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Ke CHEN

SPECIAL SUBVARIETIES OF MIXED SHIMURA VARIETIES

SOUS-VARIÉTÉS SPÉCIALES DES VARIÉTÉS DE SHIMURA MIXTES

Soutenue le 27 novembre 2009 devant la commission d'examen :

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Sous-variétés spéciales des
variétés de Shimura mixtes

Résumé. Cette thèse est dédiée à l'étude de la conjecture d'André-Oort pour les variétés de Shimura mixtes. On montre que dans une variété de Shimura mixte M définie par une donnée de Shimura mixte $(\mathbf{P}, Y)$, soient $\mathbf{C}$ un $\mathbb{Q}$-tore dans $\mathbf{P}$ et Z une sous-variété fermée quelconque dans M, alors l'ensemble des sous-variétés $\mathbf{C}$-spéciales maximales contenues dans Z est fini. La démonstration suit la stratégie de L.Clozel, E.Ullmo, et A.Yafaev dans le cas pure, qui dépend de la théorie de Ratner sur des propriétés ergodiques des flots unipotents sur des espaces homogènes. D'ailleurs, une minoration sur le degré de l'orbite sous Galois d'une sous-variété pure est montrée dans le cas mixte, adaptée du cas pure établi par E.Ullmo et A.Yafaev. Enfin, une version relative de la conjecture de Manin-Mumford est démontrée en caractéristique nul: soit A un S-schéma abélien en caractéristique nul, alors l'adhérence de Zariski d'une suite de sous-schémas de torsion dans A égale une réunion finie de sous-schémas de torsion.

Mots clefs : approximation diophantienne, variétés de Shimura mixtes, sous-variétés spéciales, conjecture d'André-Oort, conjecture de Manin-Mumford, équidistribution, theorie de Ratner.

Special Subvarieties of
Mixed Shimura Varieties

Abstract. This thesis studies the André-Oort conjecture for mixed Shimura varieties. The main result is: let M be a mixed Shimura variety defined by a mixed Shimura datum $(\mathbf{P}, Y)$, $\mathbf{C}$ a fixed $\mathbb{Q}$-torus of $\mathbf{P}$, and Z an arbitrary closed subvariety in M, then the set of maximal $\mathbf{C}$-special subvarieties of M contained in Z is finite. The proof follows the strategy applied by L.Clozel, E.Ullmo, and A.Yafaev in the pure case, which relies on Ratner's theory on ergodic properties of unipotent flows on homogeneous spaces. Besides, a minoration on the degree of the Galois orbit of a special subvariety is proved in the mixed case, adapted from the pure case established by E.Ullmo and A.Yafaev. Finally, a relative version of the Manin-Mumford conjecture is proved in characteristic zero: let A be an abelian S-scheme of characteristic zero, then the Zariski closure of a sequence of torsion subschemes in A remains a finite union of torsion subschemes.

Keywords: Diophantine approximation, mixed Shimura varieties, special subvarieties, André-Oort conjecture, Manin-Mumford conjecture, equidistribution, Ratner's theory.

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Special Subvarieties of Mixed Shimura Varieties

CHEN Ke

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Contents

Introduction

This thesis is dedicated to the study of the André-Oort conjecture for mixed Shimura varieties. The following introduction is aimed to illustrate the basic ideas and the main results of this writing through a special class of mixed Shimura varieties, which requires less preliminaries than what one might find in standard references like [Pink-0].

Recall that the pure Shimura varieties defined by P.Deligne and G.Shimura are quasi-projective varieties parameterizing rational pure Hodge structures with additional structures (e.g. abelian varieties with PEL-data). Still the group-theoretic definition is preferred: a pure Shimura datum is a pair $(\mathbf{G},X)$ subject to the axioms of P.Deligne with $\mathbf{G}$ a reductive $\mathbb{Q}$-group and X a homogeneous space under $\mathbf{G}(\mathbb{R})$ whose connected components are (non-compact) Hermitian symmetric domains associated to $\mathbf{G}(\mathbb{R})^+$; for $K\subset\mathbf{G}(\mathbb{A}^f)$ a compact open subgroup, $M_K(\mathbf{G},X)$ is a quasi-projective variety defined over the reflex field $E(\mathbf{G},X)$ whose complex locus is $\mathbf{G}(\mathbb{Q})\backslash[X\times\mathbf{G}(\mathbb{A}^f)/K]$.

The notions of mixed Shimura data of Kuga type and the associated mixed Shimura varieties can be presented in the following simplified form, which is equivalent to the definitions that appear later in Chapter 1: the data are of the form $(\mathbf{P},Y)=(\mathbf{V}\rtimes_\rho\mathbf{G},\mathbf{V}(\mathbb{R})\rtimes X)$ where $(\mathbf{G},X)$ is some pure Shimura datum, $\rho:\mathbf{G}\to\mathbf{GL}_\mathbb{Q}(\mathbf{V})$ an algebraic representation of $\mathbf{G}$ on some finite dimensional $\mathbb{Q}$-vector space $\mathbf{V}$ such that each $x:\mathbb{S}\to\mathbf{G}_\mathbb{R}$ in X induces on $\mathbf{V}$ a complex structure (namely a rational pure Hodge structure of type $\{(-1,0),(0,-1)\}$) via ρ, and $Y=\mathbf{V}(\mathbb{R})\rtimes X$ is the complex manifold whose complex structure on the fibre $\mathbf{V}(\mathbb{R})\rtimes x$ is given by $x:\mathbb{S}\to\mathbf{G}_\mathbb{R}\to\mathbf{GL}_\mathbb{R}(\mathbf{V}_\mathbb{R})$, which justifies the notation $\rtimes$ here; and the mixed Shimura varieties of Kuga type are the quasi-projective $\mathbb{Q}^{ac}$-varieties $M_K(\mathbf{P},Y)$ whose complex loci are of the form $\mathbf{P}(\mathbb{Q})\backslash[Y\times\mathbf{P}(\mathbb{A}^f)/K]$, K being compact open subgroups of $\mathbf{P}(\mathbb{A}^f)$. These varieties are equipped with canonical models over the same reflex field $E(\mathbf{G},X)$ as the corresponding pure Shimura varieties. When K is of the form $K_\mathbf{V}\rtimes K_\mathbf{G}$ for some compact open subgroups $K_\mathbf{V}\subset\mathbf{V}(\mathbb{A}^f)$ and $K_\mathbf{G}\subset\mathbf{G}(\mathbb{A}^f)$, the canonical projection $\pi:M=M_K(\mathbf{P},Y)\to S=M_{K_\mathbf{G}}(\mathbf{G},X)$ defines an abelian S-scheme over the pure Shimura variety S, which serves as the prototype of Kuga varieties.

The connected components of mixed Shimura varieties of Kuga type defined by $(\mathbf{P},Y)$ can be written in the form $M^+=\Gamma\backslash Y^+$ where Y^+ is a connected component of Y and $\Gamma\subset\mathbf{P}(\mathbb{Q})$ is some congruence subgroup. Similarly, a special subvariety of M^+ can be written as $M_1^+=(\Gamma\cap\mathbf{P}_1(\mathbb{Q}))\backslash Y_1^+$ (embedded in M^+) defined by some subdatum $(\mathbf{P}_1,Y_1)\subset(\mathbf{P},Y)$. The Γ-conjugacy class of $\mathbf{P}_1$ only depends on $M_1^+\hookrightarrow M^+$, and is called the Mumford-Tate group of M_1+; in this thesis the subdatum $(\mathbf{P}_1,Y_1)$ is usually fixed, and for simplicity $\mathbf{P}_1$ is referred to as the Mumford-Tate group of M_1^+. Note that in this mixed setting a special point is no other than a special subvariety of dimension zero, namely defined by $(\mathbf{T},x)$ with $\mathbf{T}$ a $\mathbb{Q}$-torus.

In general, mixed Shimura varieties can be viewed as torus torsors over mixed

Shimura varieties of Kuga type. They arise naturally in the theory of toroidal compactifications of (pure) Shimura varieties, whose foundation is laid down in the Ph.D thesis of R.Pink [Pink-0]. Even though non-trivial torus torsors are often encountered in toroidal compactifications, this introduction is restricted to the case of mixed Shimura varieties of Kuga type, and through them is illustrated the main subject of this thesis: the André-Oort conjecture.

The André-Oort conjecture was initially raised for pure Shimura varieties (cf. [André-4] and [Oort-2]): let M be a Shimura variety, and Σ a sequence of special subvarieties of M, then the Zariski closure of $\bigcup\Sigma$ is a finite union of special subvarieties, where by special subvariety is meant a geometrically irreducible component of the image under a Hecke correspondence of a Shimura subvariety $M' \subset M$. Replace Shimura varieties by mixed Shimura varieties of Kuga type one gets the formulation of the André-Oort-Pink conjecture in the mixed context, which has been formulated in [Pink-2].

As is pointed out in [Pink-2], [U-3] and [U-4], this formulation is analogous to the Manin-Mumford conjecture, which was first proved by M.Raynaud: let Σ be a sequence of torsion subvarieties in a complex abelian variety A, then the Zariski closure of $\bigcup\Sigma$ is a finite union of torsion subvarieties, where by torsion subvariety is meant a closed subvariety of the form $a + A'$ with $A' \subset A$ an abelian subvariety, $a+$ denoting the translation by a torsion point a. Both of the two conjectures study the distribution of a family of special sub-objects with "a lot of symmetries": in the case of Manin-Mumford, a torsion subvariety is stabilized under "a lot of" homotheties; and in the case of Andeé-Oort, a special subvariety is "stabilized" by "many" Hecke correspondences. Of course these two conjectures can be stated in different equivalent forms, and for the André-Oort conjecture the following two are preferred:

Conjecture 0.0.1 (the conjecture of André-Oort-Pink:). *(1) Let* M *be a mixed Shimura variety of Kuga type, then for any closed subvariety* $Z \subset M$, *the set* $\mathscr{S}(Z)$ *of maximal special subvarieties contained in* Z *is finite.*

(2) Let M *be a mixed Shimura variety of Kuga type, and* $(M_n)_n$ *an arbitrary sequence of special subvarieties, then the Zariski closure of* $\bigcup_n M_n$ *is weakly special, namely a finite union of special subvarieties.*

This formulation includes certain cases of the classical Manin-Mumford conjecture: for example, certain abelian varieties over number fields can be realized as a (connected) mixed Shimura variety associated to some datum $(\mathbf{P}, Y) = (\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes X)$ with $\mathbf{G}$ a $\mathbb{Q}$-torus and X a single point, and in this case the two notions of special points coincide. Note that in this case A has complex multiplication and satisfies the "Shimura condition" as in Theorem 7.44 in [Sh].

In [U-1] and [U-3], E.Ullmo sketched a program to treat these two conjectures using ideas from ergodic theory: in the two conjectures, the special sub-objects are equipped with canonical probability measures, and the conjectures would hold if one could prove that any sequence of measures associated to special sub-objects admits a convergent subsequence whose limit is again associated to some special sub-object, or equivalently, the compactness of the set of canonical probability measures associated to special subobjects. More precisely the André-Oort conjecture admits the following refinement

Conjecture 0.0.2 (the equidistribution conjecture for mixed Shimura varieties of Kuga type:). *Let* M *be a mixed Shimura variety of Kuga type, and* $(\mathrm{M}_n)_n$ *be a sequence of special subvarieties of* M *which is strict in the sense that for any special subvariety* $\mathrm{M}' \subsetneq \mathrm{M}$, $\mathrm{M}_n \not\subseteq \mathrm{M}'$ *for n large enough. Denote by*

$$\mu_n = \frac{1}{|\mathcal{O}(\mathrm{M}_n)|}\sum_{\mathrm{Y}\in\mathcal{O}(\mathrm{M}_n)} \mu_{\mathrm{Y}}$$

the average of the μ_{Y}*'s with* Y *varying over* $\mathcal{O}(\mathrm{M}_n)$, *where* $\mathcal{O}(\mathrm{M}_n)$ *denotes the* $\mathrm{Gal}_{\mathrm{E}}$*-orbit of* M_n *in* M, E *being the reflex field of* M, *and* μ_{Y} *the canonical probability measure on* $\mathrm{M}(\mathbb{C})_{\mathrm{an}}$ *associated to the complex analytic space* $\mathrm{Y}(\mathbb{C})_{\mathrm{an}}$. *Then* $(\mu_n)_n$ *converges to the canonical probability measure on* $\mathrm{M}(\mathbb{C})_{\mathrm{an}}$ *for the weak topology, and therefore* $\bigcup_n \mathrm{M}_n(\mathbb{C})$ *is dense in* $\mathrm{M}(\mathbb{C})_{\mathrm{an}}$ *for the archimedean topology.*

When specialized to the case of an abelian variety which can be realized as a mixed Shimura variety, the equidistribution conjecture predicts that the Galois orbits of torsion points are equidistributed with respect to the Haar measure on the compact complex Lie group $\mathrm{A}(\mathbb{C})_{\mathrm{an}}$, and this is a special case of the (resolved) Bogomolov conjecture.

The conjecture makes sense also in the pure case and it implies the André-Oort conjecture: if the set $\mathscr{S}(\mathrm{Z})$ is infinite for some closed subvariety $\mathrm{Z} \subsetneq \mathrm{S}$ which is Hodge generic, namely $\mathrm{Z} \not\subseteq \mathrm{S}'$ for every special subvariety $\mathrm{S}' \subsetneq \mathrm{S}$, then $\mathscr{S}(\mathrm{Z})$ forms a strict sequence (S_n), and the equidistribution conjecture predicts that the Galois orbits of the S_n's are dense in S, which contradicts the strict inclusion $\mathrm{Z} \subsetneq \mathrm{S}$. This is partially realized in [CU-3] for a strict sequence of "strongly special subvarieties" of positive dimensions inside a pure Shimura variety defined by an adjoint $\mathbb{Q}$-group, where being strongly special means that the Mumford-Tate group of the special subvariety is semi-simple. This is soon generalized in [UY-1] to the case of strict sequences of $\mathbf{C}$-special subvarieties in $\mathrm{M}_{\mathrm{K}}(\mathbf{G},\mathrm{X})$, where $\mathbf{C}$ is a fixed $\mathbb{Q}$-torus and a $\mathbf{C}$-special subvariety is nothing but a special subvariety whose Mumford-Tate group is of connected center $\mathbf{C}$. This approach is essentially ergodic-theoretic, and it does not yet permit any generalization to a sequence of special subvarieties such that infinitely many $\mathbb{Q}$-tori, non-isomorphic to each other, appear as connected centers of the Mumford-Tate groups.

The result above is reffered to as the equidistribution of homogeneous sequences of special subvarieties. Here a sequence of special subvarieties (S_n) in a (connected) pure Shimura variety S (defined by $(\mathbf{G},\mathrm{X})$) is said to be homogeneous if for some fixed $\mathbb{Q}$-torus $\mathbf{C}$ in $\mathbf{G}$ one can find subdata $(\mathbf{G}_n,\mathrm{X}_n)$ of $(\mathbf{G},\mathrm{X})$ such that S_n is defined by $(\mathbf{G}_n,\mathrm{X}_n)$ and that the $\mathbf{G}_n$'s are of common connected center $\mathbf{C}$. In the general theory of pure Shimura data, the action of the weight homomorphism factors through the connected center of the Mumford-Tate group, and the notion of homogeity here indicates that the special subvarieties "carry the same Hodge weight". Similarly, a sequence of special subvarieties $(S_n)_n$ is said to be weakly homogeneous if it is a finite union of homoeneous subsequences, namely there exist finitely many $\mathbb{Q}$-tori $\mathbf{C}_1, \dots, \mathbf{C}_r$ in $\mathbf{G}$ such that each S_n is $\mathbf{C}_i$-special for some $i \in \{1, \dots, r\}$. We apologize for this terminology that might confuse the readers with homogeneous spaces under some group action.

In [UY-1] E.Ullmo and A.Yafaev also studied the lower bound of the degree of the Galois orbit of a special subvariety S' in a pure Shimura variety S with respect to the canonical line bundle $\mathscr{L}$ defining the Baily-Borel compactification of S. Their bound is referred as the test invariant of S' with respect to S and $\mathscr{L}$. And they established a useful criterion: let $(S_n)_n$ be a sequence of special subvarieties (of arbitrary dimensions) of $S = M_K(\mathbf{G}, X)$ such that the sequence of associated test invariants $(\tau(S_n))_n$ is bounded, then $(S_n)_n$ is a weakly homogeneous sequence in the sense that there is finitely many $\mathbb{Q}$-tori $\mathbf{C}_i$ of $\mathbf{G}$ such that each S_n is $\mathbf{C}_i$-special for some i. In particular, the equidistribution result can be applied to $(S_n)_n$ to deduce that the Zariski closure of $\bigcup_n S_n$ is a finite union of special subvarieties.

