,T'} = \kappa'^{-1}_{T'} \circ \pi_1(\lambda, \sigma_0) \circ \kappa_T$. Therefore it is enough to show that the following diagram commutes :

$$\begin{tikzcd} \pi_1(G(\mathcal{Y}), \sigma_0) \arrow[r, "\pi_1(\lambda,\sigma_0)"] \arrow[d, "\pi_1(\phi_1,\sigma_0)"] & \pi_1(G'(\mathcal{Y}'), \sigma'_0) \arrow[d, "\pi_1(\phi'_1,\sigma'_0)"] \\ G \arrow[r, "\Lambda"] & G' \end{tikzcd}$$

Let $x \in \pi_1(G(\mathcal{Y}), \sigma_0)$. Then x has the form

$$x = g_{\sigma_0} e_1 g_{\sigma_1} \cdots e_n g_{\sigma_n}$$

where $(g_{\sigma_0}, e_1, g_{\sigma_1}, \ldots, e_n, g_{\sigma_n})$ is a $G(\mathcal{Y})$-loop based at $\sigma_0 = \sigma_n$. It follows that

$$\pi_1(\phi_1, \sigma_0)(x) = g_{\sigma_0} h_{e_1} g_{\sigma_1} \cdots h_{e_n} g_{\sigma_n}$$

where the elements h_{e_j} are as defined in Proposition 4.2.23. We now compute

$$\begin{aligned}
&\pi_1(\phi_1', \sigma_0') \circ \pi_1(\lambda, \sigma_0)(x) \\
&= (k_{\sigma_0}\Lambda(g_{\sigma_0})k_{\sigma_0}^{-1})(k_{\sigma_0}\Lambda(h_{e_1})k_{\sigma_1}^{-1}h_{l(e_1)}^{-1})h_{l(e_1)}(k_{\sigma_1}\Lambda(g_{\sigma_1})k_{\sigma_1}^{-1})\cdots(k_{\sigma_n}\Lambda(g_{\sigma_n})k_{\sigma_n}^{-1}) \\
&= k_{\sigma_0}\Lambda(g_{\sigma_0}h_{e_1}g_{\sigma_1}\cdots h_{e_n}g_{\sigma_n})k_{\sigma_n}^{-1} \\
&= \Lambda \circ \pi_1(\phi_1, \sigma_0)(x)
\end{aligned}$$

since $k_{\sigma_0} = k_{\sigma_n} = 1$. Thus (1) commutes.

To prove that (2) commutes, let

$$\tilde{L} = \tilde{L}_{T'} \circ L^{\lambda}_{T,T'} \circ \tilde{L}_T^{-1}.$$

We will show that $\tilde{L} = L$. By the equivariance of the morphisms of scwols used to define $\tilde{L}$, and the commutativity of (1), we have that $\tilde{L}$ is Λ-equivariant. Thus it is enough to check (for example) that $\tilde{L}(h_{i(\alpha)}\overline{\alpha}) = L(h_{i(\alpha)}\overline{\alpha})$ for all $\alpha \in \mathcal{Y} = V(\mathcal{Y}) \cup E(\mathcal{Y})$. By Proposition 4.2.23,

$$\begin{aligned}
\tilde{L}(h_{i(\alpha)}\overline{\alpha}) &= \tilde{L}_{T'} \circ L^{\lambda}_{T,T'}([1], \alpha) \\
&= \tilde{L}_{T'}([u_{i(\alpha)}], l(\alpha)) \\
&= \Lambda_{T'}(u_{i(\alpha)})h_{i(l(\alpha))}\overline{l(\alpha)}
\end{aligned}$$

Let $\pi_{i(\alpha)} = e_1e_2\cdots e_n$ be the element of $FG(\mathcal{Y})$ which corresponds to the non-backtracking path in T from σ_0 to $i(\alpha)$, and similarly for $\pi'_{i(l(\alpha))} = e_1'e_2'\cdots e_{n'}'$ in $FG'(\mathcal{Y}')$. Then

$$\begin{aligned}
\Lambda_{T'}(u_{i(\alpha)}) &= \Lambda_{T'} \circ \kappa_{T'}'^{-1}\left(F\lambda(\pi_{i(\alpha)})(\pi'_{i(l(\alpha))})^{-1}\right) \\
&= \pi_1(\phi_1', \sigma_0')\left(F\lambda(\pi_{i(\alpha)})(\pi'_{i(l(\alpha))})^{-1}\right) \\
&= \pi_1(\phi_1', \sigma_0')\left(F\lambda(e_1)F\lambda(e_2)\cdots F\lambda(e_n)e_{n'}'^{-1}\cdots e_2'^{-1}e_1'^{-1}\right) \\
&= k_{\sigma_0}\Lambda(h_{e_1})k_{\sigma_1}^{-1}k_{\sigma_1}\Lambda(h_{e_2})k_{\sigma_2}^{-1}\cdots k_{\sigma_{n-1}}\Lambda(h_{e_n})k_{\sigma_n}^{-1}h_{e_{n'}'}^{-1}\cdots h_{e_2'}^{-1}h_{e_1'}^{-1} \\
&= k_{\sigma_0}\Lambda(h_{e_1}h_{e_2}\cdots h_{e_n})k_{\sigma_n}^{-1}(h_{e_1'}h_{e_2'}\cdots h_{e_{n'}'})^{-1} \\
&= \Lambda(h_{i(\alpha)})k_{\sigma_n}^{-1}h_{i(l(\alpha))}^{-1}
\end{aligned}$$

since $k_{\sigma_0} = 1$. Substituting, we obtain finally

$$\begin{aligned}
\tilde{L}(h_{i(\alpha)}\overline{\alpha}) &= \Lambda(h_{i(\alpha)})k_{\sigma_n}^{-1}\overline{l(\alpha)} \\
&= \Lambda(h_{i(\alpha)})k_{i(\alpha)}^{-1}\overline{l(\alpha)} \\
&= \Lambda(h_{i(\alpha)})L(\overline{\alpha}) \\
&= L(h_{i(\alpha)}\overline{\alpha})
\end{aligned}$$

as desired. Note that $k_{i(\alpha)}^{-1}\overline{l(\alpha)} = L(\overline{\alpha})$ for $\alpha \in V(\mathcal{Y})$ by definition of k_σ and it holds for $\alpha \in E(\mathcal{Y})$ as well by the 'no inversion' assumption on the group action.

This completes the proof of the Main Lemma.

The following result makes precise the relationship between a developable complex of groups $G(\mathcal{Y})$ and the complex of groups induced by the action of $\pi_1(G(\mathcal{Y}), T)$ on $D(\mathcal{Y}, \iota_T)$, for some maximal tree T in $\mathcal{Y}$. It will be used to prove the Corollary to the Main Lemma below.

Lemma 4.3.11. *Let $G(\mathcal{Y})$ be a developable complex of groups over a connected scwol $\mathcal{Y}$. Choose a vertex $\sigma_0 \in V(\mathcal{Y})$ and a maximal tree T in $\mathcal{Y}$. Let $\mathbb{Z}$ be the quotient scwol*

$$\mathbb{Z} = \pi_1(G(\mathcal{Y}), T)\backslash D(\mathcal{Y}, \iota_T)$$

and let f be the canonical isomorphism of scwols

$$\begin{aligned} f : \mathcal{Y} &\to \mathbb{Z} \\ \alpha &\mapsto \pi_1(G(\mathcal{Y}), T) \cdot ([1], \alpha) \end{aligned}$$

Let $C_\bullet$ be the following data for the action of $\pi_1(G(\mathcal{Y}), T)$ on $D(\mathcal{Y}, \iota_T)$:

$$\overline{f(\alpha)} = ([1], \alpha) \quad \text{and} \quad h_{f(a)} = a^+$$

and let $G(\mathbb{Z})_{C_\bullet}$ be the complex of groups associated to this data. Then there is an isomorphism of complexes of groups

$$\theta : G(\mathcal{Y}) \to G(\mathbb{Z})$$

over f such that

$$\Lambda^\theta_{T,f(T)} = \Lambda^{-1}_{f(T)} \quad \text{and} \quad L^\theta_{T,f(T)} = \tilde{L}^{-1}_{f(T)}$$

where $f(T)$ is the image of T in $\mathbb{Z}$.

Proof. We define θ by $\theta_\sigma(g) = g$ for each $g \in G_\sigma$, and $\theta(a) = 1$ for each $a \in E(\mathcal{Y})$ (here we are identifying G_σ with its image in $\pi_1(G(\mathcal{Y}), T)$).