The precedent results of L.Clozel, E.Ullmo and A.Yafaev fit into the following strategy towards the André-Oort conjecture, namely the finiteness of $\mathscr{S}(Z)$ the set of maximal special subvarieties contained in an arbitrary closed subvariety Z of the given Shimura variety S. This set is a priori countable, and we write it as a sequence $(S_n)_n$. If this sequence is of bounded test invariants, then it is weakly special, and thus the finiteness of $\mathscr{S}(Z)$ follows from the maximality of the S_n's and the equidistribution results. On the other hand, if the sequence of test invariants is unbounded, then according to the recent work of B.Klingler and A.Yafaev [KY], for special subvarieties $S_n \subset Z$ whose test invariant is large enough, we can construct a chain of inclusions $S_n \subsetneq S'_n \subset Z$ with S'_n a special subvariety, which contradicts the maximality of $(S_n)_n$ and ends the proof.

It should be remarked that the equidistribution of $\mathbf{C}$-special subvarieties is established unconditionally. By contrast, the other ingredients in the current proof of the André-Oort conjecture make a crucial use of the effective Chebotarev theorem, which is a consequence of the Generalized Riemann Hypothesis: it appears in the estimation of the lower bound in term of the test invariant in [UY-1], and also in the proof of [KY] for the construction of the chain $S_n \subsetneq S'_n \subset Z$ that depends on the choice of a "good" prime ℓ over which one can find approapriate Hecke correspondences to confirm the existence of such a chain.

The current approach towards the André-Oort has not yet resulted in a proof of the more general equidistribution conjecture: for a sequence of special subvarieties of non-bounded test invariant, the approach in [CU-3] doesn't work, and not much is known how to treat the sequence of canonical probability measures averaged over the Galois orbits associated to them. There are positive progresses in this direction which involve more explicit calculations using automorphic forms, cf.[JLZ], but this is not touched in the current writing.

The present thesis attempts to generalize part of the strategy above to the case of mixed Shimura varieties, as follows:

• In Chapter 1 some preliminariesare are recalled on the notions of mixed Shimura data, mixed Shimura varieties, Hecke correspondences, and canonical models over the reflex fields. The notion of **C**-special subdata and **C**-special subvarieties are defined in a trivial way: for a mixed Shimura datum $(\mathbf{P},\mathrm{Y})$, fix a Levi decomposition $\mathbf{P}=\mathbf{V}\rtimes\mathbf{G}$, and write $\pi:\mathbf{P}\to\mathbf{G}$ for the canonical projection, then a subdatum $(\mathbf{P}',\mathrm{Y}')$ of $(\mathbf{P},\mathrm{Y})$ is **C**-special for some $\mathbb{Q}$-torus $\mathbf{C}\subset\mathbf{G}$ if and only if **C** equals the connected center of $\mathbf{G}'=\pi(\mathbf{P}')$; and a **C**-special subvariety is a special subvariety defined by some **C**-special subdatum of $(\mathbf{P},\mathrm{Y})$.

There are also some technical lemmas concerning the construction of mixed Shimura varieties, including the following one: if $(\mathbf{P},\mathrm{Y})$ is a mixed Shimura datum of Kuga type, y a point in Y, **Q** a $\mathbb{Q}$-subgroup of **P** such that $y(\mathbb{S})\subset\mathbf{Q}_{\mathbb{R}}$, then there exists a maximal $\mathbb{Q}$-subgroup $\mathbf{P}'$ invariant in $\mathbb{Q}$ such that $(\mathbf{P}',\mathbf{P}'(\mathbb{R})y)$ is a subdatum of Kuga type in $(\mathbf{P},\mathrm{Y})$, with $\mathbf{P}'(\mathbb{R})y=\mathbf{Q}(\mathbb{R})y$.

• Chapter 2 serves as an expanded introduction to the André-Oort-Pink conjecture. The following two equivalent formulations of the conjecture are preferred in the sequel:

(1) Let M be a mixed Shimura variety, and $(\mathrm{M}_n)_n$ a sequence of special subvarieties, then the Zariski closure of $\bigcup_n \mathrm{M}_n$ is a finite union of special subvarieties.

(2) Let M be a mixed Shimura variety, and Z a closed subvarieties in M, then the set $\mathscr{S}(\mathrm{Z})$ of maximal special subvarieties contained in Z is finite.

In this chapter is also included a detailed introduction to the main results of the thesis.

• Chapter 3 is focused on the equidistribution of homogeneous sequences of special subvarieties, where by homogeneity of a sequence is meant that the special subvarieties in the sequence are **C**-special for a fixed $\mathbb{Q}$-torus **C**. The main results can be stated in a parallel way to the conjecture of André-Oort:

(**C**-1): Let M be a mixed Shimura variety, and $(\mathrm{M}_n)_n$ be a sequence of **C**-special subvarieties in M, then the Zariski closure of $\bigcup_n \mathrm{M}_n$ is a finite union of **C**-special subvarieties;

(**C**-2): Let M be a mixed Shimura variety and $\mathrm{Z}\subset\mathrm{M}$ a closed subvariety, then the set $\mathscr{S}_{\mathbf{C}}(\mathrm{Z})$ of maximal **C**-special subvarieties in M contained in Z is finite.

Note that **C** is required to be "not of CM type", namely $y(\mathbb{S}_{\mathbb{C}})\not\subset\mathbf{C}_{\mathbb{C}}$ for any point $y\in\mathrm{Y}$. If on the contrary $y(\mathbb{S}_{\mathbb{C}})\subset\mathbf{C}_{\mathbb{C}}$ for some $y\in\mathrm{Y}$, then $(\mathbf{C},y)$ is a special subdatum. The (resolved) Manin-Mumford conjecture implies that the Zariski closure of a family of **C**-special subvarieties remains weakly special.

The strategy is as follows:

(1) The starting point is the observation that the theorem of S.Mozes and N.Shah (cf.[MS] Theorem 1.1) on a class of ergodic measures on lattice spaces already fits into the framework of mixed Shimura varieties, from which is deduced a weakened André-Oort type theorem for lattice spaces.

To be exact, for a mixed Shimura datum of Kuga type $(\mathbf{P},\mathrm{Y}) = (\mathbf{V}\rtimes\mathbf{G},\mathbf{V}(\mathbb{R})\rtimes\mathrm{X})$ and an arithmetic subgroup $\Gamma\subset\mathbf{P}(\mathbb{R})^+$, the quotient $\Omega=\Gamma\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ is referred to as the lattice space associated to $(\mathbf{P},\mathrm{Y},\Gamma)$. We first take $\mathbf{C}$ to be the connected center of $\mathbf{G}$, then we get a (countable) set of measures $\mathcal{H}_\mathbf{C}(\Omega)$, whose elements are the measures on Ω that are associated to $\mathbf{C}$-special lattice subspaces of the form $\Omega'=\Gamma\backslash\Gamma\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$, where $\mathbf{P}'$ comes from a $\mathbf{C}$-special subdatum $(\mathbf{P}',\mathrm{Y}')$ of Kuga type, namely for some (or any) Levi decomposition $\mathbf{P}'=\mathbf{V}'\rtimes\mathbf{G}'$, the connected center of $\mathbf{G}'$ is of the form $v\mathbf{C}v^{-1}$ for certain $v\in\mathbf{V}(\mathbb{Q})$. The theorem of S.Mozes and N.Shah affirms that any sequence in $\mathcal{H}_\mathbf{C}(\Omega)$ admits a subsequence which converges weakly to some probability measure μ', and by checking the involved Hodge structures one deduces that μ' lies in $\mathcal{H}_\mathbf{C}(\Omega)$, hence the compactness of $\mathcal{H}_\mathbf{C}(\Omega)$. This can be regarded as an André-Oort type theorem for sequences of $\mathbf{C}$-special lattice subspaces: if $(\Omega_n)_n$ is a sequence of $\mathbf{C}$-special lattice subspaces in Ω, then the archimedean closure of $\bigcup_n\Omega_n$ is a finite union of $\mathbf{C}$-special lattice subspaces.

(2) From $\mathbf{C}$-special lattice subspaces one can construct any $\mathbf{C}$-special subvariety in a connected mixed Shimura variety of the form $\mathrm{M}=\Gamma\backslash\mathrm{Y}^+$ as follows: let $x\in\mathrm{Y}'^+\subset\mathrm{Y}^+$, with $(\mathbf{P}',\mathrm{Y}')\subset(\mathbf{P},\mathrm{Y})$ a $\mathbf{C}$-special subdatum, then the projection $\kappa_x:\Omega\to\mathrm{M}=\Gamma\backslash\mathrm{Y}^+$, $\Gamma g\mapsto\Gamma gx$ sends $\Omega'=\Gamma\backslash\Gamma\mathbf{P}'^{\dagger}(\mathbb{R})^+$ onto $\mathrm{M}'=\Gamma\backslash\Gamma\mathrm{Y}'^+$, and the push-forward under κ_x of the canonical measure ν' on Ω supported on Ω' is exactly the canonical probability measure μ' on M supported on M'. Note that μ' only depends on $\mathbf{P}',\mathrm{Y}'^+$ and Γ, and is independent of the choice of base point $x\in\mathrm{Y}'^+$. Write $\mathcal{H}_\mathbf{C}(\mathrm{M})$ for the set of canonical measures on M defined by $\mathbf{C}$-special subvarieties. Then the above construction yields a surjective map $\mathcal{H}_\mathbf{C}(\Omega)\to\mathcal{H}_\mathbf{C}(\mathrm{M})$, but we do not know a priori whether it is continuous: this map is not merely κ_{x*} for a single fixed x.

The compactness of $\mathcal{H}_\mathbf{C}(\mathrm{M})$ follows from an argument of S.Dani and G.Margulis. Similar to the pure case treated in [CU-3], in Chapter 3 is shown that there exists a compact subset $\mathcal{C}_\mathbf{C}\subset\mathrm{Y}$ such that if $\mathrm{M}'\subset\mathrm{M}$ is a $\mathbf{C}$-special subvariety, then there is a $\mathbf{C}$-special subdatum $(\mathbf{P}',\mathrm{Y}')\subset(\mathbf{P},\mathrm{Y})$ such that $\mathrm{Y}'+\cap\mathcal{C}_\mathbf{C}\neq\emptyset$ and $\Gamma\backslash\Gamma\mathrm{Y}'^+=\mathrm{M}'$. Thus the elements in $\mathcal{H}_\mathbf{C}(\Omega)$ are of the form $\kappa_{x*}\nu'$ with ν' coming from the compact set $\mathcal{H}_\mathbf{C}(\Omega)$ and x a point in the compact set $\mathcal{C}_\mathbf{C}$, hence the compactness of $\mathcal{H}_\mathbf{C}(\mathrm{M})$. Consequently, the closure of a sequence of $\mathbf{C}$-special subvarieties is a finite union of $\mathbf{C}$-special subvarieties. Note that the closure here is taken in the archimedean topology: this is even finer than the Zariski topology, and the two topologies yield the same closure in our case. Thus the André-Oort conjecture holds for a sequence of $\mathbf{C}$-special subvarieties. .

(3) One may also replace the connected center $\mathbf{C}$ of $\mathbf{G}$ by a more general $\mathbb{Q}$-torus $\mathbf{C}'$: if for some pure subdatum $(\mathbf{G}',\mathrm{X}')\subset(\mathbf{G},\mathrm{X})$ one has $\mathbf{C}'$ equal to the connected center of $\mathbf{G}'$, then there exists only finitely many maximal $\mathbf{C}'$-special

subdata $(\mathbf{P}_i, Y_i)$ in $(\mathbf{P}, Y)$, i varying over some fixed finite index set. Thus the set of maximal $\mathbf{C}'$-special lattice subspaces in Ω is finite, and so it is with the set of maximal $\mathbf{C}'$-special subvarieties in M. In order to study the closure of a sequence of $\mathbf{C}'$-special subvarieties, one may assume that the sequence is contained in a fixed maximal $\mathbf{C}'$-special subvariety, and then the conclusion follows immediately from the arguments in (1) and (2).

The general case of non-Kuga type is slightly different from the above discussions: an intermediate class of objets called "S-spaces" is introduced in Chapter as: they serve as "real parts" of the complex mixed Shimura varieties. They carry canonical probability measures and are closed related to the lattice spaces over which the ergodic arguments are applicable. The equidistribution of $\mathbf{C}'$-special S-subspaces is proved and then taking Zariski closure yields a partial reply to the André-Oort conjecture.

- Chapter 4 is mainly an reinterpretation of an estimation of E.Ullmo and A.Yafaev in the mixed case. Recall that one of the main results in [UY-1] is a lower bound for the degree of the Galois orbit of a pure special subvariety S' in a given pure Shimura variety S with respect to the line bundle defining the Baily-Borel compactification:

$$\deg_{\mathscr{L}} \mathrm{Gal}_{E} S' \geq \tau(S') = \alpha(\mathbf{C}')\beta(\mathbf{C}', K)$$

where E is the reflex field of S, $\mathbf{C}'$ the connected center of the Mumford-Tate group of S', $\alpha(\mathbf{C}')$ a fixed power of the absolute discriminant of the splitting field of $\mathbf{C}'$ (up to some constant coefficient), K the compact open subgroup defining the Shimura variety S, and

$$\beta(\mathbf{C}', K) = \max(1, \prod_{p \in \delta(\mathbf{C}', K)} B|K^{\max}_{\mathbf{C}',p}/K_{\mathbf{C}',p}|)$$

with B a constant determined by a fixed representation of $\mathbf{G}$, and $\delta(\mathbf{C}', K)$ the finite set of rational primes p such that $K_{\mathbf{C}',p} \subsetneq K^{\max}_{\mathbf{C}',p}$ where $K_{\mathbf{C}',p}$ is the p-th component of $K_{\mathbf{C}'} = K \cap \mathbf{C}'(\mathbb{A}^{f})$, and $K^{\max}_{\mathbf{C}',p}$ is the maximal compact open subgroup of $\mathbf{C}'(\mathbb{A}^{f})$. Following the ideas of L.Clozel, E.Ullmo and A.Yafaev, in [UY-1] is established a criterion on the equidistribution of special subvarieties: let $(S_n)_n$ be a sequence of special subvarieties in S such that the associated sequence of test invariants $(\tau(S_n)_n$ is bounded when n varies, then there exists finitely many $\mathbb{Q}$-tori $\mathbf{C}_i$ such that each S_n is $\mathbf{C}_i$-special for some i, and thus the closure of $\bigcup_n S_n$ is a finite union of special subvarieties due to [CU-3] and [UY-1].

As for the mixed case, in Chapter 4 is first considered the degree of Galois orbit of a pure special subvariety M' with respect to the line bundle $\pi^*\mathscr{L}$, where π is the projection from $M = M_{K_V \rtimes K_G}(\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes X)$ onto $S = M_{K_G}(\mathbf{G}, X)$, defined over the common reflex field $E = E(\mathbf{G}, X)$, and $\mathscr{L} = \mathscr{L}(K_G)$ is the line bundle on S defining the Baily-Borel compactification. One can show that the Mumford-Tate group of M' in M is of the form $v\mathbf{G}'v^{-1}$ for some $v \in \mathbf{V}(\mathbb{Q})$ and some pure subdatum $(\mathbf{G}', X') \subset (\mathbf{G}, X)$. Then the degree to be estimated is equal to $\deg_{\mathrm{pr}_v^*\mathscr{L}} \mathrm{Gal}_E S'_v$

where S'_v is a pure special subvariety in $S_v = M_{K_G(v)}(\mathbf{G},X)$ defined by $(\mathbf{G}',X')$, $K_G(v)$ is the isotropy subgroup of $(v \mod K_V)$ in $\mathbf{V}(\mathbb{A}^f)/K_V$ under the action of K_G, and pr_v denotes the projection $M_{K_G(v)}(\mathbf{G},X) \to M_{K_G}(\mathbf{G},X)$ defined by the inclusion of compact open subgroups $K_G(v) \subset K_G$. The functorial properties of Baily-Borel compactifications imply that $pr_v^*(\mathscr{L})$ is isomorphic to the compactifying bundle $\mathscr{L}(K_G(v))$ on S_v, and that

$$\deg_{\mathscr{L}(K_G(v))} \mathrm{Gal}_E S'_v \geq \tau(S'_v) = \alpha(\mathbf{C}')\beta(\mathbf{C}', K_G(v))$$

namely $\deg_{pr^*\mathscr{L}} \mathrm{Gal}_E M' \geq \alpha(\mathbf{C}')\beta(\mathbf{C}', K_G(v))$, with $\mathbf{C}'$ the connected center of $\mathbf{G}'$.

By the same arguments as in [EY] and in [UY-1], it is shown that $\beta(\mathbf{C}', K_G(v)) \geq \prod_{p\in\delta(\mathbf{C}',K_G(v))} cp$ for some constant c independent of the choice of K_G, S'_v and v. Thereby for a sequence of pure special subvarieties $(M_n)_n$ in M whose test invariants $\tau(M_n) = \alpha(\mathbf{C}_n)\beta(\mathbf{C}_n, K_G(v_n))$ remain bounded when n varies, there exists finitely many $\mathbb{Q}$-tori $\mathbf{C}_i$ in $\mathbf{G}$ such that each M_n is $\mathbf{C}_i$-special for some i, hence the closure of $\bigcup_n M_n$ is again a finite union of speecial subvarieties.

The study of pure special subvarieties allows us to carry over part of the ideas employed in [RU] towards the Manin-Mumford conjecture for an abelian variety defined over a number field: let $T_n = a_n + A_n$ be a sequence of torsion subvarieties of the given abelian variety A, such that the torsion orders of the $a'_n s$ are bounded when n varies, then the Zariski closure of $\bigcup_n T_n$ is a finite union of torsion subvarieties, and in fact its $\mathbb{C}$-locus is given by the archimedean closure of $\bigcup_n T_n(\mathbb{C})$ in $A(\mathbb{C})_{an}$. In the case of mixed Shimura varieties of Kuga type, for a special subvariety $M' \subset M$, define the test invariant of M' to be the infinum of the test invariants of the maximal pure special subvarieties of M'. Note that M' is $\mathbf{C}$-special for some $\mathbb{Q}$-torus $\mathbb{C}$ if and only if one (or any) of its maximal pure special subvarieties is $\mathbf{C}$-special, thus the homogeneity of a sequence of special subvarieties $(M_n)_n$ is reduced to that of a sequence of pure special subvarieties. And thus the criterion above also works in the mixed case.

- Finally in Chapter 5, some variants of the Manin-Mumford conjecture are studied, inspired by the known results on the André-Oort-Pink conjecture.

Recall that an abelian S-scheme is a group S-scheme $f : A \to S$ which is proper, smooth, and of connected geometric fibers. Assume for simplicity that S is geometrically integral of characteristic zero. Then the kernels A[N] of raising to the N-th power $[N] : A \to A$ are étale torsion sheaves on S, and by taking inverse limit one obtains a continuous representation of $\pi_1(S,\bar{x})$ on $\mathbb{T}(A)_{\bar{x}}$, called the monodromy representation associated to $A \to S$ at $\bar{x}$, where $\bar{x}$ is a geometric point of S, $\pi_1(S,\bar{x})$ is the fundamental group of S, and $\mathbb{T}(A) = \varprojlim_N A[N]$ is the total Tate module (as an étale sheaf on S). In the remaining part write η for the generic point of S and take $\bar{x}$ to be the algebraic closure of η.

Start with a relative version of the Manin-Mumford conjecture for an abelian S-scheme as below:

(1) First consider the case where the monodromy representation is trivial. In this case each A[N] splits into copies of sections of $A \to S$: all the torsion sections

are defined over S. One can naturally define torsion S-subscheme of A to be an abelian S-subscheme translated by a torsion section. Then the schematic closure of a sequence of torsion S-subschemes in A is again a finite union of torsion subschemes. The proof of this claim is reduced to the generic fiber $A_\eta \to \eta$, which is the known case of the classical Manin-Mumford conjecture.

(2) In general, when the monodromy representation is no longer trivial, even the notion of torsion S-subscheme is in question: a torsion S-subscheme is only well-defined after some étale base change. It is then natural to define special S-subscheme of $A \to S$ to be the image in A of a torsion S′-subscheme B′ in $A' = A \times_S S'$ for some étale covering $S' \to S$, i.e. through some cartesian diagram

$$\begin{tikzcd} A' \arrow[r, "pr"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ S' \arrow[r, "pr"] & S \end{tikzcd}$$

with $B' = A'_1 + a'$ for some abelian S′-subscheme $A'_1 \subset A$ and torsion section a' of $f' : A' \to S'$. Then the schematic closure of a sequence of special S-subschemes in A is again a finite union of special S-subschemes. The key point is that every special S-subscheme is the image of some torsion $\hat{S}$-subscheme under $\hat{A} \to A$ given by the base change to the universal covering $\hat{S} \to S$, which reduces us to the situation treated in (1).

One might step further from these "uniform" variants of the Manin-Mumford conjecture. Define a quasi-special subscheme of the abelian S-scheme A to be a torsion S′-subscheme of the base change $f' : A' \to S'$ given by an inclusion of a closed subscheme $S' \hookrightarrow S$, i.e. via the cartesian diagram

$$\begin{tikzcd} A' \arrow[r, "\subset"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ S' \arrow[r, "\subset"] & S \end{tikzcd}$$

One is then led to a "non-uniform" version of the Manin-Mumford conjecture via the following question:

Being given $(T_n)_n$ a sequence of quasi-special subschemes of an abelian S-scheme $f : A \to S$, what is the minimal condition to be put on $(T_n)_n$ such that the Zariski closure of $\bigcup_n T_n$ is a finite union of quasi-special subschemes of A?

This question generalizes not only the Manin-Mumford conjecture but also the André-Oort-Pink conjecture for mixed Shimura varieties of Kuga type: a mixed Shimura variety of Kuga type M can be extended into an abelian S-scheme for S some pure Shimura variety, special subvarieties of M can always be described as quasi-special subschemes. However this non-uniform version might have been over-generalized from its expected form: there are already counter examples if no condition is imposed on $(T_n)_n$. Nevertheless there is still a naive criterion for the closure of $\bigcup_n T_n$ to be quasi-special. The idea already appears in the studies of the classical Manin-Mumford conjecture: if a subscheme B of an abelian S-scheme A is stabilized by some non-trivial homothety, then it would be quasi-special. This criterion is expected to be of help even in the studies of the André-Oort-Pink conjecture itself.

Notations

For a subset A of some set $\mathcal{A}$, $\mathbf{ch}_A$ always denotes the characteristic function of A as a complex valued function on $\mathcal{A}$.

For S and T two schemes, the set $S(T) = \mathrm{Hom}_{\mathbf{Sch}}(S,T)$ is called the T-locus of S. If moreover $T = \mathrm{Spec}\,R$ is affine, it is equally written as $S(R) = S(T)$ and referred as the R-locus of S.

Write $\mathbb{Q}^{ac}$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Unless explicitly mentioned, a variety is understood to be a reduced separated finite type $\mathbb{Q}^{ac}$-scheme. For Z a variety and L a subring of $\mathbb{Q}^{ac}$, Z_L means an L-model of Z, namely a reduced separated finite type L-scheme such that $Z_L \otimes_L \mathbb{Q}^{ac} \simeq Z$.

$\mathbf{i}$ denotes a fixed square root of -1 in $\mathbb{C}$. The archimedean topology on a real or complex analytic variety is the one deduced from the archimedean metric on a real or complex vector space.

For a field F, write Gal_F for the absolute Galois group of F.

For F a field, algebraic F-groups are denoted in boldface letters, like **G**, **T**, etc. and affine algebraic F-groups are abbreviated as linear F-groups (not necessarily connected for the Zariski topology). Reductive F-groups are understood to be connected. For **G** a linear F-group, write $\mathbf{C_G}$ for the neutral component of its center, and $\mathbf{T_G} = \mathbf{G}/\mathbf{G}^{der}$ its maximal abelian quotient. If moreover **G** is reductive, then the canonical map $\mathbf{G} \xrightarrow{\pi_{ab}} \mathbf{T_G}$ induces an isogeny of F-tori $\mathbf{C_G} \xrightarrow{\pi_{ab}} \mathbf{T_G}$. Normal subgroups are referred to as invariant subgroups so as to avoid possible ambiguities with the normal morphisms of schemes. For an invariant F-subgroup $\mathbf{W} \lhd \mathbf{G}$ the reduction modulo **W** is denoted by $\pi_{\mathbf{W}}$, which often appears in the situation when **W** is the (unipotent) radical of **G**. In case that $\mathbf{W} = \mathbf{Z_G}$ is the center of a reductive F-group **G**, write π_{ad} for the canonical reduction modulo $\mathbf{Z_G}$: $\mathbf{G} \longrightarrow \mathbf{G}^{ad} \cong \mathbf{G}/\mathbf{Z_G}$.

For F a field and **T** an F-torus, $\mathbf{X_T}$ resp. $\mathbf{X_T^\vee}$ is the sheaf of characters resp. of cocharacters of **T**, namely the functor which associates to any F-scheme R the abelian group $\mathrm{Hom}_{R\text{-}\mathbf{Group}}(\mathbf{T}_R, \mathbb{G}_{mR})$ resp. $\mathrm{Hom}_{R\text{-}\mathbf{Group}}(\mathbb{G}_{mR}, \mathbf{T}_R)$. They are locally constant sheaf for the étale topology, because **T** always split over some finite étale extension of the base field F. We also write $X_T = \mathbf{X_T}(\bar{F})$ and $X_T^\vee = \mathbf{X_T^\vee}(\bar{F})$ where $\bar{F}$ is a fixed separable closure of F; they are naturally equipped with an action of the Galois group of F. When F is a subfield of $\mathbb{C}$ we often identify them with the $\mathbb{C}$-locus of the respective sheaves.

For **G** a linear $\mathbb{Q}$-group, the upper scripts and lower scripts $^+$, $_+$, and $^\circ$ follow the usage of P.Deligne: $\mathbf{G}^\circ$ denotes the neutral connected component of **G** for the Zariski topology; $\mathbf{G}(\mathbb{R})^+$ is the neutral connected component of $\mathbf{G}(\mathbb{R})^+$ for the archimedean topology; and $\mathbf{G}(\mathbb{R})_+$ denotes the inverse image of $\mathbf{G}^{ad}(\mathbb{R})^+$ under the projection $\pi_{ad} : \mathbf{G} \to \mathbf{G}^{ad}$, and $\mathbf{G}(\mathbb{Q})_+ := \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{R})_+$. The linear $\mathbb{Q}$-group **G** is said to be compact if $\mathbf{G}(\mathbb{R})$ is compact as a Lie group (for the archimedean topology).

Vector spaces V over a field F are understood to be finite-dimensional, and are often identified with the associated vectorial F-group $\mathbf{V} = \mathrm{Spec}[\mathbf{Sym}\, V^\vee]$ where

$V^\vee$ is the dual of V.

For F/E a finite extension of fields and $\mathbf{G}$ a linear E-group, write $\mathbf{G}^{F/E}$ for the restriction of scalars of $\mathbf{G}_F$ over E, namely the E-group $\mathrm{Res}_{F/E}\mathbf{G}_F$, with a canonical E-homomorphism $\mathrm{Nm}_{F/E}:\mathbf{G}^{F/E}\longrightarrow\mathbf{G}$. This is of course a special case of the Weil restriction functor $\mathrm{Res}:\mathbf{Sch}_{/F}\to\mathbf{Sch}_{/E}$. In case $E=\mathbb{Q}$, write simply $\mathbf{G}^F=\mathbf{G}^{F/\mathbb{Q}}$.

$\mathbb{A}^f=\hat{\mathbb{Q}}=\mathbb{Q}\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}$ is the ring of finite adeles over $\mathbb{Q}$, and $\mathbb{A}=\mathbb{R}\times\mathbb{A}^f$ is the ring of adeles over $\mathbb{Q}$. For $\mathbf{G}$ a reductive $\mathbb{Q}$-group, write $\rho:\tilde{\mathbf{G}}\to\mathbf{G}^{der}$ for the simply connected covering of $\mathbf{G}^{der}$. Then the strong approximation theorem over global fields implies that $\mathbf{G}(\mathbb{Q})\cdot\rho\tilde{\mathbf{G}}(\mathbb{A})$ is a closed subgroup of $\mathbf{G}(\mathbb{A})$ with abelian quotient $\pi(\mathbf{G}):=\mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{Q})\cdot\rho\tilde{\mathbf{G}}(\mathbb{A})$. Put $\bar{\pi}_0\pi(\mathbf{G})$ to be the abelian quotient $\pi_0(\pi(\mathbf{G}))/\pi_0(\mathbf{G}(\mathbb{R})_+)$ with respect to the evident action of $\mathbf{G}(\mathbb{R})$ on $\mathbf{G}(\mathbb{A})$ (through the real component).

In particular, for any number field F, we have the $\mathbb{Q}$-torus $\mathbb{G}_m^F$, and the reciprocity map for F is an isomorphism of topological abelian groups $\mathbf{rec}_F:\mathrm{Gal}_F^{ab}\xrightarrow{\sim}\pi_0(\pi(\mathbb{G}_m^F))$ which associates geometric Frobenii to local uniformizers. We sometimes also denotes by $\mathbf{rec}_F$ the canonical homomorphism $\mathrm{Gal}_F^{ab}\to\bar{\pi}_0\pi(\mathbb{G}_m^F)$. Of course $\bar{\pi}_0\pi(\mathbb{G}_m^E)=\pi_0\pi(\mathbb{G}_m^E)$ when E is a CM field.

For a number field F and $\mathbf{G}$ a reductive $\mathbb{Q}$-group, we have the homomorphism $\mathrm{Nm}_F:\pi(\mathbf{G}^F)\to\pi(\mathbf{G})$ which induces $\bar{\pi}_0\pi(\mathbf{G}^F)\to\bar{\pi}_0\pi(\mathbf{G})$, and for M a $\mathbf{G}(\mathbb{Q}^{ac})$-- conjugacy class of homomorphisms $\mathbf{T}_{\mathbb{Q}^{ac}}\to\mathbf{G}_{\mathbb{Q}^{ac}}$ defined over $\mathbb{Q}$, with $\mathbf{T}\subset\mathbf{G}$ a $\mathbb{Q}$-torus, we have a homomorphism $q_M:\pi(\mathbf{T})\to\pi(\mathbf{G})$ and the induced map for the $\bar{\pi}_0\pi$-quotients.

By Shimura datum is always meant a mixed Shimura datum, usually written in the form $(\mathbf{P},Y)$, $(\mathbf{G},X)$ etc. (as will be clarified later in the preliminaries). For a such $\mathbf{P}$, we write $\mathbf{W}=\mathbf{W}_\mathbf{P}$ for its unipotent radical, $\mathbf{U}=\mathbf{U}_\mathbf{P}$ the weight -2 unipotent $\mathbb{Q}$-subgroup, and $\mathbf{V}=\mathbf{V}_\mathbf{P}=\mathbf{W}/\mathbf{U}$ the weight -1 sub-quotient. $\mathbf{U}$ and $\mathbf{V}$ are vectorial $\mathbb{Q}$-groups. A Shimura variety is understood to be attached to some mixed Shimura datum at finite level, hence a quasi-projective variety. Analytic groups and adelic groups are written in roman letters, like G, $K=\prod K_p$, etc. Given a datum $(\mathbf{P},Y)$ and $K\subset\mathbf{P}(\mathbb{A}^f)$ a compact open subgroup, we have the Shimura variety at level K defined over $\mathbb{Q}^{ac}$: $M_K=M_K(\mathbf{P},Y)$. For a fixed connected component Y^+ of Y (for the archimedean topology), define $\wp=\wp_K$ to be the projection $Y^+\times\mathbf{P}(\mathbb{A}^f)\twoheadrightarrow M_K(\mathbf{P},Y)(\mathbb{C})$, and $\wp=\wp_\Gamma:Y^+\twoheadrightarrow\Gamma\backslash Y^+$ for any discrete subgroup $\Gamma\subset\mathbf{P}(\mathbb{Q})^+$ acting on Y^+ discontinuously.