We then have

$$\Lambda^\theta_{T,f(T)} \circ \Lambda_{f(T)} = \kappa^{-1}_{f(T)} \circ \pi_1(\theta, \sigma_0) \circ \kappa_T \circ \pi_1(\phi_1, f(\sigma_0)) \circ \kappa_{f(T)}$$

We claim that

$$\pi_1(\theta, \sigma_0) \circ \kappa_T \circ \pi_1(\phi_1, f(\sigma_0)) = 1 \tag{4.2}$$

Let $g \in \pi_1(G(\mathbb{Z}), f(\sigma_0))$. Then $g = g_0 f(e_1) g_1 \cdots f(e_n) g_n$ for some $G(\mathbb{Z})$-loop $(g_0, f(e_1), g_1, \dots, f(e_n), g_n)$ based at $f(\sigma_0) = f(\sigma_n)$, and so

$$\begin{aligned}\kappa_T \circ \pi_1(\phi_1, f(\sigma_0))(g) &= \kappa_T(g_0 h_{f(e_1)} g_1 \cdots h_{f(e_n)} g_n) \\ &= \pi_{\sigma_0} g_0 \pi_{\sigma_0}^{-1} \kappa_T(h_{f(e_1)}) \pi_{\sigma_1} g_1 \pi_{\sigma_1}^{-1} \cdots \kappa_T(h_{f(e_n)}) \pi_{\sigma_n} g_n \pi_{\sigma_n}^{-1}\end{aligned}$$

where π_σ is associated to the unique non-backtracking path in T from σ_0 to σ. Now, applying $h_{f(a)} = a^+$ and $\kappa_T(a^+) = \pi_{t(a)} a^+ \pi_{i(a)}^{-1}$, as well as $\pi_{\sigma_0} = \pi_{\sigma_n} = 1$, we have

$$\begin{aligned}\pi_1(\theta, \sigma_0) \circ \kappa_T \circ \pi_1(\phi_1, f(\sigma_0))(g) &= \pi_1(\theta, \sigma_0)(g_0 e_1 g_1 \cdots e_n g_n) \\ &= g_0 f(e_1) g_1 \cdots f(e_n) g_n \\ &= g\end{aligned}$$

and so Equation (4.2) holds. Thus $\Lambda^\theta_{T,f(T)} \circ \Lambda_{f(T)} = 1$. By conjugating Equation (4.2), we obtain

$$\Lambda_{f(T)} \circ \Lambda^\theta_{T,f(T)} = 1$$

and conclude that $\Lambda^\theta_{T,f(T)} = \Lambda^{-1}_{f(T)}$.

To show that $L^\theta_{T,f(T)} = \tilde{L}^{-1}_{f(T)}$, let

$$u_\sigma = \kappa^{-1}_{f(T)} \left\{ F\theta(\pi_\sigma) \left(\pi'_{f(\sigma)} \right)^{-1} \right\}$$

be the elements of $\pi_1(G(\mathbb{Z}), f(T))$ with respect to which $L^\theta_{T,f(T)}$ is defined. Here π_σ denotes the non-backtracking path in T from σ_0 to σ, and similarly for $\pi'_{f(\sigma)}$ and $f(T)$. By definition of θ,

$$F\theta(\pi_\sigma) = \pi'_{f(\sigma)}$$

hence $u_\sigma = 1$ for all $\sigma \in V(\mathcal{Y})$. Also, for each $\alpha \in \mathcal{Y}$, the element $h_{i(f(\alpha))} \in \pi_1(G(\mathcal{Y}), T)$ with respect to which $\tilde{L}_{f(T)}$ is defined is a product of oriented edges $a^\pm$ with $a \in T$. Hence $h_{i(f(\alpha))} = 1$.

Applying these facts, we have, for $g \in \pi_1(G(\mathcal{Y}), T)$ and $\alpha \in \mathcal{Y}$,

$$\begin{aligned}\tilde{L}_{f(T)} \circ L^\theta_{T,f(T)}([g], \alpha) &= \tilde{L}_{f(T)}([\Lambda^\theta_{T,f(T)}(g)], f(\alpha)) \\ &= \Lambda_{f(T)} \circ \Lambda^\theta_{T,f(T)}(g) h_{i(f(\alpha))} \overline{f(\alpha)} \\ &= g([1], \alpha) \\ &= ([g], \alpha)\end{aligned}$$

and

$$\begin{aligned} L^{\theta}_{T,f(T)} \circ \tilde{L}_{f(T)}([g], f(\alpha)) &= L^{\theta}_{T,f(T)}(\Lambda_{f(T)}(g)\overline{f(\alpha)}) \\ &= L^{\theta}_{T,f(T)}([\Lambda_{f(T)}(g)], \alpha) \\ &= ([\Lambda^{\theta}_{T,f(T)} \circ \Lambda_{f(T)}(g)], f(\alpha)) \\ &= ([g], f(\alpha)) \end{aligned}$$

Thus $L^{\theta}_{T,f(T)} = \tilde{L}^{-1}_{f(T)}$.

The following result corresponds to Corollary 4.5, [Ba].

Corollary 4.3.12. *Let $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ be developable complexes of groups over connected scwols $\mathcal{Y}$ and $\mathcal{Y}'$, and choose maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively. Suppose $L : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$ is a morphism of scwols which is equivariant with respect to some homomorphism of groups $\Lambda : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$. If there is a $\sigma_0 \in V(\mathcal{Y})$ such that*

$$L([1], \sigma_0) = ([1], \sigma_0')$$

for some $\sigma_0' \in V(\mathcal{Y}')$, then there exists a morphism $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ of complexes of groups such that $L = L^{\lambda}_{T,T'}$ and $\Lambda = \Lambda^{\lambda}_{T,T'}$.

Proof. Let the quotient scwol $\mathbb{Z}$, the isomorphism $f : \mathcal{Y} \to \mathbb{Z}$, the data $C_\bullet$, the complex of groups $G(\mathcal{Y})_{C_\bullet}$ and the isomorphism $\theta : G(\mathcal{Y}) \to G(\mathbb{Z})$ be as in the statement of Lemma 4.3.11 above, and similarly for $\mathbb{Z}'$, f', $C'_\bullet$, $G'(\mathcal{Y}')_{C'_\bullet}$ and θ'. Let $l : \mathbb{Z} \to \mathbb{Z}'$ be the map of quotient scwols induced by L and Λ. By definition of l, $C_\bullet$ and $C'_\bullet$, and by the assumption on L, we have

$$L(\overline{\sigma_0}) = \overline{l(\sigma_0)}$$

so we may choose $N_\bullet$ with $k_{\sigma_0} = 1$. Let

$$\mu = \mu_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathbb{Z})_{C_\bullet} \to G'(\mathbb{Z}')_{C'_\bullet}$$

be the induced morphism of complexes of groups.

Let

$$\lambda = \theta'^{-1} \circ \mu \circ \theta : G(\mathcal{Y}) \to G'(\mathcal{Y}')$$

We claim that $\Lambda = \Lambda^{\lambda}_{T,T'}$ and $L = L^{\lambda}_{T,T'}$. By Corollary 4.3.2, it is enough to show that

$$\Lambda = (\Lambda^{\theta'}_{T',f'(T')})^{-1} \circ \Lambda^{\mu}_{f(T),f'(T')} \circ \Lambda^{\theta}_{T,f(T)}$$

and

$$L = (L^{\theta'}_{T',f'(T')})^{-1} \circ L^{\mu}_{f(T),f'(T')} \circ L^{\theta}_{T,f(T)}$$

The result follows from the Main Lemma applied to μ, and Lemma 4.3.11 above. □

4.3.5 Coverings and developability

This section considers the relationship between the existence of a covering and developability.

Lemma 4.3.13. *Let $G(\mathcal{Y})$ and $G'(\mathcal{Y}')$ be complexes of groups over nonempty, connected scwols $\mathcal{Y}$ and $\mathcal{Y}'$. Assume there is a covering $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$. If $G'(\mathcal{Y}')$ is developable, then $G(\mathcal{Y})$ is developable.*

Proof. Let $\iota' : G'(\mathcal{Y}') \to FG'(\mathcal{Y}')$ be the natural morphism defined after Definition 4.2.14 in Section 4.2.4. By Proposition 4.2.15, since $G'(\mathcal{Y}')$ is developable, ι' is injective on the local groups. Thus, as ϕ is a covering, the composite morphism $\iota' \circ \phi : G(\mathcal{Y}) \to FG'(\mathcal{Y}')$ is injective on the local groups. Hence, by Proposition 4.2.11, the complex of groups $G(\mathcal{Y})$ is developable. □

We do not know if the converse to Lemma 4.3.13 holds in general. (According to Haefliger, the converse is true by a functorial $1-1$ correspondence between the coverings of an étale groupoid and the coverings of its classifying space.) However, in the presence of nonpositive curvature, we have the following partial converse to Lemma 4.3.13. Recall that an M_κ-polyhedral complex is a polyhedral complex with n-dimensional cells isometric to polyhedra in the simply connected Riemannian n-manifold of constant sectional curvature κ.

Lemma 4.3.14. *Let $\phi : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups, over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Suppose that for some $\kappa \leq 0$, $\mathcal{Y}$ and $\mathcal{Y}'$ are the scwols associated to connected M_κ-polyhedral complexes with finitely many isometry classes of cells K and K' respectively, and that $|l| : |\mathcal{Y}| \to |\mathcal{Y}'|$ is a local isometry on each simplex. If $G(\mathcal{Y})$ has nonpositive curvature (thus is developable), then $G'(\mathcal{Y}')$ also has nonpositive curvature, thus $G'(\mathcal{Y}')$ is developable.*

Proof. By Lemma 4.2.26, to show that $G'(\mathcal{Y}')$ is nonpositively curved, it suffices to show that for each vertex τ' of K', the geometric link of $\tilde{\tau}'$ in the local development $\mathrm{st}(\tilde{\tau}')$, with the induced spherical structure, is CAT(1). We first show, using the following lemma, that if τ' is a vertex of K', then $\tau' = f(\tau)$ for some vertex τ of K.