For a Shimura datum $(\mathbf{P},Y)$ and any number field F containing the reflex field $E=E(\mathbf{P},Y)$, write $M_{K,F}=M_K(\mathbf{P},Y)_F$ for the quasi-canonical F-model of $M_K=M_K(\mathbf{P},Y)$, i.e. the base change of the canonical model of M_K to F, such that for any special subdatum $(\mathbf{T},x)\hookrightarrow(\mathbf{P},Y)$ of reflex field E_x, putting $F_x=E_x\cdot F$, the point $[x,tK]\in M_K(\mathbb{C})$ for any $t\in\mathbf{T}(\mathbb{A}^f)$ is a F_x^{ab}-point, whose Galois conjugates are described by

$$\sigma[x,tK]=[x,\mathbf{rec}_x^F(\sigma)tK],\ \forall\sigma\in\mathrm{Gal}_{F_x}.$$

Here F_x is the composite of F with E_x, and the homomorphism $\mathbf{rec}_x^F$ is the composition of either row of the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Gal}_{F_x} \arrow[r, "\mathbf{rec}_{F_x}"] \arrow[d] & \bar{\pi}_0\pi(\mathbb{G}_m^{F_x}) \arrow[r, "\mu_x"] \arrow[d, "\mathrm{Nm}_{F_x/E_x}"'] & \bar{\pi}_0\pi(T^{F_x}) \arrow[r, "\mathrm{Nm}_{F_x/Q}"] \arrow[d, "\mathrm{Nm}_{F_x/E_x}"'] & \bar{\pi}_0\pi(T) \arrow[d, "="] \\
\mathrm{Gal}_{E_x} \arrow[r, "\mathbf{rec}_{E_x}"] & \bar{\pi}_0\pi(\mathbb{G}_m^{E_x}) \arrow[r, "\mu_x"] & \bar{\pi}_0\pi(T^{E_x}) \arrow[r, "\mathrm{Nm}_{E_x/Q}"] & \pi_{\mathbb{A}}(T)
\end{tikzcd}$$

where by abuse of notations, write $\mathbf{rec}_F$ for the composition

$$\mathrm{Gal}_L \twoheadrightarrow \mathrm{Gal}_L^{ab} \cong \pi_0\pi(\mathbb{G}_m^L) \twoheadrightarrow \bar{\pi}_0\pi(\mathbb{G}_m^L)$$

L being either E_x or F_x.

Chapter 1

Preliminaries

1.1 Shimura data and Shimura varieties

We denote by $\mathbb{S}$ the Deligne torus, namely the $\mathbb{R}$-group $\mathbb{G}_{\mathrm{m}}^{\mathbb{C}/\mathbb{R}}$, and we write w for the weight homomorphism $w : \mathbb{G}_{\mathrm{m}\mathbb{R}} \to \mathbb{S}$, $t \mapsto t^{-1}$, and μ for the Hodge homomorphism $\mu : \mathbb{G}_{\mathrm{m}\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $\mu(z) = (z, 1)$ with respect to the canonical isomorphism of $\mathbb{C}$-tori $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{\mathrm{m}\mathbb{C}} \times \mathbb{G}_{\mathrm{m}\mathbb{C}}$.

Definition 1.1.1. Variation of polarizable Hodge structures (cf. [Deligne3] 2.1 and 2.3;[Pink0] Chap.1, 1.1):

(1) A homogeneous rational Hodge structure is a pair (V, ρ) where V is a finite dimensional $\mathbb{Q}$-vector space and ρ is a homomorphism of $\mathbb{R}$-groups $\mathbb{S} \xrightarrow{\rho} \mathbf{GL}_{\mathbb{R}}(\mathrm{V}_{\mathbb{R}})$ such that $\rho \circ w$ is the central character $t \mapsto t^n \mathbf{id}_{\mathrm{V}}$ for some $n \in \mathbb{Z}$; n is referred to as the weight of (V, ρ). Equivalently this is characterized by a bi-grading $\mathrm{V}_{\mathbb{C}} = \bigoplus_{p \in \mathbb{Z}} \mathrm{V}^{p,q}$ of $\mathbb{C}$-vector spaces such that $\mathrm{V}^{p,q} \neq 0$ implies $p + q = n$ and that $c(\mathrm{V}^{p,q}) = \mathrm{V}^{q,p}$, where c denotes the complex conjugation on $\mathrm{V}_{\mathbb{C}}$: $v \otimes z \mapsto v \otimes \bar{z}$, $\forall v \in \mathrm{V}$, $z \in \mathbb{C}$. $\mathrm{V}^{p,q}$ is the subspace of $\mathrm{V}_{\mathbb{C}}$ where $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$ acts through the scalar multiplication $z_1^{-p} z_2^{-q}$. The set $\{(p, q) | \mathrm{V}^{p,q} \neq 0\}$ is called the type of (V, ρ).

For $m \in \mathbb{Z}$, we have the m-th Tate twist $\mathbb{Q}(m)$: this is the Hodge structure $(\mathbb{Q}(2\pi\mathbf{i})^m, \tau_m)$ of type $(-m, -m)$ and weight $2m$, where $\tau_m : \mathbb{S} \to \mathbb{G}_{\mathrm{m}\mathbb{R}} = \mathbf{GL}_{\mathbb{R}}(\mathbb{R})$, $z \mapsto (z\bar{z})^m$. For a rational Hodge structure (V, ρ), its m-th Tate twist is $\mathrm{V}(m) = (\mathrm{V} \otimes_{\mathbb{Q}} \mathbb{Q}(m), \rho \otimes \tau_m)$ with τ_m acting through the center of $\mathbf{GL}_{\mathbb{R}}(\mathrm{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}(m))$.

The set of homogeneous rational Hodge structures is endowed with natural operations like the tensor product and the internal Hom. When we allow further non-homogeneous objects as finite direct sum of homogeneous ones, we get a $\mathbb{Q}$-linear neutral Tannakian category with neutral element $\mathbb{Q}(0)$, called the category of pure rational Hodge structures, denoted as $\mathbf{HS}_{\mathbb{Q}}$.

For a given rational Hodge structure (V, ρ) of weight n, the dual representation $(\mathrm{V}^\vee, \rho^\vee)$ is canonically a rational Hodge structure of weight $-n$.

A rational Hodge structure (V, ρ) of weight n is polarizable if it admits a polarization, namely a homomorphism of Hodge structures $\langle\ ,\ \rangle : V \otimes V \to \mathbb{Q}(n)$ such that

$$x \otimes y \mapsto (2\pi \mathbf{i})^{-n} \langle x, \rho(\mathbf{i}) y \rangle$$

defines a definite symmetric bi-linear form on $V_{\mathbb{R}}$.

The (rational) Mumford-Tate group of (V, ρ), denoted by $\mathbf{MT}(\rho)$, is the smallest $\mathbb{Q}$-subgroup of $\mathbf{GL}_{\mathbb{Q}}(V)$ whose real locus contains the image of ρ, or equivalently, whose complex locus contains the image of $\rho \circ \mu$. The Tannakian subcategory of $\mathbf{HS}_{\mathbb{Q}}$ generated by (V, ρ) is canonically isomorphic to $\mathbf{Rep}(\mathbf{MT}(\rho)/\mathbb{Q})$. Note that the Mumford-Tate group of a polarizable rational Hodge structure is always a reductive $\mathbb{Q}$-group.

Actually we could have started with other subring of $\mathbb{R}$ instead of $\mathbb{Q} \subset \mathbb{R}$. For example, by replacing $\mathbb{Q}$-vector spaces by free $\mathbb{Z}$-modules of finite type, we get the notion of polarizable integral Hodge structure. In this case $\mathbb{Z}(1)$ is exactly the kernel of $\exp : \mathbb{C} \to \mathbb{C}^{\times}$.

(2) A rational mixed Hodge structure consists of a pair (V, ρ) where V is a finite dimensional $\mathbb{Q}$-vector space and $\rho : \mathbb{S}_{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ a homomorphism of $\mathbb{C}$-groups such that $\rho \circ w$ is defined over $\mathbb{Q}$ and that the weight filtration $W_\bullet = (W_m)_m$, namely the ascending filtration associated to $\rho \circ w$, gives rise to a rational Hodge structure of weight m on $\mathbf{gr}_m W_\bullet = W_m / W_{m-1}$ for all $m \in \mathbb{Z}$. Equivalently this is characterized by a bi-grading $V_{\mathbb{C}} = \oplus_{p,q \in \mathbb{Z}} V^{p,q}$ such that $W_m = \oplus_{p+q \leq m} V^{p,q}$ is defined over $\mathbb{Q}$ with $c(V^{p,q}) \subset V^{q,p} + W_{p+q-1}$ for all $p, q \in \mathbb{Z}$.

The Hodge filtration of (V, ρ) is the descending filtration associated to $\rho \circ \mu$, namely $F^\bullet = (F^n)_n$ with $F^n = \oplus_{p \geq n} V^{p,q}$. Actually (V, ρ) can be recovered from the triple $(V, W_\bullet, F^\bullet)$, just as ρ is determined by $\rho \circ w$ and $\rho \circ \mu$. Note that the Hodge filtration is in general not defined over $\mathbb{Q}$: in fact it is rarely stable under the complex conjugation.

The (rational) Mumford-Tate group of a rational mixed Hodge structure (V, ρ) is the smallest $\mathbb{Q}$-subgroup of $\mathbf{GL}_{\mathbb{Q}}(V)$ whose complex locus contains the image of ρ. Similar to the pure case, the category of rational Hodge structures, denoted as $\mathbf{MHS}_{\mathbb{Q}}$, is neutral Tannakian, and the Tannakian subcategory generated by (V, ρ) is canonically isomorphic to $\mathbf{Rep}(\mathbf{MT}(\rho)/\mathbb{Q})$.

A rational mixed Hodge structure is polarizable if so it is with each graded quotient $\mathbf{gr}_m W_\bullet$. The type of (V, ρ) is the finite set $\{(p, q) | V^{p,q} \neq 0\}$. Note that the notion of polarization is only defined via the successive quotients, hence even if the mixed Hodge structure is polarizable, the Mumford-Tate group might fail to be reductive.

In the same way we could replace $\mathbb{Q}$ by other subrings of $\mathbb{R}$, e.g. we get the notion of polarizable integral mixed Hodge structures when replacing $\mathbb{Q}$-vector spaces by free $\mathbb{Z}$-modules of finite type.

(3) For S a complex analytic variety, a variation of mixed rational Hodge structures on S is a local system (i.e. locally constant sheaf) of $\mathbb{Q}$-vector spaces $\mathcal{V}$ together with a descending filtration $F^\bullet$ and an ascending one $W_\bullet$ of coherent $\mathcal{O}_S$-submodules of $\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_S$, such that for any $s \in S$, the fiber $(\mathcal{V}_s, F(s), W(s))$ is a rational mixed Hodge structure, and $d(F^p) \subset F^{p-1} \otimes \Omega^1_S$ for any $p \in \mathbb{Z}$, where $d : \mathcal{V} \otimes_{\mathbb{Q}_S} \mathcal{O}_S \to \mathcal{V} \otimes_{\mathbb{Q}_S} \Omega^1_S$ is induced from the classical differential $\mathcal{O}_S \to \Omega^1_S$. The variation is homogeneous if the weight filtration is trivial, which will also be referred to as a variation of pure rational Hodge structure (of some weight).

Let $(\mathcal{V}, F^\bullet, W_\bullet)$ be a variation of mixed rational hodge structures over S. It is said to be polarizable if each graded quotient $\mathbf{gr}_n = W_n/W_{n-1}$ admits a polarization as a variation of rational Hodge structures, namely a homomorphism of local systems $\mathbf{gr}_n \otimes_{\mathbb{Q}_S} \mathbf{gr}_n \longrightarrow \mathbb{Q}(n)_S$ which induces a polarization on each fiber of $\mathbf{gr}_n$, $\forall n \in \mathbb{Z}$.

The notion of variation of (polarizable) integral mixed Hodge structures is clear from the context.

Remark 1.1.2. Note that we have defined the Mumford-Tate group of (V, ρ) to be some $\mathbb{Q}$-subgroup of $\mathbf{GL}_{\mathbb{Q}}(V)$, which is the automorphism group of the neutral Tannakian category generated by (V, ρ). Sometimes it is necessary to study further the automorphism group of the Tannakian category generated by (V, ρ) and $\mathbb{Q}(1)$, which is also referred to as the Mumford-Tate group of (V, ρ), realized as a $\mathbb{Q}$-subgroup of $\mathbf{GL}_{\mathbb{Q}}(V) \times \mathbb{G}_m$. This latter one won't be considered in the present treatment.

Definition 1.1.3. Shimura data and Shimura varieties, pure and mixed (cf.[D-2] 2.1.1, [Pink0] Chap.2 2.1 and Chap.3 3.1):

(1) A pure Shimura datum consists of a pair $(\mathbf{G}, X)$ with $\mathbf{G}$ a reductive $\mathbb{Q}$-group with $\mathbf{G}^{ad}$ having no compact $\mathbb{Q}$-factors, X a $\mathbf{G}(\mathbb{R})$-orbit in $\mathfrak{X}(\mathbf{G}) = \mathrm{Hom}_{\mathbf{Gr}/\mathbb{R}}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$, such that for some (or equivalently, every) $x \in X$:

(P1) $\mathrm{Ad} \circ x : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \longrightarrow \mathbf{GL}_{\mathbb{R}}(\mathbf{Lie}\mathbf{G}_{\mathbb{R}})$ induces on $\mathbf{Lie}\mathbf{G}$ a pure rational Hodge structure of type $\{(-1,1), (0,0), (1,-1)\}$;

(P2) $\mathbf{Int}(x(\mathbf{i}))$ induces on $\mathbf{G}^{ad}_{\mathbb{R}}$ a Cartan involution.

Note that from the definition X is a Hermitian symmetric domain of noncompact type, and each of its connected components is homogeneous under $\mathbf{G}(\mathbb{R})^+$, of stabilizer $\mathbf{G}(\mathbb{R})_+$.

(2) A mixed Shimura datum consists of a tuple $(\mathbf{P}, \mathbf{U}, Y)$ where $\mathbf{P}$ is a connected linear $\mathbb{Q}$-group of unipotent radical $\mathbf{W}$, $\mathbf{U}$ an invariant $\mathbb{Q}$-subgroup of $\mathbf{P}$ contained in $\mathbf{W}$, Y a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-orbit in $\mathfrak{Y}(\mathbf{P}) = \mathrm{Hom}_{\mathbb{C}\text{-}\mathbf{Group}}(\mathbb{S}_{\mathbb{C}}, \mathbf{P}_{\mathbb{C}})$ such that for some (or every) $y \in Y$:

(M1) $\mathrm{Ad} \circ y : \mathbb{S}_{\mathbb{C}} \to \mathbf{P}_{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathbf{Lie}\mathbf{P}_{\mathbb{C}})$ induces on $\mathbf{Lie}\mathbf{P}$ a rational mixed Hodge structure whose Hodge types is contained in the set

$$\{(-1,-1), (-1,0), (0,-1), (-1,1), (0,0), (1,-1)\}$$

with rational weight filtration $W_{-2} = \mathbf{Lie\,U}$, $W_{-1} = \mathbf{Lie\,W}$, $W_0 = \mathbf{Lie\,P}$;

(M2) $\pi_{\mathbf{U}} \circ y : \mathbb{S}_{\mathbb{C}} \to \mathbf{P}_{\mathbb{C}} \to (\mathbf{P}/\mathbf{U})_{\mathbb{C}}$ is defined over $\mathbb{R}$, $\pi_{\mathbf{U}}$ being the reduction modulo $\mathbf{U}$;

(M3) Write $\mathbf{G} = \mathbf{P}/\mathbf{W}$, then $\pi_{\mathbf{W}} \circ \rho : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ satisfies the conditions defining pure Shimura data, namely the conjugation by $\pi_{\mathrm{ad}} \circ \pi_{\mathbf{W}} \circ y(\mathbf{i})$ is a Cartan involution on $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$, and that $\mathbf{G}^{\mathrm{ad}}$ has no compact $\mathbb{Q}$-factors;

(M4) Write $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ to be any Levi decomposition of $\mathbf{P}$, then the center of $\mathbf{G}$ acts on $\mathbf{W}$ through a $\mathbb{Q}$-torus isogeneous to a $\mathbb{Q}$-torus $\mathbf{C} \times \mathbb{G}_{\mathrm{m}}^r$ for some compact $\mathbb{Q}$-torus $\mathbf{C}$.

It is clear that mixed Shimura data generalize the notion of pure Shimura data, and in what follows we simply refer to the mixed ones as Shimura data. We often write $(\mathbf{P}, \mathrm{Y})$ for a Shimura datum, and put $\mathbf{U} = \mathbf{U_P}$ resp. $\mathbf{V} = \mathbf{V_P}$ to be the unipotent part of weight -2 resp. the weight -1 subquotient of $\mathbf{P}$. $(\mathbf{P}, \mathrm{Y})$ is pure if and only if $\mathbf{U_P} = 0 = \mathbf{V_P}$.

(3) For $(\mathbf{P}, \mathrm{Y})$ a Shimura datum, K a compact open subgroup of $\mathbf{P}(\mathbb{A}^{\mathrm{f}})$, the complex Shimura variety associated to $(\mathbf{P}, \mathrm{Y})$ at level K is the complex analytic variety

$$\mathrm{M_K}(\mathbf{P}, \mathrm{Y})_{\mathbb{C}} = (\mathbf{P}(\mathbb{Q}) \backslash [\mathrm{Y} \times \mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}])^{\mathrm{an}}$$

and the complex Shimura scheme associated to $(\mathbf{P}, \mathrm{Y})$ is the projective limit

$$\mathrm{M}(\mathbf{P}, \mathrm{Y})_{\mathbb{C}} = \varprojlim_{\mathrm{K}} \mathrm{M_K}(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$$

with K running through the compact open subgroups of $\mathbf{P}(\mathbb{A}^{\mathrm{f}})$.

It has been proved, essentially after a reduction to the pure case which relies on the Baily-Borel compactification of locally symmetric Hermitian domains (cf. [BB], [Bo-1], [Bo-2], [D2], [Mo]), that each finite level object $\mathrm{M_K}(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$ is a quasi-projective algebraic $\mathbb{C}$-variety, and is defined over $\mathbb{Q}^{\mathrm{ac}}$, denoted as $\mathrm{M_K}(\mathbf{P}, \mathrm{Y})$. Over $\mathbb{Q}^{\mathrm{ac}}$ also descend the transition maps in the projective limit, hence we get a pro-scheme over $\mathbb{Q}^{\mathrm{ac}}$, written as $\mathrm{M}(\mathbf{P}, \mathrm{Y})$.

Note that the existence of canonical model claimed in [Milne-0] was not established correctly, as is kindly pointed out by the referee. The reader is referred to [Mo] Section 2 for details.

Example 1.1.4 (mixed Shimura varieties of Kuga type). Let $(\mathbf{G}, \mathrm{X})$ be a pure Shimura datum, and $\rho : \mathbf{G} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{V})$ an algebraic representation of $\mathbf{G}$ on a finite dimensional $\mathbb{Q}$-vector space $\mathbf{V}$ such that every $x \in \mathrm{X}$ induces on $\mathbf{V}$ through $\rho \circ x$ a pure Hodge structure of type $\{(-1,0),(0,-1)\}$ (namely a complex structure). Then $(\mathbf{P}, \mathrm{Y}) := (\mathbf{V} \rtimes_\rho \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes \mathrm{X})$ is a mixed Shimura datum, without unipotent part of weight 2. Here we write $\mathbf{V}(\mathbb{R}) \rtimes \mathrm{X}$ to indicate that the complex structure on the product $\mathbf{V}(\mathbb{R}) \times \mathrm{X}$ is such that on the fiber over $x \in \mathrm{X}$ of the canonical projection $\pi : \mathrm{Y} \to \mathrm{X}$ is induced the complex structure defined by $\rho \circ x$. Note that π comes from the canonical projection of Shimura data $(\mathbf{P}, \mathrm{Y}) \to (\mathbf{G}, \mathrm{X})$, and it admits a section defined by the inclusion of $\mathbf{G}$ into $\mathbf{P}$: $(\mathbf{G}, \mathrm{X}) \hookrightarrow (\mathbf{P}, \mathrm{Y})$.

We refer to data of this form as *mixed Shimura data of Kuga type*, and the associated mixed Shimura varieties at finite levels are called *mixed Shimura varieties of Kuga type*. Very often for mixed Shimura varieties of Kuga type we require the finite level K to be of the for $K_V \rtimes K_G$, with compact open subgroups $K_V \subset \mathbf{V}(\mathbb{A}^f)$ and $K_G \subset \mathbf{G}(\mathbb{A}^f)$ such that K_G stabilizes K_V. In this way the morphism of Shimura varieties $\pi : M = M_K(\mathbf{P}, Y) \to S = M_{K_G}(\mathbf{G}, X)$ admits a section $M_{K_G}(\mathbf{G}, X) \hookrightarrow M_K(\mathbf{P}, Y)$, induced by the section $(\mathbf{G}, X) \hookrightarrow (\mathbf{P}, Y)$. We can check easily that $\pi : M \to S$ is an abelian S-scheme, i.e. a proper group scheme over S with connected smooth geometric fibers.

In [Ku], Kuga studied certain fiber varieties over symmetric domains whose fibers are abelian varieties. Our notion of mixed Shimura varieties of Kuga type naturally fits into this setting. Note that general mixed Shimura varieties, such as those encountered in the theory of toroidal compactification of pure Shimura varieties, are not necessarily of Kuga type: the unipotent radical of the defining $\mathbb{Q}$-group $\mathbf{P}$ may contain some central part of weight 2, and it results that a general mixed Shimura variety can be realized as a $\mathbf{T}$-torsor over some mixed Shimura variety of Kuga type, $\mathbf{T}$ being a torus defined over a number field.

Proposition 1.1.5 (Universal property of the domain Y, cf.[Pink0] Chap.1). *Let $(\mathbf{P}, Y)$ be a mixed Shimura datum. Then for any algebraic representation $\mathbf{P} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{V})$, the constant sheaf $\mathcal{V}$ on Y of fiber $\mathbf{V}$ admits a structure of a variation of polarized rational mixed Hodge structures on Y.*

Let $\Gamma \subset \mathbf{P}(\mathbb{R})$ be a neat discrete subgroup, then the quotient $\Gamma\backslash(\mathbf{V} \times Y)$ gives rise to a variation of rational polarizable Hodge structures on $\Gamma\backslash Y$ of fiber $\mathbf{V}$. If moreover Γ stabilizes a lattice $\mathbf{V}_{\mathbb{Z}} \subset \mathbf{V}$, then $\Gamma\backslash(\mathbf{V}_{\mathbb{Z}} \times Y)$ defines a variation of polarized integral mixed Hodge structure $\mathcal{V}_{\mathbb{Z}}$ on $\Gamma\backslash Y$ of fiber $\mathbf{V}_{\mathbb{Z}}$.

More details will be mentioned later on canonical models of Shimura varieties (or schemes).

Definition 1.1.6. (cf.[Pink-0] Chap.2, 2.1 and Chap.3 3.1) We define the morphism between Shimura data and between Shimura varieties in the evident way:

(1) A morphism of Shimura data $(\mathbf{P}_1, Y_1) \to (\mathbf{P}_2, Y_2)$ is a pair (f, f_*) where $f : \mathbf{P}_1 \to \mathbf{P}_2$ is a homomorphism of $\mathbb{Q}$-groups and f_* is the induced map $\mathfrak{Y}(\mathbf{P}_1) \to \mathfrak{Y}(\mathbf{P}_2)$ $y \to y \circ f$, such that $f_*(Y_1) \subset Y_2$. Note that this require f to respect the mixed Hodge structures: at the level of Lie algebras $\mathbf{Lie} f$ is a morphism of rational mixed Hodge structures. In particular it sends $W_i(\mathbf{Lie}\,\mathbf{P}_1)$ into $W_i(\mathbf{Lie}\,\mathbf{P}_2)$, $i \in \{-2, -1, 0\}$.

(2) Let $(\mathbf{P}_1, Y_1) \xrightarrow{(f, f_*)} (\mathbf{P}_2, Y_2)$ be a morphism of Shimura data, then we have the morphism of Shimura schemes $f : M(\mathbf{P}_1, Y_1) \to M(\mathbf{P}_2, Y_2)$. Let $K_i \subset \mathbf{P}_i(\mathbb{A}^f)$ $(i = 1, 2)$ be compact open subgroups such that $f(K_1) \subset K_2$. Then we have the morphism of Shimura varieties at finite levels $f : M_{K_1}(\mathbf{P}_1, Y_1) \to M_{K_2}(\mathbf{P}_2, Y_2)$.

Moreover, if $g \in \mathbf{P}_2(\mathbb{A}^f)$ is given, then we can translate f by g, namely $g * f : \mathrm{M}(\mathbf{P}_1, \mathrm{Y}_1) \to \mathrm{M}(\mathbf{P}_2, \mathrm{Y}_2)$, $[x, a] \mapsto [f(x), f(a)g]$. At finite level we have $g * f : \mathrm{M}_{\mathrm{K}_1}(\mathbf{P}_1, \mathrm{Y}_1) \to \mathrm{M}_{\mathrm{K}_2}(\mathbf{P}_2, \mathrm{Y}_2)$, $[x, a\mathrm{K}_1] \mapsto [f(x), f(a)g\mathrm{K}_2]$, provided that $f(\mathrm{K}_1) \subset g\mathrm{K}_2 g^{-1}$.

(3) In particular a subdatum of $(\mathbf{P}, \mathrm{Y})$ is a morphism $(\mathbf{P}_1, \mathrm{Y}_1) \xrightarrow{(f, f_*)} (\mathbf{P}, \mathrm{Y})$ with $f : \mathbf{P}_1 \hookrightarrow \mathbf{P}$ an inclusion of subgroup and f_* the corresponding inclusion $\mathrm{Y}_1 \hookrightarrow \mathrm{Y}$. In this case, one can easily verify that $\mathrm{W}_i(\mathbf{Lie}\mathbf{P}_1) = \mathbf{Lie}\mathbf{P}_1 \cap \mathrm{W}_i(\mathbf{Lie}\mathbf{P})$, $i \in \{-2, -1, 0\}$. We'll come back to subdata and special subvarieties in Section 1.3 (cf.Definition 1.3.3).

N.B. We have been rather vague about the fields of definition for the morphisms described above. This will be clarified when the notion of canonical model is introduced in Section 1.2.

Remark 1.1.7. Our definitions differ slightly from that of R.Pink (cf.[Pink-0] Chap.2), and is modeled on the pure case studied by P.Deligne (cf.[D-2] 2.1.1): the data $(\mathbf{P}, \mathbf{U}, \mathrm{Y})$ in this thesis consists of a $\mathbb{Q}$-group $\mathbf{P}$ and $\mathbf{U} \lhd \mathbf{P}$ together with a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-orbit in $\mathfrak{Y}(\mathrm{P})$; on the other hand the definition of R.Pink can be equivalently formulated as triples $(\mathbf{P}, \mathbf{U}, h : \mathcal{Y} \to \mathrm{Y})$ where $(\mathbf{P}, \mathbf{U}, \mathrm{Y})$ is a Shimura datum in the sense of Definition.1.1.3, $\mathcal{Y}$ a homogeneous space under $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$ and h is a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-equivariant finite covering. In this thesis, Shimura data and Shimura varieties is understood to be in the sense of P.Delinge as in Definition.1.1.3, and the counterparts defined in [Pink-0] Chap.2 are called Shimura data resp. Shimura varieties in the sense of R.Pink. The category of Shimura data in the sense of P.Deligne is a full subcategory of that of R.Pink.

Definition 1.1.8. Among morphisms of Shimura data/varieties we single out the following notions:

(1) (Fibration over a pure section:) For $(\mathbf{P}, \mathrm{Y})$ a Shimura datum, a Levi decomposition $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ gives rise to a pure section $(\mathbf{G}, \mathrm{X}) \underset{\pi}{\overset{i}{\rightleftarrows}} (\mathbf{P}, \mathrm{Y})$ where $\mathrm{X} = \mathbf{G}(\mathbb{R})x$ for some $x \in \mathrm{Y}$ with $x(\mathbb{S}) \subset \mathbf{G}_{\mathbb{R}}$, i induced by the inclusion $i : \mathbf{G} \hookrightarrow \mathbf{P}$, and π induced by $\pi_{\mathbf{W}}$ the reduction modulo $\mathbf{W}$ the unipotent radical of $\mathbf{P}$. Note that pure sections do exist: start with an arbitrary $y \in \mathrm{Y}$, then $x = (i \circ \pi)(y)$ is in Y and maps $\mathbb{S}_{\mathbb{R}}$ into $\mathbf{G}_{\mathbb{R}}$; it remains to take $\mathrm{X} = \mathbf{G}(\mathbb{R})x$.

Let $(\mathbf{G}, \mathrm{X}) \subset (\mathbf{P}, \mathrm{Y})$ be a pure section, $\mathrm{K}_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}^f)$ and $\mathrm{K}_{\mathbf{W}} \subset \mathbf{W}(\mathbb{A}^f)$ be compact open subgroups, and put $\mathrm{K} = \mathrm{K}_{\mathbf{W}} \rtimes \mathrm{K}_{\mathbf{G}} \subset \mathbf{P}(\mathbb{A}^f)$. Then we have morphisms of mixed Shimura varieties

$$\mathrm{M}_{\mathrm{K}_{\mathbf{G}}}(\mathbf{G}, \mathrm{X}) \underset{\pi}{\overset{i}{\rightleftarrows}} \mathrm{M}_{\mathrm{K}}(\mathbf{P}, \mathrm{Y})$$

with π surjective and i injective. The fibers of π are geometrically integral varieties that are torsors over some abelian varieties under some tori, referred to as torus bundles over abelian varieties in what follows.

(2) (Geometrically connected components:) For the finite level Shimura variety $\mathrm{M}_{\mathrm{K}}(\mathbf{P}, \mathrm{Y})$, write $\wp = \wp_{\mathrm{K}}$ for the projection $\mathrm{Y} \times \mathbf{P}(\mathbb{A}^f) \twoheadrightarrow \mathrm{M}_{\mathrm{K}}(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$, $(x, a) \mapsto$

$[x, aK]$. Fix a connected component Y^+ of Y, $\mathbf{P}(\mathbb{Q})_+$ the subgroup of $\mathbf{P}(\mathbb{Q})$ stabilizing Y^+, and a finite set $\mathfrak{R}_K^{\mathbf{P}}$ representing the double quotient $\mathbf{P}(\mathbb{Q})_+\backslash\mathbf{P}(\mathbb{A}^f)/K$, we have the isomorphism

$$\gamma = \gamma_K^{\mathbf{P}} : M_K(\mathbf{P}, Y)(\mathbb{C}) \longrightarrow \coprod_{g\in\mathfrak{R}_K^{\mathbf{P}}} \Gamma_K(g)\backslash Y^+, \ \ [x, gK] \mapsto \Gamma_K(g)x$$

with $\Gamma_K(g) = \mathbf{P}(\mathbb{Q})_+ \cap gKg^{-1}$ which induces $\gamma\circ\wp(Y^+ \times gK) = \Gamma_K(g)\backslash Y^+$. The images $\wp(Y^+ \times gK)$'s are referred to as (complex) connected Shimura varieties, and are often identified with $\Gamma_K(g)\backslash Y^+$. The set of geometrically connected components of $M_K(\mathbf{P}, Y)$ is thus identified with $\mathfrak{R}_K^{\mathbf{P}}$.

Passing to the projective limit indexed by the compact open subgroups of $\mathbf{P}(\mathbb{A}^f)$, we get the connected Shimura scheme, whose complex locus is $\varprojlim_\Gamma \Gamma\backslash Y^+$ with Γ running through congruence subgroups of $\mathbf{P}(\mathbb{Q})_+$. Any geometrically connected component of $M(\mathbf{P}, Y)$ is of this form.

We'll need the following result on pure sections.