Lemma 4.3.15. *The nondegenerate morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$ associated to the covering ϕ : $G(\mathcal{Y}) \to G'(\mathcal{Y}')$ surjects onto the set of vertices of $\mathcal{Y}'$.*

Proof. Let $\sigma \in V(\mathcal{Y})$ and $l(\sigma) = \sigma' \in V(\mathcal{Y}')$. From the definitions of nondegenerate morphism of scwols and covering of complexes of groups, it follows that every vertex of $\mathcal{Y}'$ which is incident to an edge meeting σ' lies in the image of l. Since $\mathcal{Y}'$ is connected, we conclude that l surjects onto $V(\mathcal{Y}')$. □

Let τ' be a vertex of K'. By Lemma 4.3.15, $\tau' = l(\tau)$ for some $\tau \in V(\mathcal{Y})$. Suppose τ is not a vertex of K. Then there is an $a \in E(\mathcal{Y})$ such that $i(a) = \tau$. It follows that $i(l(a)) = l(i(a)) = \tau'$, so $l(a) \in E(\mathcal{Y}')$ has initial vertex τ'. This contradicts τ' a vertex of K'. Hence τ is a vertex of K.

Since $G(\mathcal{Y})$ is nonpositively curved, the geometric link of $\tilde{\tau}$ in the local development $\mathrm{st}(\tilde{\tau})$, with the induced spherical structure, is CAT(1). By Lemma 4.2.24, there is a ϕ_τ-equivariant bijection $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tilde{\tau}')$. We claim this bijection is an isometry in the induced metric, which completes the proof.

By definition of the induced metric, the action of G_τ on $\mathrm{st}(\tilde{\tau})$ induces a simplicial map $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tau)$ which is a local isometry on each simplex. Similarly, the action of $G_{\tau'}$ on $\mathrm{st}(\tilde{\tau}')$ induces $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tau)$ which is a local isometry on each simplex. By assumption, the restriction of $|l|$ to $\mathrm{st}(\tau)$ is a local isometry on each simplex. Hence, the bijection $\mathrm{st}(\tilde{\tau}) \to \mathrm{st}(\tilde{\tau}')$ is a local isometry on each simplex, and thus an isometry. □

4.4 Coverings and overgroups

In this section we prove Theorem 4.1.3, stated in the Introduction. We first define the notion of isomorphism of coverings. In Section 4.4.1 we define a map from overgroups to coverings, and in Section 4.4.2 a map from coverings to overgroups. Then in Section 4.4.3 we conclude the proof of Theorem 4.1.3 by showing that these maps are mutual inverses.

Definition 4.4.1 (isomorphism of coverings). *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ and $\lambda' : G(\mathcal{Y}) \to G''(\mathcal{Y}'')$ be coverings of developable complexes of groups over connected scwols. Fix $\sigma_0 \in V(\mathcal{Y})$. We say that λ and λ' are isomorphic coverings if for any choice of maximal trees T, T' and T'' in $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{Y}''$ respectively, there exists an isomorphism $\lambda'' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ of complexes of groups such that the following diagram of morphisms of universal covers (defined in Proposition 4.3.1) commutes*

$$\begin{tikzcd} D(\mathcal{Y},\iota_T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\lambda'}_{T,T''}"'] & D(\mathcal{Y}',\iota_{T'}) \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'',\iota_{T''}) \end{tikzcd}$$

Note that by Corollary 4.3.2, this diagram commutes for one triple (T, T', T'') if and only if it commutes for all triples (T, T', T''). By Proposition 4.3.3, since λ and λ' are coverings, $L^{\lambda}_{T,T'}$ and $L^{\lambda'}_{T,T''}$ are isomorphisms. By Proposition 4.3.9, since λ'' is an isomorphism, the map $L^{\lambda''}_{T',T''}$ is an isomorphism. Hence, two coverings are isomorphic if and only if they induce a commutative diagram of isomorphisms of universal covers.

For the remainder of Section 4.4, we fix the following data :

- $\mathcal{X}$, the scwol associated to a simply connected polyhedral complex K,
- Γ, a subgroup of $\mathrm{Aut}(K)$ which acts on $\mathcal{X}$, with quotient $\mathcal{Y} = \Gamma\backslash\mathcal{X}$,
- a vertex $\sigma_0 \in V(\mathcal{Y})$, and
- a set of choices $C_\bullet = (\overline{\sigma}, h_a)$ giving rise to a complex of groups $G(\mathcal{Y})_{C_\bullet} = (G_\sigma, \psi_a, g_{a,b})$ induced by the action of Γ on $\mathcal{X}$.

Let $\mathrm{Over}(\Gamma)$ be the set of overgroups of Γ which act without inversions, that is, the set of subgroups of $\mathrm{Aut}(K)$ containing Γ which act without inversions. Let $\mathrm{Cov}(G(\mathcal{Y}))$ be the set of isomorphism classes of coverings of faithful, developable complexes of groups by $G(\mathcal{Y})$.

4.4.1 The map from overgroups to coverings

In this section we construct a map

$$\underline{a} : \mathrm{Over}(\Gamma) \to \mathrm{Cov}(G(\mathcal{Y}))$$

We first show in Lemma 4.4.2 that an overgroup induces a covering of complexes of groups. Then in Lemma 4.4.3 we show that, without loss of generality, we may apply the Main Lemma to this covering. In Lemma 4.4.4, we define $\underline{a}$ and show that $\underline{a}$ is well-defined on isomorphism classes of coverings.

Lemma 4.4.2. *Let Γ' be an overgroup of Γ acting without inversions. Let $G'(\mathcal{Y}')_{C'_\bullet}$ be a complex of groups over $\mathcal{Y}' = \Gamma'\backslash\mathcal{X}$ induced by the action of Γ' on $\mathcal{X}$, for some choices $C'_\bullet$. Let $L = Id : \mathcal{X} \to \mathcal{X}$ and let $\Lambda : \Gamma \hookrightarrow \Gamma'$ be the inclusion, inducing $l : \mathcal{Y} \to \mathcal{Y}'$. For some choices $N_\bullet$, let*

$$\lambda = \lambda_{C_\bullet, C'_\bullet, N_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G'(\mathcal{Y}')_{C'_\bullet}$$

be the morphism of complexes of groups over l induced by L and Λ (see Definition 4.2.13). Then λ is a covering.

Proof. By definition, $\lambda_\sigma = \mathrm{Ad}(k_\sigma)$, up to inclusion, where $k_\sigma\overline{\sigma} = \overline{l(\sigma)}$. The local maps λ_σ are thus injective.

We write $[g]_a$ for the coset of $g \in G_{t(a)}$ in $G_{t(a)}/\psi_a(G_{i(a)})$, and similarly for $[g']_{a'}$ when $g' \in G'_{t(a')}$. It now suffices to show that for every $a' \in E(\mathcal{Y}')$ with $t(a') = \sigma' = l(\sigma) \in V(\mathcal{Y})$, the map on cosets

$$\coprod_{\substack{a\in l^{-1}(a')\\ t(a)=\sigma}} \Gamma_{\overline{\sigma}}/h_a(\Gamma_{\overline{i(a)}})h_a^{-1} \longrightarrow \Gamma'_{\overline{l(\sigma)}}/h'_{a'}(\Gamma'_{\overline{i(a')}})h'^{-1}_{a'}$$

$$[g]_a \longmapsto [\lambda_\sigma(g)\lambda(a)]_{a'}$$

is bijective. Suppose $[\lambda_\sigma(g)\lambda(a)]_{a'} = [\lambda_\sigma(h)\lambda(b)]_{a'}$. Then by definition of λ,

$$h'_{a'}k_{i(b)}h_b^{-1}k_{t(b)}^{-1}k_\sigma h^{-1}gk_\sigma^{-1}k_{t(a)}h_ak_{i(a)}^{-1}(h'_{a'})^{-1} \in h'_{a'}\Gamma'_{\overline{i(a')}}(h'_{a'})^{-1}$$

hence

$$k_{i(b)}h_b^{-1}h^{-1}gh_ak_{i(a)}^{-1} \in \Gamma'_{\overline{i(a')}}$$

Since $k_{i(a)}$ and $k_{i(b)}$ send $\overline{i(a)}$ and $\overline{i(b)}$ respectively to $\overline{i(a')}$, the element $h_b^{-1}h^{-1}gh_a$ in Γ sends $\overline{i(a)}$ to $\overline{i(b)}$. Since $l(a) = l(b)$, this implies that $a = b$. Hence $h^{-1}g$ maps $\overline{i(a)}$ to itself, thus $[h]_a = [g]_a$. Therefore the map on cosets is injective.