Lemma 1.1.9. *Let $f : (\mathbf{P}_1, Y_1) \to (\mathbf{P}_2, Y_2)$ be a morphism of Shimura data, defined by a homomorphism of $\mathbb{Q}$-group $f : \mathbf{P}_1 \to \mathbf{P}_2$ together with the map $f_* : Y_1 \to Y_2$ induced from $f_* : \mathfrak{Y}(\mathbf{P}_1) \to \mathfrak{Y}(\mathbf{P}_2)$. Then*

(1) If $(\mathbf{G}_1, X_1)$ is a pure section of $(\mathbf{P}_1, Y_1)$, then $(\mathbf{P}_2, Y_2)$ admits a pure section $(\mathbf{G}_2, X_2)$ such that $(f(\mathbf{G}_1), f_*(X_1))$ is a subdatum of $(\mathbf{G}_2, X_2)$

Conversely, if $(\mathbf{G}_2, X_2)$ is a pure section of $(\mathbf{P}_2, Y_2)$, then $(\mathbf{P}_1, Y_1)$ has a pure section $(\mathbf{G}_1, X_1)$ such that for some $w \in \mathbf{W}_2(\mathbb{Q})$ $(f(\mathbf{G}_1), f_*(X_1))$ is a subdatum of $(w\mathbf{G}_2w^{-1}, w \rtimes X_2)$.

(2) In particular, consider the case where f is an inclusion of subdatum. If $(\mathbf{G}_2, X_2)$ is a pure section of $(\mathbf{P}_2, X_2)$, then any pure section of $(\mathbf{P}_1, Y_1)$ can be written in the form $(w\mathbf{G}_1w^{-1}, w \rtimes X_1)$ for some $w \in \mathbf{W}_2(\mathbb{Q})$ and $(\mathbf{G}_1, X_1)$ some pure subdatum of $(\mathbf{G}_2, X_2)$; or equivalently, if $(\mathbf{G}_1, X_1)$ is a pure section of $(\mathbf{P}_1, Y_1)$, and $(\mathbf{G}_2, X_2)$ a pure section of $(\mathbf{P}_2, Y_2)$, then there exists $w \in \mathbf{W}_2(\mathbb{Q})$ such that $(\mathbf{G}_1, X_1) \subset (w\mathbf{G}_2w^{-1}, w \rtimes X_2)$.

Proof. Recall (cf.Definition 1.1.6) that as a morphism of Shimura data, f respects the mixed Hodge structure on the Lie algebras, i.e. $\mathbf{Lie}(f) : \mathbf{Lie}\,\mathbf{P}_1 \to \mathbf{Lie}\,\mathbf{P}_2$ is a morphism of mixed Hodge structures. In particular it preserves the weight filtration, thus $f(\mathbf{W}_1) \subset \mathbf{W}_2$ and $f(\mathbf{U}_1) \subset \mathbf{U}_2$.

(1) $f(\mathbf{W}_1) \subset \mathbf{W}_2$ implies a commutative diagram

$$\begin{array}{ccc} (\mathbf{P}_1, Y_1) & \xrightarrow{f} & (\mathbf{P}_2, Y_2) \\ \downarrow{\scriptstyle \pi_{\mathbf{W}_1}} & & \downarrow{\scriptstyle \pi_{\mathbf{W}_2}} \\ (\mathbf{G}_1, X_1) & \xrightarrow{\bar{f}} & (\mathbf{G}_2, X_2) \end{array}$$

A Levi $\mathbb{Q}$-subgroup of $\mathbf{P}_2$ is a maximal reductive $\mathbb{Q}$-group of $\mathbf{P}_2$; they are conjugate under $\mathbf{W}_2(\mathbb{Q})$. It is clear that $f(\mathbf{G}_1)$ is a reductive $\mathbb{Q}$-subgroup of $\mathbf{P}_2$, and it is contained in some maximal one, i.e. contained in some Levi $\mathbb{Q}$-subgroup $\mathbf{G}_2$. Put $X_2 = \mathbf{G}_2(\mathbb{R})x_2$ where $x_2 \in f_*(X_1)$. Then $(\mathbf{G}_2, X_2)$ is a pure section of $(\mathbf{P}_2, Y_2)$.

The converse is proved in the same way: let $(\mathbf{G}_2', X_2')$ be a pure section of $(\mathbf{P}_2, Y_2)$, which contains the pure subdatum $(f(\mathbf{G}_1), f(X_1))$. Then any given pure section $(\mathbf{G}_2, X_2)$ of $(\mathbf{P}_2, Y_2)$ is conjugated to $(\mathbf{G}_2', X_2')$ by some $w \in \mathbf{W}_2(\mathbb{Q})$.

(2) For a linear $\mathbb{Q}$-group $\mathbf{P}$ with a Levi-decomposition $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$, all its Levi $\mathbb{Q}$-subgroups are of the form $w\mathbf{G}w^{-1}$ for $w \in \mathbf{W}(\mathbb{Q})$. Hence the conclusion. ■

We would also like to remark that, if we are given an inclusion of subdatum $(\mathbf{P}', Y') \subset (\mathbf{P}, Y)$ where $(\mathbf{P}, Y)$ admits a pure section $(\mathbf{G}, X)$, then $\pi_{\mathbf{W}}(\mathbf{P}', Y') = (\mathbf{G}', X')$ is a subdatum of $(\mathbf{G}, X)$, and a pure section of $(\mathbf{P}', Y')$ is of the form $(w\mathbf{G}'w^{-1}, w \rtimes X')$ with $w \in \mathbf{W}(\mathbb{Q})$ depending on $\mathbf{W}' = \mathbf{W} \cap \mathbf{P}'$ and $\mathbf{G}' = \pi_{\mathbf{W}}(\mathbf{P}')$.

1.1.10 Fibers over a pure section

Consider a Shimura datum $(\mathbf{P}, Y)$ with a pure section $(\mathbf{G}, X)$, and assume that $0 \neq \mathbf{U} \subsetneq \mathbf{W}$ corresponding to the weight filtration of degree not exceeding -1. By calculating the Hodge types we see that

$$0 \to \mathbf{U} \to \mathbf{W} \to \mathbf{V} \to 0$$

is a central extension of vectorial $\mathbb{Q}$-groups, determined by a unique alternating bi-linear map $\psi : \mathbf{V} \times \mathbf{V} \to \mathbf{U}$ in the sense that:

(1) Identify $\mathbf{W}$ with the $\mathbb{Q}$-variety $\mathbf{U} \times \mathbf{V}$, then the group law of $\mathbf{W}$ becomes

$$(u_1, v_1)(u_2, v_2) = (u_1 + u_2 + \psi(v_1, v_2), v_1 + v_2),$$

with neutral element $(0,0)$ and inverse map $(u, v) \mapsto (-u, -v)$. In particular $(u, v)^n = (nu, nv)$, $\forall n \in \mathbb{Z}$.

(2) $\mathbf{Lie W}$ is the central extension of $\mathbf{Lie V}$ by $\mathbf{Lie U}$ as Lie algebra, whose Lie bracket is

$$[(u_1, v_1), (u_2, v_2)] = (2\psi(v_1, v_2), 0).$$

We also get the formula $(u_1, v_1)(u_2, v_2)(u_1, v_1)^{-1} = (2\psi(v_1, v_2) + u_2, v_2)$.

Then the projection $\pi_{\mathbf{W}} : Y \twoheadrightarrow X$ is of fiber $F = \mathbf{U}(\mathbb{C})\mathbf{W}(\mathbb{R})$ and $Y \cong F \times X$ as real analytical varieties. The isomorphism no longer holds when the complex structure is taken into consideration: even in the simpler case where $\mathbf{U} = 1$, the complex structure on the fiber of $\mathbf{V}(\mathbb{R}) \times X \to X$ at $x \in X$ is defined by $x : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{GL}_{\mathbb{R}}(\mathbf{V}_{\mathbb{R}})$, and in general we write $Y = F \rtimes X$ to indicate the twisted complex structure on Y.

Consider a congruence lattice $\Gamma_{\mathbf{W}} \subset \mathbf{W}(\mathbb{Q}) \subset \mathbf{W}(\mathbb{R})$ which is the central extension of lattices $0 \to \Gamma_{\mathbf{U}} \to \Gamma_{\mathbf{W}} \to \Gamma_{\mathbf{V}} \to 0$ determined by $\psi : \Gamma_{\mathbf{V}} \times \Gamma_{\mathbf{V}} \to \Gamma_{\mathbf{U}}$ which is, by abuse of notations, the restriction of $\psi : \mathbf{V} \times \mathbf{V} \to \mathbf{U}$ to the lattices. Then the pro-finite completion of $\Gamma_{\mathbf{W}}$ is a compact open subgroup $K_{\mathbf{W}} \subset \mathbf{W}(\mathbb{A}^f)$. Take a compact open subgroup $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}^f)$ stabilizing $K_{\mathbf{W}}$ with respect to the Levi decomposition, then we have the fibration $M_K(\mathbf{P},Y)_{\mathbb{C}} \to M_{K_{\mathbf{G}}}(\mathbf{G},X)_{\mathbb{C}}$ by spaces of the form $\Gamma\backslash F$. More precisely, for $\Gamma = K_{\mathbf{G}} \cap \mathbf{G}(\mathbb{Q})_+$, the fibers over the component $\Gamma\backslash X^+$ are of the form $\Gamma_{\mathbf{W}}\backslash F$, whose complex structures vary with the base points. In particular, the fibers of the fibration $M_K(\mathbf{P},Y)_{\mathbb{C}} \to M_{K_{\mathbf{G}}}(\mathbf{G},X)_{\mathbb{C}}$ are geometrically connected. Thus the fibration induces a bijection between their sets of connected components:

$$\pi_0(M(\mathbf{P},Y)_{\mathbb{C}}) \cong \varprojlim_{K} \mathfrak{R}_K^{\mathbf{P}} = \pi_0(M(\mathbf{G},X)_{\mathbb{C}}) = \varprojlim_{K_{\mathbf{G}}} \mathfrak{R}_{K_{\mathbf{G}}}^{\mathbf{G}} \cong \mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}^f)/Z_{\mathbf{G}}^{\wedge}$$

where $K_{\mathbf{G}}$ runs through the compact open subgroups of $\mathbf{G}(\mathbb{A}^f)$, $\mathfrak{R}_{K_{\mathbf{G}}}^{\mathbf{G}}$ represents the finite set $\mathbf{G}(\mathbb{R})_+\backslash\mathbf{G}(\mathbb{A}^f)/K_{\mathbf{G}}$, and $Z_{\mathbf{G}}^{\wedge}$ is the projective limit $\varprojlim_{K_{\mathbf{G}}} Z_{\mathbf{G}}(\mathbb{A}^f)/[Z_{\mathbf{G}}(\mathbb{Q})\cdot K_{\mathbf{G}} \cap Z_{\mathbf{G}}(\mathbb{A}^f)]$. We refer the readers to [Del-1], [Del-2], [Mil-0] for the last identity characterizing $\pi_0(M(\mathbf{G},X)_{\mathbb{C}})$ in terms of $Z_{\mathbf{G}}^{\vee}$.

1.1.11 Special sections

Given a Shimura datum $(\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}, Y = F \rtimes X)$ with pure section $(\mathbf{G},X)$, we have the fibration of Shimura varieties $M_K(\mathbf{P},Y) \to M_{K_{\mathbf{G}}}(\mathbf{G},X)$ with section, where $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}^f)$, $K = K_{\mathbf{W}} \rtimes K_{\mathbf{G}} \subset \mathbf{P}(\mathbb{A}^f)$ are compact open subgroups. For any $w = (u,v) \in \mathbf{W}(\mathbb{Q})$, we have the Levi decomposition $\mathbf{P} = \mathbf{W} \rtimes (w\mathbf{G}w^{-1})$, hence a second pure section $(w\mathbf{G}w^{-1}, w \rtimes X)$ of $(\mathbf{P},Y)$, and an immersion

$$i_w : M_{K_w}(w\mathbf{G}w^{-1}, w \rtimes X) \hookrightarrow M_K(\mathbf{P},Y)$$

where $K_w = K \cap wK_{\mathbf{G}}w^{-1}$. Note that the inclusion $K_w \subset wK_{\mathbf{G}}w^{-1}$ is in general not a equality, and the composition $\pi_{\mathbf{W}} \circ i_w$ is a finite covering $M_{K_w}(w\mathbf{G}w^{-1}, w \rtimes X) \to M_{K_{\mathbf{G}}}(\mathbf{G},X)$. It is étale of degree $[K_{\mathbf{G}} : K_{\mathbf{G}}(w)]$ whenever $K_{\mathbf{G}}$ is torsion free, where

$$K_{\mathbf{G}}(w) = w^{-1}K_w w = \{g \in K_{\mathbf{G}} : wgw^{-1}g^{-1} \in K_{\mathbf{W}}\}.$$

We call $M_{K_w}(w\mathbf{G}w^{-1}, w \rtimes X)$ the special section by $w \in \mathbf{W}(\mathbb{Q})$, denoted as $M(w)$ where $M = M_K(\mathbf{P},Y)$. It is a section of $\pi_{\mathbf{W}}$ as long as $K_{\mathbf{G}}$ stabilizes the compact open subgroup generated by $K_{\mathbf{W}}$ and w, namely $K_w = wK_{\mathbf{G}}w^{-1}$.

We have seen that $w^n = (u,v)^n = (nu, nv)$. Now that $K_{\mathbf{W}}$ is determined by the lattices $\Gamma_{\mathbf{U}} \subset \mathbf{U}(\mathbb{Q})$ and $\Gamma_{\mathbf{V}} \subset \mathbf{V}(\mathbb{Q})$, we define the torsion order of $M(w) \subset M \twoheadrightarrow S$, where $M = M_K(\mathbf{P},Y)$ and $S = M_{K_{\mathbf{G}}}(\mathbf{G},X)$, to be smallest integer $n > 0$ such that $w^n = (nu, nv)$ lies in $\Gamma_{\mathbf{W}} = K \cap \mathbf{W}(\mathbb{Q})$. If n_u is the order of the reduction of $u \in \mathbf{U}(\mathbb{Q})$ modulo $\Gamma_{\mathbf{U}}$, and n_v that for $v \in \mathbf{V}(\mathbb{Q})$ modulo $\Gamma_{\mathbf{V}}$, then the torsion order n of (u,v) is the least common multiple of n_u and n_v.

In particular, if we are in the case of mixed Shimura varieties of Kuga type, namely $\mathbf{U} = 0$, $\mathbf{W} = \mathbf{V}$, then $\pi_{\mathbf{V}}$ is an abelian S-scheme $A \to S = M_{K_{\mathbf{G}}}(\mathbf{G},X)$, whose geometric fibers are abelian varieties of complex loci $\Gamma_{\mathbf{V}}\backslash\mathbf{V}(\mathbb{R})$. For each $n \in \mathbb{N}_{>0}$,

$A[n]$ is a finite étale S-group, locally isomorphic to the constant group $(\mathbb{Z}/n)^{2g}$, g being the relative dimension of $\pi_{\mathbf{W}}$. If we take $\beta = \{v\} \subset \mathbf{V}(\mathbb{Q})$ lifting a Drinfel'd basis of level n, namely whose reduction modulo $\Gamma_{\mathbf{V}}$ forms a basis of $(\Gamma_{\mathbf{V}}\backslash\mathbf{V}(\mathbb{Q}))[n]$ over $\mathbb{Z}/n\mathbb{Z}$, and put $K_{\mathbf{G}}(\beta) = \bigcap_{v\in\beta} K_{\mathbf{G}}(v)$, then we have a finite Galois covering of pure Shimura varieties $S' = M_{K_{\mathbf{G}}(\beta)}(\mathbf{G},X) \to S$, and the base change $A' \to S'$ of $A \to S$ is an abelian S'-scheme with a level-n structure. For $v \in \beta$, the special section $A'(v)$ is a torsion section of $A' \to S'$ of order n, and this justifies the name of torsion order in the general case of mixed Shimura varieties, where group structure hardly occurs for the fibration, but multiples of special sections make sense.

Lemma 1.1.12. *Let* $(\mathbf{P},Y)$ *be a Shimura datum with a pure section* $(\mathbf{G},X)$, $K = K_{\mathbf{W}} \rtimes K_{\mathbf{G}} \subset \mathbf{P}(\mathbb{A}^f)$ *compact open subgroup, and* $\pi : M_K(\mathbf{P},Y) \to M_{K_{\mathbf{G}}}(\mathbf{G},X)$ *the associated Shimura variety with the pure section. The the union of special sections of* π *is dense in* $M_K(\mathbf{P},Y)$ *for the Zariski topology. If moreover* $\mathbf{U_P} = 1$, *then the union is dense for the archimedean topology.*

Proof. First note that $\mathbf{P}(\mathbb{Q}) \subset \mathbf{P}(\mathbb{C})$ is dense for the complex Zariski topology, and so it is with $\bigcup_{w\in\mathbf{W}(\mathbb{Q})} w \rtimes X$ is $Y = \mathbf{U}(\mathbb{C})\mathbf{W}(\mathbb{R})\cdot X$, hence the first conclusion.

If moreover $\mathbf{U_P} = 1$, then $\mathbf{P}(\mathbb{Q})$ is dense in $\mathbf{P}(\mathbb{R})$ for the archimedean topology, and so it is with $\bigcup_{q\in\mathbf{V}(\mathbb{Q})} q \rtimes X$ in Y. ∎

Example 1.1.13. (1) Fix $n \in \mathbb{N}$ nonzero. Let $V_n = \mathbb{G}_{a\mathbb{Q}}^{2n}$ be a rational Hodge structure of type $\{(-1,0),(0,-1)\}$ and of dimension $2n$ over $\mathbb{Q}$, equipped with the standard non-degenerate alternating form $\psi_n : V_n \otimes_{\mathbb{Q}} V_n \to U = \mathbb{Q}(1)$, and $G_n = \mathrm{CSp}_{2n}$ the $\mathbb{Q}$-group of automorphisms of V_n preserving ψ_n up to a nonzero scalar. The pure Shimura datum $(G_n,X_n) = (\mathrm{CSp}_{2n},\mathscr{H}_n^{\pm})$ gives rise to the Siegel moculi varieties of genus n parameterizing principally polarized abelian schemes of dimension g (with certain level structures), where $X_n = \mathscr{H}_n^{\pm}$ is the double Siegel space of genus n (with two connected components).

Let W_n be the extension of V_n by $U = \mathbb{Q}(1)$ via ψ_n, equipped with the obvious action of $G_n = \mathrm{CSp}_{2n}$, and

$$(Q_n,A_n) = (V_n \rtimes G_n, V_n(\mathbb{R}) \rtimes X_n)$$

$$(P_n,Y_n) = (W_n \rtimes G_n, U(\mathbb{C})W_n(\mathbb{R}) \rtimes X_n)$$

the induced mixed Shimura data. Consider the two step fibration

$$\pi_W : (P_n,Y_n) \xrightarrow{\pi_U} (Q_n,A_n) \xrightarrow{\pi_V} (G_n,X_n).$$

where the data are all of reflex field $\mathbb{Q}$. Take $K \subset P(\mathbb{A}^f)$ a compact open subgroup, we have morphisms of Shimura varieties of reflex field $\mathbb{Q}$:

$$M_K(P_n,Y_n) \xrightarrow{\pi_U} M_{\pi_U(K)}(Q_n,A_n) \xrightarrow{\pi_V} M_{\pi_W(K)}(G_n,X_n)$$

where π_U is a $\mathbb{G}_m$-torsor and π_V is an abelian scheme of relative dimension n.

Let $L_n \subset V_n$ be an integral Hodge structure which is principally polarized, i.e. the polarization map $\psi : L_n \otimes L_n \to \mathbb{Z}(1)$ is onto, then G_n also admits a smooth $\mathbb{Z}$-model. Write $K_V \subset V_n(\mathbb{A}^f)$ for the profinite completion of L_n in $V_n(\mathbb{A}^f)$ which is a free $\hat{\mathbb{Z}}$-module of full rank, $K_G[N] = \mathrm{Ker}(G_n(\hat{\mathbb{Z}}) \to G_n(\mathbb{Z}/N))$ with $N \in \mathbb{N}$ and $N \geq 3$, and $K_Q = K_V \rtimes K_G[N] \subset Q_n(\mathbb{A}^f)$. Then

$$\pi_V : M_{K_Q}(Q_n, A_n) \longrightarrow M_{K_G[N]}(G_n, X_n)$$

is the universal abelian scheme over the modular scheme of principally polarized abelian schemes with level-N structures. It admits N^{2n} distinct torsion sections annihilated by N, each of which is a morphism of Shimura varieties.

(2) We also include the case when $n = 0$. Consider

$$(G_0, X_0) = (\mathbb{G}_{m\mathbb{Q}}, \{\pm\} = \pi_0(\mathscr{H}_1^{\pm}))$$

$$(P_0, Y_0) = (U \rtimes G_0, \mathbb{C} \rtimes X_0)$$

where G_0 acts on U by multiplication, which identifies U with $\mathbb{Q}(1)$ via $\mathrm{Nm} : \mathbb{S} \to \mathbb{G}_{m\mathbb{R}}$. Each finite level Shimura variety $M_K(P_0, Y_0)$ is a split torus over the zero-dimensional variety $M_{\pi(K)}(G_0, X_0)$ with respect to the projection $\pi : (P_0, Y_0) \to (G_0, X_0)$.

We usually identify P_0 with the upper triangular mirabolic subgroup of $GL_{2,\mathbb{Q}}$. But (P_0, Y_0) is not a Shimura subdatum of $(GL_{2,\mathbb{Q}}, \mathscr{H}_1^{\pm})$; instead it is a rational boundary component of the latter, which is the simplest case in the theory of toroidal compactification of mixed Shimura varieties developped by R.Pink in his thesis [Pink-0].

We present a lemma concerning the group action on the unipotent parts, which prepares us for the later Reduction Lemma

Lemma 1.1.14. *Let* $(\mathbf{P}, Y)$ *be a mixed Shimura datum with pure section* $(\mathbf{G}, X)$ *defined by a Levi decomposition* $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$*. Write* $\mathbf{U}$ *for the weight -2 part of* $\mathbf{W}$*,* $\mathbf{V} = \mathbf{W}/\mathbf{U}$ *the weight -1 subquotient. Denote by* ρ *resp.* $\rho_{\mathbf{U}}$ *resp.* $\rho_{\mathbf{V}}$ *the action of* $\mathbf{G}$ *on* $\mathbf{W}$ *resp.* $\mathbf{U}$ *resp.* $\mathbf{V}$ *defined by conjugation in* $\mathbf{P}$*.*

(1) $\mathbf{W}$ *is the central extension of* $\mathbf{V}$ *by* $\mathbf{U}$*, characterized by an alternating bilinear map* $\psi : \mathbf{V} \otimes_{\mathbb{Q}} \mathbf{V} \to \mathbf{U}$*, such that if* $\mathbf{W}$ *is identified with* $\mathbf{U} \times \mathbf{V}$ *as* $\mathbb{Q}$*-variety, then the group law on* $\mathbf{W}$ *is given by* $(u_1, v_1)(u_2, v_2) = (u_1 + u_2 + \psi(v_1, v_2), v_1 + v_2)$*. Moreover* ψ *is* $\mathbf{G}$*-invariant with respect to* $\rho_{\mathbf{V}}$ *and* $\rho_{\mathbf{U}}$*.*

(2) For any $x \in X \subset Y$*, the induced map* $x : \mathbb{S} \to \mathbf{GL}_{\mathbb{R}}(\mathbf{V}_{\mathbb{R}})$ *is a polarizable rational Hodge structure of type* $\{(-1,0),(0,-1)\}$*. Write* $\lambda : \mathbf{V} \otimes_{\mathbb{Q}} \mathbf{V} \to \mathbb{Q}(1)$ *for such a polarization, then* $\rho_{\mathbf{V}}$ *preserves* λ *up to scalars, i.e.* $\rho_{\mathbf{V}}$ *factors through* $\mathrm{CSp}(\lambda) \hookrightarrow \mathbf{GL}_{\mathbb{Q}}(\mathbf{V})$*.*

(3) For any $x \in X \subset Y$*, the induced map* $x : \mathbb{S} \to \mathbf{GL}_{\mathbb{R}}(\mathbf{U})$ *is central. Furthermore, if* $\mathbf{G}$ *is the generic Mumford-Tate group of* X*, then* $\mathbf{G}$ *acts on* $\mathbf{U}$ *through some split* $\mathbb{Q}$*-torus.*

Proof. (1) This is already established in 1.1.7.

(2) From the universal properties of (pure or mixed) Shimura datum, we know that $\mathbf{G} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{V})$ induces a variation of polarizable rational Hodge structure $\mathcal{V}$ on X. X is simply connected, hence $\mathcal{V}$ is a constant sheaf. In particular the polarization λ gives a polarization on the fiber of $\mathcal{V}$ at $x \in \mathrm{X}$, namely a polarization of the Hodge structure $x : \mathbb{S} \to \mathbf{GL}_{\mathbb{R}}(\mathbf{V}_{\mathbb{R}})$. It has been encoded in the definition of mixed Hodge data that this is of type $\{(-1,0),(0,-1)\}$, and that $\mathbf{G}$ respects λ, which gives rise to the factorization $\mathbf{G} \to \mathrm{CSp}(\lambda) \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{V})$.

(3) Because $\mathbf{U}$ is of one single Hodge type (-1,-1), by definition we see that $\mathbb{S}$ acts on $\mathbf{U}_{\mathbb{R}}$ by scalar multiplications, which are central.

Consider the action of $\mathfrak{g} := \mathbf{Lie}\mathbf{G}$ on $\mathfrak{u} := \mathbf{Lie}\mathbf{U}$, with the Hodge structures defined by $x \in \mathrm{X} \subset \mathrm{Y}$. It is easy to see that this induces a homomorphism of Hodge structures $\mathfrak{g} \otimes_{\mathbb{Q}} \mathfrak{u} \to \mathfrak{u}$, i.e. $\mathbf{Lie}\rho_{\mathbf{U}} : \mathfrak{g} \to \mathrm{End}_{\mathbb{Q}}(\mathfrak{u})$, where the Hodge structure on $\mathrm{End}_{\mathbb{Q}}(\mathfrak{u})$ is induced from $\mathfrak{u} \otimes_{\mathbb{Q}} \mathfrak{u}^{\vee}$. Therefore $\mathrm{End}_{\mathbb{Q}}(\mathfrak{u})$ is of Hodge type (0,0). Now that $\mathbf{Lie}\rho_{\mathbf{U}}$ is homomorphism of Hodge structures, $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$ must lie in the kernel of $\mathbf{Lie}\rho_{\mathbf{U}}$.

Let $\mathbf{H}$ be a non-commutative simple $\mathbb{Q}$-factor of $\mathbf{G}$. Since $\mathbf{H}$ is invariant in $\mathbf{G}$ and in particular stabilized by $\mathbb{S}$-conjugation, we see that $\mathfrak{h} := \mathbf{Lie}\mathbf{H}$ is a direct summand of the Hodge structure on $\mathfrak{g}$ (defined by x). By definition of pure Shimura data we know that the action of $\mathbb{S}$ on $\mathbf{H}$ is nontrivial, therefore $\mathfrak{h}^{-1,1} \neq 0 \neq \mathfrak{h}^{1,-1}$, and thus

$$0 \neq \mathfrak{h} \cap \mathfrak{g}^{1,-1} \subset \mathfrak{h} \cap \mathrm{Ker}(\mathbf{Lie}\rho_{\mathbf{U}}).$$

$\mathrm{Ker}\,\mathbf{Lie}\rho_{\mathbf{U}}$ is an ideal of $\mathfrak{g}$, hence it contains minimal ideal $\mathfrak{h}$.

From the above calculation we conclude that $\mathfrak{g}^{\mathrm{der}}$ lies in the kernel of $\mathbf{Lie}\rho_{\mathbf{U}}$, and $\mathbf{G}$ acts on $\mathbf{U}$ through the $\mathbb{Q}$-torus $\mathbf{T} = \mathbf{G}/\mathbf{G}^{\mathrm{der}}$. Because the action of $\mathbb{S}$ on $\mathbf{GL}_{\mathbb{R}}(\mathbf{U}_{\mathbb{R}})$ is given by the composition $\rho_{\mathbf{U}} \circ x : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{GL}_{\mathbb{R}}(\mathbf{U}_{\mathbb{R}})$, we see that this action is equal to $\rho'_{\mathbf{U}} \circ x'$, where $x' : \mathbb{S} \to \mathbf{T}_{\mathbb{R}}$ is induced by x modulo $\mathbf{G}^{\mathrm{der}}$, and $\rho'_{\mathbf{U}}$ is homomorphism of $\mathbb{Q}$-groups $\mathbf{T} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{U})$ through which $\rho_{\mathbf{U}}$ factors.

It remains to prove that this action factors through some split $\mathbb{Q}$-torus in case $\mathbf{G} = \mathbf{MT}(\mathrm{X})$. We may assume that $\mathbf{G}^{\mathrm{der}} = 1$ and $\mathbf{G}$ is itself a $\mathbb{Q}$-torus $\mathbf{T}$, and $(\mathbf{G},\mathrm{X}) = (\mathbf{T},x)$. As has been encoded in the definition, $\mathbf{T}$ is the almost direct product of $\mathbf{C}$ with $\mathbf{T}'$ where $\mathbf{C}$ is a compact $\mathbb{Q}$-torus and $\mathbf{T}'$ is a split torus. Because $\mathbf{T}$ is the Mumford-Tate group of $x : \mathbb{S} \to \mathbb{T}_{\mathbb{R}}$, we see that $\mathbb{T}_{\mathbb{C}}$ is generated by the $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-conjugates of the image of $x_{\mathbb{C}}$. The weight cocharacter $w : \mathbb{G}_{\mathrm{m}\mathbb{R}} \to \mathbb{S} \to \mathbb{T}$ is defined over $\mathbb{Q}$, and its image is invariant under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Note that $\mathbb{S}$ is the almost direct product of $\mathbb{G}_{\mathrm{m}\mathbb{R}}$ with $\mathbb{S}^1$, we conclude that $\mathbf{C}$ is generated by the $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-conjugates of $x_{\mathbb{C}}(\mathbb{S}^1_{\mathbb{C}})$. Calculating the Hodge weights we see that $\mathbb{S}^1$ acts on $\mathbf{U}_{\mathbb{R}}$ trivially, and so it is with the $\mathbf{C}$-action on it. Hence $\mathbf{T} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{U})$ factors through $\mathbf{T}/\mathbf{C}$ which is a split $\mathbb{Q}$-torus. ■

Lemma 1.1.15 (Reduction lemma). *(cf.[Pink0] Chap.2 Reduction Lemma 2.26) Let* $(\mathbf{P},\mathrm{Y})$ *be a Shimura datum, and write* $\mathbf{V} = \mathbf{W}/\mathbf{U}$ *for the weight -1 subquotient* *of* $\mathbf{P}$*. Suppose that* $\mathbf{P}$ *is the generic Mumford-Tate group of* Y*. Then there exists a morphism of Shimura data*

$$(\mathbf{P},Y) \longrightarrow (\tilde{\mathbf{P}},\tilde{Y}) = (\mathbf{H},X_{\mathbf{H}}) \times \prod_{n=1}^{N}(\mathsf{Q}_n,\mathsf{A}_n)^{a_n} \times \prod_{n=0}^{N}(\mathsf{P}_n,\mathsf{Y}_n)^{b_n}$$

such that $\mathrm{Ker}(\mathbf{P}\to\tilde{\mathbf{P}})$ *is finite, where* $\mathbf{H}$ *is a reductive* $\mathbb{Q}$*-group, and* $(\mathsf{Q}_n,\mathsf{A}_n)^{a_n}$ *resp.* $(\mathsf{P}_n,\mathsf{Y}_n)^{b_n}$ *is the* a_n*- resp.* b_n*-fold direct product of* $(\mathsf{Q}_n,\mathsf{A}_n)$ *resp.* $(\mathsf{P}_n,\mathsf{Y}_n)$*, with* $\mathrm{N}, a_n, b_n \in \mathbb{N}$*.*

Moreover, the component $(\mathbf{H},X_{\mathbf{H}})$ *can be chosen in such a way that if* ϕ *is a morphism from* $(\mathbf{H},X_{\mathbf{H}})$ *towards some mixed datum of the form* $(\mathsf{Q}_n,\mathsf{A}_n)$ *or* $(\mathsf{P}_n,\mathsf{Y}_n)$*, then* $\mathrm{Ker}\,\phi \supset \mathbf{H}^{\mathrm{der}}$*.*

Proof. We fix a Levi decomposition $\mathbf{P}=\mathbf{W}\rtimes\mathbf{G}$, with induced actions $\rho:\mathbf{G}\to\mathbf{Aut}_{\mathbb{Q}}(\mathbf{W})$, $\rho_{\mathbf{V}}:\mathbf{G}\to\mathbf{GL}_{\mathbb{Q}}(\mathbf{V})$, and $\rho_{\mathbf{U}}:\mathbf{G}\to\mathbf{GL}_{\mathbb{Q}}(\mathbf{U})$. The bi-linear map $\psi:\mathbf{V}\otimes_{\mathbb{Q}}\mathbf{V}\to\mathbf{U}$ is $\mathbf{G}$-equivariant.