Let us show that the map on cosets is surjective. Let $[h']_{a'}$ be an element of the target set. Let $b' = k_\sigma^{-1}h'h'_{a'}\overline{a'}$. Since $h' \in \Gamma'_{\overline{\sigma'}}$, we have $t(b') = \overline{\sigma}$. Let $c = p(b')$, where p is the natural projection $\mathcal{X} \to \mathcal{Y} = \Gamma\backslash\mathcal{X}$. Let $g \in \Gamma_{\overline{\sigma}}$ be such that $gh_c\overline{c} = b'$. We claim that $[g]_c$ maps to $[h']_{a'}$, that is,

$$h'^{-1}k_\sigma gh_ck_{i(c)}^{-1}h'^{-1}_{a'} \in h'_{a'}\Gamma'_{\overline{i(a')}}h'^{-1}_{a'}$$

Since $k_{i(c)}^{-1}$ sends $\overline{i(a')}$ to $\overline{i(c)}$, and the element $k_\sigma gh_c$ sends $\overline{i(c)}$ to $i(k_\sigma b') = i(h'h'_{a'}(\overline{a'}))$, it follows that $h'^{-1}_{a'}h'^{-1}k_\sigma gh_ck_{i(c)}^{-1}$ fixes $\overline{i(a')}$, which proves the claim. □

We now show that every covering λ induced by an overgroup, as in Lemma 4.4.2, is isomorphic to a covering λ' to which the Main Lemma may be applied. More precisely :

Lemma 4.4.3. *With the notation of Lemma 4.4.2, fix a vertex $\sigma_0 \in V(\mathcal{Y})$. Then there is a choice $C''_\bullet$ of data for Γ' acting on $\mathcal{X}$ such that $\overline{\sigma_0} = \overline{l(\sigma_0)}$, and a choice $N'_\bullet = \{k'_\sigma\}$ such that $k'_{\sigma_0} = 1$, so that λ is isomorphic to the covering*

$$\lambda' = \lambda'_{C_\bullet, C''_\bullet, N'_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$$

where $G''(\mathcal{Y}'')_{C''_\bullet}$ is the complex of groups induced by $C''_\bullet$.

Proof. By definition of l, there is a choice $C''_\bullet$ so that $\overline{\sigma_0}$, determined by $C_\bullet$, equals $\overline{l(\sigma_0)}$ determined by $C''_\bullet$. We now define a collection $N'_\bullet = \{k'_\sigma\}$ such that $k'_\sigma\overline{\sigma} = \overline{l(\sigma)}$ for all $\sigma \in V(\mathcal{Y})$.

Choose a section $s : V(\mathcal{Y}') \to V(\mathcal{Y})$ for l. That is, for each $\sigma' \in V(\mathcal{Y}')$, choose $s(\sigma') \in V(\mathcal{Y})$ such that $l(s(\sigma')) = \sigma'$. In particular, if $\sigma'_0 = l(\sigma_0)$, let $s(\sigma'_0) = \sigma_0$.

For each $s(\sigma') \in V(\mathcal{Y})$, choose an element $k'_{s(\sigma')} \in \Gamma'$ such that $k'_{s(\sigma')}\overline{s(\sigma')} = \overline{\sigma'}$, where $\overline{s(\sigma')}$ is determined by $C_\bullet$ and $\overline{\sigma'}$ by $C''_\bullet$. Since $s(\sigma'_0) = \sigma_0$, and by choice of $C''_\bullet$, we have $k'_{\sigma_0}\overline{\sigma_0} = \overline{l(\sigma_0)} = \overline{\sigma_0}$, so we may choose $k'_{\sigma_0} = 1$. For all other $\sigma \in V(\mathcal{Y})$, let

$$k'_\sigma = k'_{s(l(\sigma))}k^{-1}_{s(l(\sigma))}k_\sigma \tag{4.3}$$

where $N_\bullet = \{k_\sigma\}$. Note that

$$k'_\sigma\overline{\sigma} = k'_{s(l(\sigma))}k^{-1}_{s(l(\sigma))}k_\sigma\overline{\sigma} = k'_{s(l(\sigma))}k^{-1}_{s(l(\sigma))}\overline{l(\sigma)} = k'_{s(l(\sigma))}\overline{s(l(\sigma))} = \overline{l(s)}$$

This defines a collection $N'_\bullet = \{k'_\sigma\}$ with $k'_{\sigma_0} = 1$. Let $\lambda' : G(\mathcal{Y})_{C_\bullet} \to G''(\mathcal{Y}'')_{C'_\bullet}$ be the covering induced by $N'_\bullet$.

We now construct an isomorphism of complexes of groups $\mu : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ such that the following diagram commutes

$$\begin{tikzcd} G(\mathcal{Y}) \arrow[r, "\lambda"] \arrow[dr, "\lambda'"'] & G'(\mathcal{Y}') \arrow[d, "\mu"] \\ & G''(\mathcal{Y}'') \end{tikzcd} \tag{4.4}$$

By Corollary 4.3.2, it follows that λ is isomorphic to λ'.

Let $f : \mathcal{Y}' \to \mathcal{Y}''$ be the identity map (both $\mathcal{Y}'$ and $\mathcal{Y}''$ are the quotient $\Gamma'\backslash\mathcal{X}$). We choose a collection $N''_\bullet = \{k''_{\sigma'}\}$ of elements of Γ' such that $k''_{\sigma'}\overline{\sigma'} = \overline{f(\sigma')}$ as follows. By Equation (4.3), if $l(\sigma_1) = l(\sigma_2)$ then $k'_{\sigma_1}k^{-1}_{\sigma_1} = k'_{\sigma_2}k^{-1}_{\sigma_2}$. Given $\sigma' \in V(\mathcal{Y}')$, it is thus well-defined to put

$$k''_{\sigma'} = k'_\sigma k^{-1}_\sigma$$

for any $\sigma \in l^{-1}(\sigma')$. We check

$$k''_{\sigma'}\overline{\sigma'} = k'_\sigma k_\sigma^{-1}\overline{\sigma'} = k'_\sigma\overline{\sigma} = \overline{\sigma'} = \overline{f(\sigma')}$$

as required. Define $\mu = \mu_{C'_\bullet, C''_\bullet, N''_\bullet} : G'(\mathcal{Y}')_{C'_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$. Since $G'(\mathcal{Y}')$ and $G''(\mathcal{Y}'')$ are both associated to the action of Γ' on $\mathcal{X}$, μ is an isomorphism.

By definition of composition of morphisms, for $g \in G_\sigma$ we have

$$\begin{aligned}(\mu \circ \lambda)_\sigma(g) &= \mu_{l(\sigma)} \circ \lambda_\sigma(g)\\ &= \mathrm{Ad}(k''_{l(\sigma)}) \circ \mathrm{Ad}(k_\sigma)(g)\\ &= \mathrm{Ad}(k''_{l(\sigma)}k_\sigma)(g)\\ &= \mathrm{Ad}(k'_\sigma)(g)\\ &= \lambda'_\sigma(g)\end{aligned}$$

and for $a \in E(\mathcal{Y})$

$$\begin{aligned}(\mu \circ \lambda)(a) &= \mu_{l(t(a))}(\lambda(a))\mu(l(a))\\ &= \mathrm{Ad}(k''_{l(t(a))})(k_{t(a)}h_a k_{i(a)}^{-1}h_{l(a)}^{-1})k''_{t(l(a))}h_{l(a)}(k''_{i(l(a))})^{-1}h_{f(l(a))}^{-1}\\ &= k''_{l(t(a))}k_{t(a)}h_a k_{i(a)}^{-1}(k''_{i(l(a))})^{-1}h_{f(l(a))}^{-1}\\ &= k'_{t(a)}h_a(k'_{i(a)})^{-1}h_{f(l(a))}^{-1}\\ &= \lambda'(a)\end{aligned}$$

hence the diagram at (4.4) commutes. □

Lemma 4.4.4. *Let Γ' be an overgroup of Γ. Let $C'_\bullet$, $N_\bullet$ and $C''_\bullet$, $N'_\bullet$ be any two choices as in Lemma 4.4.2, and let*

$$\lambda_{C_\bullet,C'_\bullet,N_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G'(\mathcal{Y}')_{C'_\bullet} \quad \text{and} \quad \lambda'_{C_\bullet,C''_\bullet,N'_\bullet} : G(\mathcal{Y})_{C_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$$

be the associated coverings. Then λ and λ' are isomorphic coverings. Thus the map

$$\underline{a} : \mathrm{Over}(\Gamma) \to \mathrm{Cov}(G(\mathcal{Y}))$$

taking an overgroup Γ' of Γ to the isomorphism class of the covering λ is well-defined.

Proof. Fix a vertex $\sigma_0 \in V(\mathcal{Y})$ and let $\sigma'_0 = l(\sigma_0)$. By Lemma 4.4.3, we may without loss of generality assume that the Main Lemma may be applied to λ and λ'. As in the proof of Lemma 4.4.3, choose a collection $N''_\bullet = \{k''_{\sigma'}\}$ with $k''_{\sigma'_0} = k'_{\sigma_0}k_{\sigma_0}^{-1} = 1$. Then we may apply the Main Lemma to the isomorphism of complexes of groups

$$\lambda'' = \lambda_{C'_\bullet,C''_\bullet,N''_\bullet} : G'(\mathcal{Y}')_{C'_\bullet} \to G''(\mathcal{Y}'')_{C''_\bullet}$$

Choose maximal trees T, T' and T'' in $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{Y}''$ respectively. We need to check that the triangle

$$\begin{tikzcd} D(\mathcal{Y}, \iota_T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\lambda'}_{T,T''}"'] & D(\mathcal{Y}', \iota_{T'}) \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'', \iota_{T''}) \end{tikzcd} \tag{4.5}$$

commutes. Using the Main Lemma three times, we obtain the diagram

$$\begin{tikzcd} D(\mathcal{Y}, \iota_T) \arrow[r, "L^{\lambda}_{T,T''}"] \arrow[drr, "L^{\lambda'}_{T,T''}"'] \arrow[dd, "\tilde{L}_T"'] & D(\mathcal{Y}', \iota_{T'}) \arrow[dr, "L^{\lambda''}_{T',T''}"] \arrow[dd, "\tilde{L}_{T'}"] & \\ & & D(\mathcal{Y}'', \iota_{T''}) \arrow[dd, "\tilde{L}_{T''}"'] \\ \mathcal{X} \arrow[r, "Id"] \arrow[drr, "Id"'] & \mathcal{X} \arrow[dr, "Id"] & \\ & & \mathcal{X}, \end{tikzcd}$$

and see that the commutativity of (4.5) is equivalent to the commutativity of the tautological triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "Id"] \arrow[dr, "Id"'] & \mathcal{X} \arrow[d, "Id"] \\ & \mathcal{X} \end{tikzcd}$$

which is obvious. □

4.4.2 The map from coverings to overgroups

We now show that there is a map

$$\underline{b} : \mathrm{Cov}(G(\mathcal{Y})) \to \mathrm{Over}(\Gamma).$$

Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of complexes of groups, where $G'(\mathcal{Y}')$ is faithful and developable. For any maximal subtrees T and T' of $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, let $\Lambda_{T,T'} : \pi_1(G(\mathcal{Y}), T) \to \pi_1(G'(\mathcal{Y}'), T')$ be the associated group monomorphism, and $L^{\lambda}_{T,T'} : D(\mathcal{Y}, \iota_T) \to D(\mathcal{Y}', \iota_{T'})$ be the associated $\Lambda_{T,T'}$-equivariant isomorphism of scwols. Composition with the isomorphism $\tilde{L}_T^{-1}$ (see Proposition 4.2.23) yields an isomorphism of scwols

$$L_{\lambda,T'} = L^{\lambda}_{T,T'} \circ \tilde{L}_T^{-1} : \mathcal{X} \to D(\mathcal{Y}', \iota_{T'})$$

which is equivariant with respect to $\Lambda_{T,T'} \circ \Lambda_T^{-1} : \Gamma \to \pi_1(G'(\mathcal{Y}'), T')$. We set $\underline{b}(\lambda)$ to be the group

$$\underline{b}(\lambda) = L_{\lambda,T'}^{-1}(\pi_1(G'(\mathcal{Y}'), T'))L_{\lambda,T'}$$

which acts on $\mathcal{X}$. Since $G'(\mathcal{Y}')$ is faithful, $\pi_1(G'(\mathcal{Y}'), T')$ acts faithfully on $D(\mathcal{Y}', \iota_{T'})$. Hence we may identify $\underline{b}(\lambda)$ with a subgroup of $\mathrm{Aut}(K)$ which acts on $\mathcal{X}$. As $\Lambda_{T,T'}$ is injective, $\underline{b}(\lambda)$ is an overgroup of Γ.

Lemma 4.4.5 below shows that $\underline{b}$ is well-defined, that is, only depends on the isomorphism class of the covering λ.

Lemma 4.4.5. *Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ and $\lambda' : G(\mathcal{Y}) \to G''(\mathcal{Y}'')$ be isomorphic coverings of complexes of finite groups, with $G'(\mathcal{Y}')$ and $G''(\mathcal{Y}'')$ faithful and developable. Then $\underline{b}(\lambda) = \underline{b}(\lambda')$.*

Proof. By definition, there exists an isomorphism $\lambda'' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ such that, for any choice of maximal trees, we have a commuting triangle

$$\begin{tikzcd} D(\mathcal{Y}, \iota_T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[dr, "L^{\lambda'}_{T,T''}"'] & D(\mathcal{Y}', \iota_{T'}) \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'', \iota_{T''}) \end{tikzcd}$$

and thus, composing with $\tilde{L}_T^{-1}$, a commuting triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "L_{\lambda,T'}"] \arrow[dr, "L_{\lambda',T''}"'] & D(\mathcal{Y}', \iota_{T'}) \arrow[d, "L^{\lambda''}_{T',T''}"] \\ & D(\mathcal{Y}'', \iota_{T''}) \end{tikzcd}$$

Since λ'' is an isomorphism, by Proposition 4.3.9 the group homomorphism $\Lambda_{T',T''} : \pi_1(G'(\mathcal{Y}'), T') \to \pi_1(G''(\mathcal{Y}''), T'')$ is an isomorphism. Thus, as $L^{\lambda''}_{T',T''}$ is $\Lambda_{T',T''}$-equivariant,

$$\begin{aligned} \underline{b}(\lambda') &= L_{\lambda',T''}^{-1}(\pi_1(G''(\mathcal{Y}''), T''))L_{\lambda',T''}^{-1} \\ &= L_{\lambda,T'}^{-1}(L^{\lambda''}_{T',T''})^{-1}(\pi_1(G''(\mathcal{Y}''), T''))L^{\lambda''}_{T',T''}L_{\lambda,T'} \\ &= L_{\lambda,T'}^{-1}(\pi_1(G'(\mathcal{Y}'), T'))L_{\lambda,T'} \\ &= \underline{b}(\lambda) \end{aligned}$$

Therefore $\underline{b}$ is well-defined. □

4.4.3 Proof of Theorem 4.1.3

We now complete the proof of Theorem 4.1.3. Let $\underline{a} : \mathrm{Over}(\Gamma) \to \mathrm{Cov}(G(\mathcal{Y}))$ be as defined in Section 4.4.1 and $\underline{b} : \mathrm{Cov}(G(\mathcal{Y})) \to \mathrm{Over}(\Gamma)$ be as defined in Section 4.4.2.

Proposition 4.4.6. *The maps $\underline{a}$ and $\underline{b}$ are mutually inverse bijections.*

Proof. We first prove that $\underline{b} \circ \underline{a} = 1$. For this, let Γ' be an overgroup of Γ acting without inversions, and let $\underline{a}(\Gamma') = \lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be an associated covering over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. By Lemma 4.4.3, we may assume that we can apply the Main Lemma to λ. For any maximal subtrees T and T' of $\mathcal{Y}$ and $\mathcal{Y}'$ respectively, we have then a commuting diagram of (equivariant) isomorphisms of scwols

$$\begin{tikzcd} D(\mathcal{Y}, \iota_T) \arrow[r, "L^{\lambda}_{T,T'}"] \arrow[d, "\tilde{L}_T"'] & D(\mathcal{Y}', \iota_{T'}) \arrow[d, "\tilde{L}_{T'}"] \\ \mathcal{X} \arrow[r, "L=Id"'] & \mathcal{X} \end{tikzcd}$$

Thus

$$\begin{aligned} \underline{b}(\lambda) &= L^{-1}_{\lambda,T'}(\pi_1(G'(\mathcal{Y}'),T'))L_{\lambda,T'} \\ &= (L^{\lambda}_{T,T'} \circ \tilde{L}_T^{-1})^{-1}(\pi_1(G'(\mathcal{Y}'),T'))L^{\lambda}_{T,T'} \circ \tilde{L}_T^{-1} \\ &= \tilde{L}_{T'}(\pi_1(G'(\mathcal{Y}'),T'))\tilde{L}_{T'}^{-1} \\ &= \Gamma' \end{aligned}$$

since $\tilde{L}_{T'}$ is equivariant with respect to the isomorphism $\Lambda_{T'} : \pi_1(G'(\mathcal{Y}'),T') \to \Gamma'$. We conclude that $\underline{b}\ \underline{a}(\Gamma') = \Gamma'$.

We now prove that $\underline{a} \circ \underline{b} = 1$. Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering of a faithful developable complex of groups $G'(\mathcal{Y}')$ over a morphism of scwols $l : \mathcal{Y} \to \mathcal{Y}'$. Choose a vertex $\sigma_0 \in V(\mathcal{Y})$ and maximal trees T and T' in $\mathcal{Y}$ and $\mathcal{Y}'$ respectively. Without loss of generality, we identify $G'(\mathcal{Y}')$ with the complex of groups induced by the action of $\pi_1(G'(\mathcal{Y}'),T')$ on $D(\mathcal{Y}', \iota_{T'})$, using the isomorphism θ' defined in Lemma 4.3.11 above. By abuse of notation, we write λ for $\theta' \circ \lambda$. Let $\Gamma' = \underline{b}(\lambda)$.

Let $\mu = \underline{a}(\Gamma')$ be a covering $\mu : G(\mathcal{Y}) \to G''(\mathcal{Y}'')_{C''_\bullet}$ over a morphism of scwols $l' : \mathcal{Y} \to \mathcal{Y}''$, where $G''(\mathcal{Y}'')$ is a complex of groups induced by the action of Γ' on $\mathcal{X}$. By Lemma 4.4.3, we may assume that $\overline{\sigma_0} = \overline{l'(\sigma_0)}$ so that we can apply the Main Lemma to μ. We now show that λ and $\mu = \underline{a}\,\underline{b}(\lambda)$ are isomorphic coverings.

The map $\underline{b}$ induces a group isomorphism

$$\Lambda_{\underline{b}} : \pi_1(G'(\mathcal{Y}'), T') \to \underline{b}(\lambda)$$

with, for each $g' \in \pi_1(G'(\mathcal{Y}'), T')$ and each $\alpha \in \mathcal{X}$,

$$\Lambda_{\underline{b}}(g') \cdot \alpha = L^{-1}_{\lambda,T'}(g' \cdot L_{\lambda,T'}(\alpha)).$$

By construction, $L^{-1}_{\lambda,T'} : D(\mathcal{Y}', \iota_{T'}) \to \mathcal{X}$ is $\Lambda_{\underline{b}}$-equivariant. Let $f : \mathcal{Y}' \to \mathcal{Y}''$ be the induced map of the quotient scwols

$$\mathcal{Y}' = \pi_1(G'(\mathcal{Y}'), T')\backslash D(\mathcal{Y}', \iota_{T'}) \quad \text{and} \quad \mathcal{Y}'' = \Gamma'\backslash\mathcal{X}$$

Since $\Lambda_{\underline{b}}$ and $L^{-1}_{\lambda,T'}$ are both isomorphisms, f is an isomorphism of scwols. We claim that the following diagram of morphisms of scwols commutes :

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "l"] \arrow[dr, "l'"'] & \mathcal{Y}' \arrow[d, "f"] \\
 & \mathcal{Y}''.
\end{tikzcd}$$

Let $\alpha \in \mathcal{Y}$. Then $\alpha = \Gamma\overline{\alpha}$ with $\overline{\alpha} \in \mathcal{X}$. We identify $l(\alpha) \in \mathcal{Y}'$ with the orbit $\pi_1(G'(\mathcal{Y}'), T')([1], l(\alpha)) = \pi_1(G'(\mathcal{Y}'), T')([u_{i(\alpha)}], l(\alpha))$. Then

$$f(l(\alpha)) = \Gamma' L^{-1}_{\lambda,T'}([u_{i(\alpha)}], l(\alpha)) = \Gamma' h_{i(\alpha)}\overline{\alpha} = \Gamma'\overline{\alpha} = l'(\alpha)$$

proving the claim.

We next choose elements $k_{\sigma'} \in \Gamma'$ such that, for each $\sigma' \in V(\mathcal{Y}')$,

$$k_{\sigma'} L^{-1}_{\lambda,T'}([1], \sigma') = \overline{f(\sigma')}.$$

We claim that $L^{-1}_{\lambda,T'}([1], l(\sigma_0)) = \overline{f(l(\sigma_0))}$. Now

$$L_{\lambda,T'}(\overline{f(l(\sigma_0))}) = L^{\lambda}_{T,T'} \circ \tilde{L}^{-1}_{T}(\overline{f(l(\sigma_0))}) = L^{\lambda}_{T,T'}([1], \sigma_0) = ([1], l(\sigma_0))$$

since $h_{i(f(l(\sigma_0)))} = 1$ and $u_{\sigma_0} = 1$, which proves the claim. Hence we may, and do, choose $k_{\sigma'_0} = 1$.

The elements $k_{\sigma'}$ then induce a morphism $\phi : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$ over f, given by $\phi_{\sigma'}(g') = k_{\sigma'}\Lambda_{\underline{b}}(g')k^{-1}_{\sigma'}$ for $g' \in G'_{\sigma'}$, and $\phi(a') = k_{t(a')}\Lambda_{\underline{b}}(a'^{+})k^{-1}_{i(a')}h^{-1}_{f(a')}$ for $a' \in E(\mathcal{Y}')$. Since $\Lambda_{\underline{b}}$ and f are isomorphisms, ϕ is an isomorphism of complexes of groups. Moreover, the following diagram commutes up to a homotopy from $\Lambda_{\underline{b}}\iota'_{T'}$ to $\phi''_1\phi$, given by the elements $\{k_{\sigma'}\}$:

$$\begin{tikzcd}
G'(\mathcal{Y}') \arrow[r, "\iota'_{T'}"] \arrow[d, "\phi"'] & \pi_1(G'(\mathcal{Y}'), T') \arrow[d, "\Lambda_{\underline{b}}"] \\
G''(\mathcal{Y}'') \arrow[r, "\phi''_1"] & \Gamma'
\end{tikzcd}$$

Hence, by Proposition 4.2.20, there is a $\Lambda_{\underline{b}}$-equivariant isomorphism of scwols

$$L_{\underline{b}} : D(\mathcal{Y}', \iota_{T'}) \to D(\mathcal{Y}'', \phi_1'')$$

given explicitly by

$$([g'], \alpha') \mapsto ([\Lambda_{\underline{b}}(g')k_{i(\alpha')}^{-1}], f(\alpha'))$$

We now choose a maximal subtree T'' of $\mathcal{Y}''$ and compose $L_{\underline{b}}$ with the isomorphism $L_{T''}^{-1}$: $D(\mathcal{Y}'', \phi_1'') \to D(\mathcal{Y}'', T'')$ to obtain an isomorphism of scwols

$$L : D(\mathcal{Y}', \iota_{T'}) \to D(\mathcal{Y}'', \iota_{T''})$$

which is equivariant with respect to the composition of group isomorphisms

$$\Lambda_{T''}^{-1} \circ \Lambda_{\underline{b}} : \pi_1(G'(\mathcal{Y}'), T') \to \Gamma' \to \pi_1(G''(\mathcal{Y}''), T'')$$

Since $k_{\sigma_0'} = 1$ and $h_{f(\sigma_0')} = 1$,

$$L_{\underline{b}}([1], \sigma_0') = ([k_{\sigma_0'}], f(\sigma_0')) = ([h_{f(\sigma_0')}], f(\sigma_0')) = L_{T''}([1], f(\sigma_0'))$$

hence $L([1], \sigma_0') = ([1], f(\sigma_0'))$. We may thus apply the Corollary to the Main Lemma to L. We now have $L = L_{T',T''}^{\lambda'}$ for some morphism $\lambda' : G'(\mathcal{Y}') \to G''(\mathcal{Y}'')$. By Proposition 4.3.9, since L is an isomorphism of scwols which is equivariant with respect to an isomorphism of groups, λ' is an isomorphism of complexes of groups.

To complete the proof, it now suffices to show that the following diagram commutes :

$$\begin{tikzcd} D(\mathcal{Y}, \iota_T) \arrow[r, "L_{T,T'}^{\lambda}"] \arrow[dr, "L_{T,T''}^{\mu}"'] & D(\mathcal{Y}', \iota_{T'}) \arrow[d, "L=L_{T',T''}^{\lambda'}"] \\ & D(\mathcal{Y}'', \iota_{T''}) \end{tikzcd}$$

By definition of L, it suffices to show that

$$L_{\underline{b}} \circ L_{T,T'}^{\lambda} = L_{T''} \circ L_{T,T''}^{\mu}$$

Let $g \in \pi_1(G(\mathcal{Y}), T)$ and $\alpha \in \mathcal{Y}$. We write $u_{i(\alpha)}^{\lambda}$ for the element of $\pi_1(G'(\mathcal{Y}'), T')$ with respect to which $L_{T,T'}^{\lambda}$ is defined, and similarly for $u_{i(\alpha)}^{\mu} \in \pi_1(G''(\mathcal{Y}''), T'')$. Then

$$L_{\underline{b}} \circ L_{T,T'}^{\lambda}([g], \alpha) = \left(\left[\Lambda_{\underline{b}}\left\{\Lambda_{T,T'}(g)u_{i(\alpha)}^{\lambda}\right\} k_{i(l(\alpha))}^{-1}\right], f(l(\alpha))\right)$$

and

$$L_{T''} \circ L^{\mu}_{T,T''}([g],\alpha) = \left(\left[\Lambda_{T''}\left\{\Lambda_{T,T''}(g)u^{\mu}_{i(\alpha)}\right\} h_{i(l'(\alpha))}\right], l'(\alpha)\right)$$

Since $f \circ l = l'$, it suffices to show that

$$\Lambda_{\underline{b}}\left\{\Lambda_{T,T'}(g)u^{\lambda}_{i(\alpha)}\right\} k^{-1}_{i(l(\alpha))}\overline{f(l(\alpha))} = \Lambda_{T''}\left\{\Lambda_{T,T''}(g)u^{\mu}_{i(\alpha)}\right\} h_{i(l'(\alpha))}\overline{l'(\alpha)} \tag{4.6}$$

By definition of the elements $k_{\sigma'}$, the left-hand side of (4.6) equals

$$\begin{aligned}
&\Lambda_{\underline{b}}\left\{\Lambda_{T,T'}(g)u^{\lambda}_{i(\alpha)}\right\} L^{-1}_{\lambda,T'}([1], l(\alpha)) \\
&= L^{-1}_{\lambda,T'}\left(\Lambda_{T,T'}(g)u^{\lambda}_{i(\alpha)} \cdot ([1], l(\alpha))\right) \quad \text{since } L^{-1}_{\lambda,T'} \text{ is } \Lambda_{\underline{b}}\text{-equivariant} \\
&= L^{-1}_{\lambda,T'}\left(\Lambda_{T,T'}(g) \cdot ([u^{\lambda}_{i(\alpha)}], l(\alpha))\right) \\
&= L^{-1}_{\lambda,T'}\left(\Lambda_{T,T'}(g) \cdot L^{\lambda}_{T,T'}([1],\alpha)\right) \\
&= L^{-1}_{\lambda,T'} \circ L^{\lambda}_{T,T'}([g],\alpha) \quad \text{since } L^{\lambda}_{T,T'} \text{ is } \Lambda_{T,T'}\text{-equivariant} \\
&= \tilde{L}_T([g],\alpha) \quad \text{by definition of } L_{\lambda,T'}
\end{aligned}$$

On the right-hand side of (4.6), we have, by definition of $\tilde{L}_{T''}$,

$$\begin{aligned}
&\Lambda_{T''}\left\{\Lambda_{T,T''}(g)u^{\mu}_{i(\alpha)}\right\} \tilde{L}_{T''}([1], l'(\alpha)) \\
&= \tilde{L}_{T''}\left(\Lambda_{T,T''}(g)u^{\mu}_{i(\alpha)} \cdot ([1], l'(\alpha))\right) \quad \text{since } \tilde{L}_{T''} \text{ is } \Lambda_{T''}\text{-equivariant} \\
&= \tilde{L}_{T''}\left(\Lambda_{T,T''}(g) \cdot ([u^{\mu}_{i(\alpha)}], l'(\alpha))\right) \\
&= \tilde{L}_{T''}\left(\Lambda_{T,T''}(g) \cdot L^{\mu}_{T,T''}([1],\alpha)\right) \\
&= \tilde{L}_{T''} \circ L^{\mu}_{T,T''}([g],\alpha) \quad \text{since } L^{\mu}_{T,T''} \text{ is } \Lambda_{T,T''}\text{-equivariant}
\end{aligned}$$

But by the Main Lemma applied to μ, we have a commuting square

$$\begin{tikzcd}
D(\mathcal{Y}, \iota_T) \arrow[r, "L^{\mu}_{T,T''}"] \arrow[d, "\tilde{L}_T"'] & D(\mathcal{Y}'', \iota_{T''}) \arrow[d, "\tilde{L}_{T''}"] \\
\mathcal{X} \arrow[r, "Id"'] & \mathcal{X}
\end{tikzcd}$$

hence equation (4.6) holds. □

4.5 Counting overlattices

We now apply Theorem 4.1.3 to obtain estimates for the number of overlattices of a given lattice Γ. We first establish a bijection between n-sheeted coverings and overlattices of index n.

Corollary 4.5.1. *Let K be a simply connected, locally finite polyhedral complex, and let Γ be a cocompact lattice in* $\mathrm{Aut}(K)$ *(acting without inversions) which induces a complex of groups $G(\mathcal{Y})$. Then there is a bijection between the set of overlattices of Γ of index n (acting without inversions) and the set of isomorphism classes of n-sheeted coverings of faithful developable complexes of groups by $G(\mathcal{Y})$.*

Proof. By the definition of n-sheeted covering, the bijection of Theorem 4.1.3 sends an isomorphism class of finite-sheeted coverings to an overgroup containing Γ with finite index.

Since Γ is cocompact, the quotient scwol $\mathcal{Y}$ is finite and the local groups G_σ of $G(\mathcal{Y})$ are finite groups. Let $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a finite-sheeted covering, where $G'(\mathcal{Y}')$ is a faithful, developable complex of groups. Then $\mathcal{Y}'$ is finite by Lemma 4.3.15, and the local groups $G'_{\sigma'}$ are finite since λ is finite-sheeted. It follows that the overgroup $\underline{b}(\lambda)$ is a cocompact lattice acting without inversions on K.

It remains to show that the bijection $\underline{a}$ sends an overlattice Γ' of index n to an n'-sheeted covering, with $n = n'$. Let $\lambda = \underline{a}(\Gamma') : G(\mathcal{Y}) \to G'(\mathcal{Y}')$ be a covering associated to Γ', over the morphism of quotient scwols $l : \Gamma\backslash\mathcal{X} \to \Gamma'\backslash\mathcal{X}$. Then

$$n = [\Gamma' : \Gamma] = \frac{\mathrm{Vol}(\Gamma\backslash\backslash V(\mathcal{X}))}{\mathrm{Vol}(\Gamma'\backslash\backslash V(\mathcal{X}))} = \frac{\sum\limits_{\sigma \in V(\mathcal{Y})} \frac{1}{|G_\sigma|}}{\sum\limits_{\sigma' \in V(\mathcal{Y}')} \frac{1}{|G'_{\sigma'}|}}$$
$$= \frac{\sum\limits_{\sigma' \in V(\mathcal{Y}')} \sum\limits_{\sigma \in l^{-1}(\sigma')} \frac{1}{|G_\sigma|}}{\sum\limits_{\sigma' \in V(\mathcal{Y}')} \frac{1}{|G'_{\sigma'}|}} = \frac{\sum\limits_{\sigma' \in V(\mathcal{Y}')} \frac{n'}{|G'_{\sigma'}|}}{\sum\limits_{\sigma' \in V(\mathcal{Y}')} \frac{1}{|G'_{\sigma'}|}} = n'$$

as required. □

4.5.1 Upper bound

Let K be a simply connected, locally finite polyhedral complex. In this section, we establish an upper bound on the number of overlattices of a cocompact lattice in $\mathrm{Aut}(K)$, using deep results of finite group theory.

Suppose G is a group of order n. Let $n = \prod_{i=1}^{t} p_i^{k_i}$ be the prime decomposition of n and let $\mu(n) = \max\{k_i\}$. We denote by $d(G)$ the minimum cardinality of a generating set for G, and by $f(n)$ the number of isomorphism classes of groups of order n. By results of Lucchini [Luc], Guralnick [Gur] and Sims [Si],

$$d(G) \leq \mu(n) + 1$$

and by work of Pyber [P] and Sims [Si], we obtain

$$f(n) \leq n^{\frac{2}{27}\mu(n)^2+\frac{1}{2}\mu^{5/3}(n)+75\mu(n)+16}$$

Let $g(n) = \frac{2}{27}\mu(n)^2 + \frac{1}{2}\mu^{5/3}(n) + 75\mu(n) + 16$, so that $f(n) \leq n^{g(n)}$.

Theorem 4.5.2. *Let Γ be a cocompact lattice acting on $\mathcal{X}$, where $\mathcal{X}$ is the scwol associated to a simply-connected, locally finite polyhedral complex. Then there are some positive constants C_0 and C_1, depending only on Γ, such that*

$$\forall n > 1, \qquad u_\Gamma(n) \leq (C_0 n)^{C_1 \log^2(n)}$$

Proof. Fix a quotient complex of groups $G(\mathcal{Y})_{C_\bullet}$ for the action of Γ on $\mathcal{X}$. By Lemma 4.3.15, since $\mathcal{Y}$ is finite there exist only finitely many scwols $\mathcal{Y}'$ such that a covering may be defined over a morphism $\mathcal{Y} \to \mathcal{Y}'$. Thus it is enough to show the upper bound for the number of overlattices with a fixed quotient scwol. We count the n-sheeted coverings of complexes of groups $\lambda : G(\mathcal{Y}) \to G'(\mathcal{Y}') = (G'_{\sigma'}, \psi_{a'}, g_{a',b'})$ over morphisms $l : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}'$ is fixed. Note that we do not insist on the complex of groups $G'(\mathcal{Y}')$ being faithful or developable.

For $\sigma \in V(\mathcal{Y})$, let $c_\sigma = |G_\sigma|$, and for $\sigma' \in V(\mathcal{Y}')$ let

$$c_{\sigma'} = \left(\sum_{\sigma \in l^{-1}(\sigma')} c_\sigma^{-1} \right)^{-1}$$

By definition of an n-sheeted covering, the cardinality $|G'_{\sigma'}|$ is equal to $nc_{\sigma'}$.

There are at most $\prod_{\sigma' \in V(\mathcal{Y}')} (c_{\sigma'} n)^{g(c_{\sigma'} n)}$ isomorphism classes of groups $G'_{\sigma'}$. There are at most $\prod_{a' \in E(\mathcal{Y}')} (c_{t(a')} n)^{\mu(c_{i(a')} n)+1}$ monomorphisms $\psi_{a'} : G'_{i(a')} \to G'_{t(a')}$ and at most $\prod_{\sigma \in V(\mathcal{Y})} (c_{l(\sigma)} n)^{\mu(c_\sigma)+1}$ injections $\lambda_\sigma : G_\sigma \to G'_{l(\sigma)}$. There are $|G'_{t(l(a))}| = nc_{t(l(a))}$ choices for each $\lambda(a)$.

Let $M = \max_{\sigma \in V(\mathcal{Y})} \max\{c_\sigma, c_{l(\sigma)}\}$ and $\mu = \mu(Mn)$. Let $c_0 = |V(\mathcal{Y})| \geq |V(\mathcal{Y}')|$ and $c_1 = |E(\mathcal{Y})| \geq |E(\mathcal{Y}')|$. The number $u_\Gamma(n)$ is at most the product of the number of isomorphism classes of groups $G'_{\sigma'}$, the number of monomorphisms $\psi_{a'}$, the number of twisting elements $g_{a',b'}$, the number of local maps λ_σ, and the number of elements $\lambda(a)$. Combining all the estimates above, we get the following upper bound for $u_\Gamma(n)$:

$$\begin{aligned}
u_\Gamma(n) \leq & \prod_{\sigma'\in V(\mathcal{Y}')} (c_{\sigma'}n)^{g(c_{\sigma'}n)} \prod_{a'\in E(\mathcal{Y}')} (c_{t(a')}n)^{\mu(c_{i(a')}n)+1} \prod_{a'\in E(\mathcal{Y}')} |G'_{t(a')}| \\
& \prod_{\sigma\in V(\mathcal{Y})} (c_{l(\sigma)}n)^{\mu(c_\sigma)+1} \prod_{a\in E(\mathcal{Y})} nc_{t(l(a))} \\
\leq & \prod_{\sigma'\in V(\mathcal{Y}')} (Mn)^{g(Mn)} \prod_{a'\in E(\mathcal{Y}')} (Mn)^{\mu(Mn)+1+1} \prod_{\sigma\in V(\mathcal{Y})} (Mn)^{\mu(M)+1} \prod_{a\in E(\mathcal{Y})} nM \\
\leq & (Mn)^{c_0 g(Mn)+c_1(\mu(Mn)+2)+c_0(\mu(M)+1)+c_1} \leq (Mn)^{C_1\mu^2} \leq (C_0 n)^{C_1'(\log n)^2}
\end{aligned}$$

where $C_1 = c_0(2/27 + 1/2 + 75 + c_0 + c_1 + 16 + 3c_0 + c_1)$ and $C_1' = C_1/\log 2$. □

4.5.2 Lower bound for right-angled buildings

In this section we establish a lower bound on the number of overlattices, for certain right-angled hyperbolic buildings. See Theorem 4.5.4 below for a precise statement.

We first define right-angled hyperbolic buildings. Let P be a compact convex polyhedron in $\mathbb{H}^n$, with all dihedral angles $\frac{\pi}{2}$, and let (W, I) be the right-angled Coxeter group generated by reflections in the $(n-1)$-dimensional faces of P. Each face of P then has the type of a unique subset $J \subseteq I$ such that W_J, the subgroup of W generated by $j \in J$, is finite. In particular, each $(n-1)$-dimensional face of P has the type of a unique $i \in I$, and so we will refer to the corresponding $(n-1)$-dimensional face of P as an i-face.

A *hyperbolic building* of type (W, I) is a polyhedral complex X equipped with a maximal family of subcomplexes, called *apartments*. Each apartment is polyhedrally isometric to the tesselation of $\mathbb{H}^n$ by the images of P under W, and these images are called *chambers*. The apartments and chambers of X satisfy the usual axioms for Bruhat–Tits buildings :

- each chamber is contained in an apartment ; and
- for each pair of apartments A and A', there exists a polyhedral isometry from A onto A' which fixes $A \cap A'$.

For $i \in I$, an $\{i\}$*-residue* of X is the connected subcomplex consisting of all chambers which meet in a given i-face of X.

An example of a right-angled hyperbolic building is Bourdon's 2-dimensional building $I_{p,q}$ (see [Bo2]). Here, P is a regular right-angled hyperbolic p-gon and each $\{i\}$-residue consists of q copies of P, glued together along a common edge. Right-angled buildings exist only in dimensions 2, 3 and 4 (see [PV]).

The following result classifies right-angled hyperbolic buildings.

Proposition 4.5.3 (Proposition 1.2, [HP]). *Let (W, I) be a right-angled Coxeter system and $\{q_i\}$ a family of positive integers ($q_i \geq 2$). Then, up to isometry, there exists a unique building X of type (W, I), such that for each $i \in I$, the $\{i\}$-residue of X has cardinality q_i.*

In the 2-dimensional case, this result is due to Bourdon [Bo2]. According to [HP], Proposition 4.5.3 was proved by M. Globus, and known also to M. Davis, T. Januszkiewicz and J. Świątkowski.

Let $T = T_{2p}$ be the $2p$-regular tree, where p is prime. In [L1], Lim constructed many non-isomorphic coverings of faithful graphs of groups with universal cover T, of the form shown in Figure 4.1.

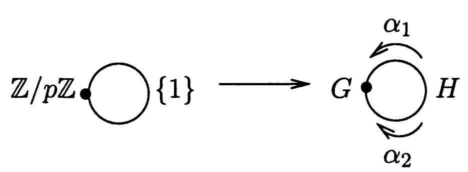


FIG. 4.1 – Coverings of graphs of groups

If Γ is the cocompact lattice in $\mathrm{Aut}(T)$ associated to the left-hand graph of groups, this yields the lower bound $u_\Gamma(n) \geq n^{\frac{1}{2}(k-3)}$, for $n = p^k$ and $k \geq 3$.

We now explain how to use these constructions to prove the following :

Theorem 4.5.4. *Let X be a right-angled hyperbolic building of type (W, I), with chambers P and parameters $\{q_i\}$. Assume that for some $i_1, i_2 \in I$, $i_1 \neq i_2$,*

1. *$q_{i_1} = q_{i_2} = 2p$ where p is prime; and*
2. *the i_1- and i_2-faces of P are non-adjacent (equivalently, $m_{i_1,i_2} = \infty$ in the Coxeter system associated to X).*

Then there is a cocompact lattice Γ, acting without inversions on X, such that for $n = p^k$, and $k \geq 3$,

$$u_\Gamma(n) \geq n^{\frac{k}{50} - \frac{27}{10}}$$

Proof. First, we take the "double cover" of the graphs of groups in Figure 1 above to obtain coverings of faithful graphs of groups with universal cover T, of the form

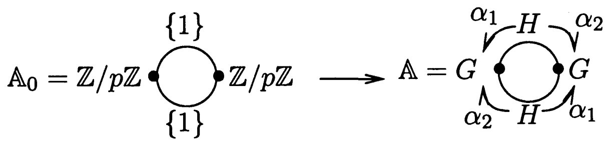


FIG. 4.2 – "Double covers" of Figure 4.1

We now carry out a special case of the Functor Theorem, [T3]. Let A be the graph with two edges underlying the graphs of groups in Figure 2. Let P and P' be two copies of P. Glue the i_1-face of P to the i_1-face of P' in a type-preserving manner, and similarly with the i_2-faces, and let the resulting polyhedral complex be Y. If $\mathcal{Y}$ is the scwol associated to Y, then each edge and each vertex of A may be identified to a vertex of $\mathcal{Y}$. Also, each face of P and P' may be identified to a vertex of $\mathcal{Y}$, so that the vertices of $\mathcal{Y}$ now have types J with W_J finite.

Let $\mathbb{A}_0$ and $\mathbb{A}$ be as in Figure 2. Then $\mathbb{A}$ induces a complex of groups $G(\mathcal{Y})$ over $\mathcal{Y}$, as follows (the construction for $\mathbb{A}_0$ is similar). First fix the local groups induced by the identification of A with some of the vertices of $\mathcal{Y}$. Each map from edge to vertex groups in $\mathbb{A}$ then induces a monomorphism ψ_a along an edge a of $\mathcal{Y}$. For each $i \in I$, let G_i be a group of order q_i.

Let J be a subset of I such that W_J is finite. If J does not contain i_1 or i_2, then the local group at the vertices of $\mathcal{Y}$ of type J is

$$H \times \prod_{j \in J} G_j$$

The monomorphisms between such local groups are natural inclusions. Now consider J containing one of i_1 and i_2 (since $m_{i_1,i_2} = \infty$, J cannot contain both i_1 and i_2). Without loss of generality suppose J contains i_1. Then the face of type J in Y is contained in the glued i_1-face, and the local group at the vertex of $\mathcal{Y}$ of type J is

$$G \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

The monomorphism from G to this local group is inclusion onto the first factor. For each $J' \subset J$ with $i_1 \in J$, the monomorphism

$$G \times \prod_{\substack{j \in J' \\ j \neq i_1}} G_j \to G \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

is the natural inclusion. For each $J' \subset J$ with $i_1 \notin J$, the monomorphism

$$H \times \prod_{j \in J'} G_j \to G \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

is a monomorphism $H \to G$ from the graph of groups $\mathbb{A}$ on the first factor, and natural inclusions on the other factors. Put all $g_{a,b} = 1$ and we have a complex of groups $G(\mathcal{Y})$.

Let $G(\mathcal{Y})_0$ be the complex of groups induced in this way by $\mathbb{A}_0$. It is not hard to verify that $G(\mathcal{Y})_0$ has nonpositive curvature and is thus developable, and that its universal cover is the scwol associated to the hyperbolic building X. Also, every covering as in Figure 2 induces a covering of the associated complexes of groups $G(\mathcal{Y})_0 \to G(\mathcal{Y})$. By Lemma 4.3.14, since $G(\mathcal{Y})_0$ has nonpositive curvature, each $G(\mathcal{Y})$ is developable. The arguments used to show faithfulness of the graphs of groups, together with Proposition 4.3.8, imply that each $G(\mathcal{Y})$ is faithful. Moreover, by Lim's construction, non-isomorphic coverings of the form in Figure 2 induce non-isomorphic coverings $G(\mathcal{Y})_0 \to G(\mathcal{Y})$. By Theorem 4.1.3, this completes the proof. $\square$

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