Step 1: we may assume that $\mathbf{U}$ is one-dimensional as long as it is non-zero. The idea is that the actions of $\mathbb{S}$ and of $\mathbf{G}$ on $\mathbf{U}$ both factor through split tori, and $\mathbf{U}$ is decomposed into a direct sum $\mathbf{U}=\oplus_i\mathbf{U}_i$, and ψ is decomposed into $\psi=\oplus_i\psi_i$. Write $\mathbf{W}_i$ the extension of $\mathbf{V}$ by $\mathbb{G}_a$ through ψ_i and $\mathbf{P}_i=\mathbf{W}_i\rtimes\mathbf{G}$, then we have an embedding $\mathbf{P}\hookrightarrow\prod_i\mathbf{P}_i$ in an obvious way. Write Y_i for the image of Y under $\mathfrak{Y}(\mathbf{P})\to\mathfrak{Y}(\mathbf{P}_i)$, then it is easy to prove that $(\mathbf{P}_i,Y_i)$ is a mixed Shimura datum, and we have an embedding $(\mathbf{P},Y)\to\prod_i(\mathbf{P}_i,Y_i)$. Note that each $\mathbf{P}_i$ remains the generic Mumford-Tate group of Y_i, and it suffices to prove the lemma for each $(\mathbf{P}_i,Y_i)$.

Step 2: we may assume that $\mathbf{V}$ is irreducible as a representation of $\mathbf{G}$. Since $\mathbf{G}$ is reductive we decompose $\mathbf{V}$ into a direct sum of irreducible representations $\mathbf{V}=\oplus_i\mathbf{V}_i$. Then we can embed $\mathbf{W}$ into $\prod_i\mathbf{W}_i$ where $\mathbf{W}_i$ is the extension of $\mathbf{V}_i$ by $\mathbf{U}(=\mathbb{G}_a)$ by through $\psi_{|\mathbf{V}_i}$. Here we allow $\mathbf{U}$ to be zero so that $\mathbf{W}=\mathbf{V}$ is included as the degenerated case corresponding to $\psi:\mathbf{V}\otimes_{\mathbb{Q}}\mathbf{V}\to 0$. In the same way as in Step 1 we embed $(\mathbf{P},Y)$ into $\prod_i(\mathbf{P}_i,Y_i)$ with $\mathbf{P}_i=\mathbf{W}_i\rtimes\mathbf{G}$, and it suffices to treat each $(\mathbf{P}_i,Y_i)$ separately.

Step 3: we may assume that the action of $\mathbf{G}$ on $\mathbf{W}$ is faithful. In fact write $\mathbf{H}_0$ for the kernel of $\mathbf{G}\to\mathbf{Aut}_{\mathbb{Q}}(\mathbf{W})$, then $\mathbf{G}=\mathbf{H}_0\tilde{\times}\mathbf{H}_1$ for some reductive $\mathbb{Q}$-subgroup, and by putting $\mathbf{G}_0=\mathbf{G}/\mathbf{H}_1$, $\mathbf{G}_1=\mathbf{G}/\mathbf{H}_0$ we get Shimura data $(\mathbf{G}_i,X_i)$ with $\mathbf{G}_i=\mathbf{MT}(X_i)$, and $(\mathbf{P}_1=\mathbf{W}\rtimes\mathbf{G},Y_1)$ with $\mathbf{P}_1=\mathbf{MT}(Y_1)$. Then the canonical map $(\mathbf{P},Y)\to(\mathbf{G}_0,X_0)\times(\mathbf{P}_1,Y_1)$ is of finite kernel. We need to show that the two factors both satisfy the Reduction Lemma.

For $(\mathbf{G}_0,X_0)$ the construction goes as follows. A representation $\mathbf{M}$ of $\mathbf{G}_0$ is of abelian Kuga type if $\mathbb{S}\to\mathbf{G}_{0,\mathbb{R}}\to\mathbf{GL}_{\mathbb{R}}(\mathbf{M}_{\mathbb{R}})$ induces on $\mathbf{M}$ a rational Hodge structure of type $\{(-1,0),(0,-1)\}$ for some (or any) $x\in X_0$. Note that this is equivalent to the existence of a morphism of Shimura data $(\mathbf{G}_0,X_0)\to(\mathsf{G}_n,\mathsf{X}_n)$ for some $n>0$. If such representations do not exist, then $(\mathbf{G}_0, X_0)$ serves as the part $(\mathbf{H}, X_{\mathbf{H}})$ as is stated in the Lemma. Otherwise there exists some representation $\mathbf{M}$ of abelian type and we have a nontrivial morphism $(\mathbf{G}_0, X_0) \to (\mathrm{G}_n, \mathrm{X}_n)$ for some $n>0$. Take $\mathbf{G}'_0$ to be the kernel of $\mathbf{G}_0 \to \mathrm{G}_n$, we may apply again the argument in the last paragraph and get a morphism $(\mathbf{G}_0, X_0) \to (\mathbf{H}', X') \times (\mathrm{G}_n, \mathrm{X}_n)$ of finite kernel with $\dim \mathbf{H}' < \dim \mathbf{G}$, and by induction the lemma holds for $(\mathbf{G}_0, X_0)$.

Step 4: the faithful action of $\mathbf{G}$ on $\mathbf{W}$.
If $\psi = 0$ then $\mathbf{W} = \mathbb{G}_a \oplus \mathbf{V}$ and we have naturally

$$(\mathbf{P}, Y) \hookrightarrow (\mathbb{G}_a \rtimes \mathbf{G}, \mathbb{C} \rtimes X) \times (\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes X)$$

and it suffices to treat each factor separately. For $(\mathbb{G}_a \rtimes \mathbf{G}, \mathbb{C} \rtimes X)$ the reduction in Step 3 reduces us to the case where $\mathbf{G}$ acts on $\mathbb{G}_a$ faithfully, whence $\mathbf{G}$ is isomorphic to $\mathbb{G}_m$ and the datum becomes $(\mathrm{P}_0, \mathrm{Y}_0)$. For $(\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes X)$ we are reduced to the case where $\mathbf{G}$ acts on $\mathbf{V}$ faithfully, whence an embedding

$$(\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes X) \hookrightarrow (\mathrm{Q}_n, \mathrm{A}_n)$$

for some $n>0$.

If $\psi \neq 0$, then we first put $\mathbf{V}_0$ to be the kernel of ψ. ψ is $\mathbf{G}$-invariant thus $\mathbf{V}_0$ is a subrepresentation of $\mathbf{V}$, which has to be trivial because we are already reduced to the case that $\mathbf{V}$ is irreducible and ψ non-zero. In this case $\mathbf{G}$ preserves ψ implies a non-trivial morphism

$$(\mathbf{P}, Y) \to (\mathrm{P}_n, \mathrm{Y}_n)$$

some $n>0$ where P_n is actually given by $\mathbf{W} \rtimes \mathrm{CSp}(\psi)$. The kernel of $\mathbf{P} \to \mathrm{P}_n$ is a reductive $\mathbb{Q}$-subgroup of $\mathbf{G}$, and it suffices to apply the precedent reductions recurrently. ■

Remark 1.1.16. The Reduction Lemma of R.Pink treats the two case $\mathbf{W} = \mathbf{U}$ and $\mathbf{W} = \mathbf{V}$ separately, and what we have done is essentially combining them into the general case.

To end this section we introduce the notion of real part for mixed Shimura data. They will come back later in the chapter on the equidistribution of special subvarieties.

Definition 1.1.17. Let $(\mathbf{P}, Y)$ be a mixed Shimura datum, and $\mathbf{U}$ the weight -2 $\mathbb{Q}$-subgroup of $\mathbf{P}$.

(1) The real part of $(\mathbf{P}, Y)$ is the pair $(\mathbf{P}, Y_{\mathbb{R}})$ where $Y_{\mathbb{R}} := \{y \in Y | y : \mathbb{S}_{\mathbb{C}} \to \mathbf{P}_{\mathbb{C}} \text{ is defined over } \mathbb{R}\}$.

(2) The imaginary part of $(\mathbf{P}, Y)$ is $\mathbb{I}(\mathbf{U}) = \mathbf{U}(\mathbb{R})(-1) := 2\pi \mathbf{i}\mathbf{U}(\mathbb{R}) \subset \mathbf{U}(\mathbb{C})$, viewed as a real vector space of dimension $\dim_{\mathbb{Q}} \mathbf{U}$ inside $\mathbf{U}(\mathbb{C})$ "orthogonal" to $\mathbf{U}(\mathbb{R})$. Note that it is stable under the action of $\mathbf{P}$ on $\mathbf{U}$ by conjugation $\mathbf{P}(\mathbb{R}) \times \mathbf{U}(\mathbb{C}) \to \mathbf{U}(\mathbb{C})$.

Proposition 1.1.18. *Let* $(\mathbf{P},\mathrm{Y})$ *and* $\mathbf{U}\subset\mathbf{P}$ *be as in the definition above.*

(1) Let $(\mathbf{G},\mathrm{X})\subset(\mathbf{P},\mathrm{Y})$ *be a pure section, then* $\mathrm{Y}_{\mathbb{R}}=\mathbf{P}(\mathbb{R})x$ *for any* $x\in\mathrm{X}$*;* $\mathrm{Y}_{\mathbb{R}}$ *is indpendent of the choice of pure section* $(\mathbf{G},\mathrm{X})$.

(2) There exists a surjection $\mathrm{im}:\mathrm{Y}\to\mathbb{I}(\mathbf{U})$ *such that for any* $y\in\mathrm{Y}$, $\mathrm{im}(y)$ *is the unique element in* $\mathbb{I}(\mathbf{U})$ *with the property that* $\mathbf{Int}(\mathrm{im}(y)^{-1})\circ y:\mathbb{S}_{\mathbb{C}}\to\mathbf{P}_{\mathbb{C}}$ *is defined over* $\mathbb{R}$.

Note that $\mathbf{Int}(\mathrm{im}(y)^{-1})$ *is the conjugation by* $\mathrm{im}(y)^{-1}$*. If no confusion is caused, we write the conjugation of* $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$ *on* Y *as left translation, then* $\mathbf{Int}(\mathrm{im}(y)^{-1})\circ y=\mathrm{im}(y)^{-1}y$.

(3) We have a bijection

$$\mathrm{Y}\to\mathbb{I}(\mathbf{U})\times\mathrm{Y}_{\mathbb{R}},\ y\mapsto(\mathrm{im}(y),\mathrm{im}(y)^{-1}y)$$

Moreover this map is $\mathbf{P}(\mathbb{R})$*-equivariant, where* $\mathbf{P}(\mathbb{R})$ *acts on* Y *and* $\mathrm{Y}_{\mathbb{R}}$ *both via conjugation (written as left translation though), and on* $\mathbb{I}(\mathbf{U})$ *via the conjugation in* $\mathbf{P}(\mathbb{C})$.

Proof. (1) Take $x\in\mathrm{X}\subset\mathrm{Y}$. Then $x:\mathbb{S}\to\mathbf{G}_{\mathbb{R}}\subset\mathbf{P}_{\mathbb{R}}$ is defined over $\mathbb{R}$, hence $x\in\mathrm{Y}_{\mathbb{R}}$ and $\mathrm{X}\subset\mathrm{Y}_{\mathbb{R}}$. For $q\in\mathbf{P}(\mathbb{R})$, qx (the conjugate of x by q) remains a homomorphism defined over $\mathbb{R}$: $\mathbb{S}\to\mathbf{P}_{\mathbb{R}}$, thus $\mathrm{Y}_{\mathbb{R}}\supset\mathbf{P}(\mathbb{R})x$.

Take $y\in\mathrm{Y}_{\mathbb{R}}$, then $y(\mathbb{S})\subset\mathbf{P}_{\mathbb{R}}$ is an $\mathbb{R}$-torus, and is contained in a maximal reductive $\mathbb{R}$-subgroup of $\mathbf{P}_{\mathbb{R}}$, namely a Levi $\mathbb{R}$-subgroup $\mathbf{L}$ of $\mathbf{P}_{\mathbb{R}}$. Then $\mathbf{L}$ is conjugate to $\mathbf{G}_{\mathbb{R}}$ by some $u\in\mathbf{W}(\mathbb{R})$ such that $u\mathbf{L}u^{-1}=\mathbf{G}_{\mathbb{R}}$, hence $x:=uy\in\mathrm{X}$, and $y\in\mathbf{W}(\mathbb{R})x\subset\mathbf{P}(\mathbb{R})x$. Therefore $\mathrm{Y}_{\mathbb{R}}\subset\mathbf{P}(\mathbb{R})x$, hence the equality.

(2) This is taken from [Pink0] Chap.4, 4.14. We briefly reproduce the construction as follows.

Fix $(\mathbf{G},\mathrm{X})\subset(\mathbf{P},\mathrm{Y})$ a pure section, x a point in X, and identify $\mathbf{W}$ as the product of $\mathbb{Q}$-varieties $\mathbf{W}=\mathbf{U}\times\mathbf{V}$ with group law described in 1.1.10 of this chapter. An element $y\in\mathrm{Y}=\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})x$ can be written as $y=uqx$ for $u\in\mathbf{U}(\mathbb{C})$ and $q\in\mathbf{P}(\mathbb{R})$. We regard the complex vector space $\mathbf{U}(\mathbb{C})$ as an orthogonal direct sum of $\mathbf{U}(\mathbb{R})$ with $\mathbb{I}(\mathbf{U})=2\pi\mathbf{i}\mathbf{U}(\mathbb{R})\subset\mathbf{U}(\mathbb{C})$, and we have $u=u(y)\cdot u'$ for some $u(y)\in\mathbb{I}(\mathbf{U})$ and $u'\in\mathbf{U}(\mathbb{R})$, and $y=u(x)q(x)x$ for $u(x)\in\mathbb{I}(\mathbf{U})$ and some $q(x)\in\mathbf{P}(\mathbb{R})$

Let's check that $y\mapsto u(y)$ is a well-defined map. If $y=u_1q_1x=u_2q_2x$ for some $u_1,u_2\in\mathbb{I}(\mathbf{U})$ and $q_1,q_2\in\mathbf{P}(\mathbb{R})$, it turns out that $u_2^{-1}u_1q_1x=q_2x$, or simply $ux_1=qx_1$ where $u\in\mathbb{I}(\mathbf{U})$, $x_1=q_1x$, $q=q_2q_1^{-1}\in\mathbf{P}(\mathbb{R})$. Hence uq^{-1} fixes x_1. But for $x_1=q_1x$, its isotropy subgroup in $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$ is $q_1\mathrm{G}_xq_1^{-1}$, where G_x is the isotropy subgroup of $x\in\mathrm{X}$ in $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$. Calculating the Hodge structure on the Lie algebra we find that G_x is a Lie subgroup of $\mathbf{G}(\mathbb{R})$. And thus $q_1\mathrm{G}_xq_1^{-1}\subset\mathbf{P}(\mathbb{R})$ and $u\in\mathbf{P}(\mathbb{R})$. By definition $\mathbb{I}(\mathbf{U})\cap\mathbf{P}(\mathbb{R})=1$, therefore $u=1$ and $u_1=u_2$.

We also check that $y\mapsto u(y)$ is independent of the choice of base point $x\in\mathrm{Y}_{\mathbb{R}}$: let $x'=q'^{-1}x\in\mathrm{Y}_{\mathbb{R}}$ be another point, then $y=u(y)qx=u(y)qq'x'$, $qq'\in\mathbf{P}(\mathbb{R})$ and we get the same element $u(y)\in\mathbb{I}(\mathbf{U})$.

The map $y\mapsto u(y)$ is the morphism im in the proposition. It is also clear that $u(y)^{-1}y\in\mathrm{Y}_{\mathbb{R}}$.

(3) Note that the conjugation by $\mathbf{P}(\mathbb{R})$ on $\mathbf{U}(\mathbb{C})$ respects the orthogonal decomposition $\mathbf{U}(\mathbb{C}) = \mathbf{U}(\mathbb{R}) \oplus \mathbb{I}(\mathbf{U})$.

We then check that $Y \to Y_{\mathbb{R}}$ $y \mapsto \mathrm{im}(y)^{-1}y$ is well-defined. We may fix $x \in X$ a point in a pure section $(\mathbf{G}, X)$ of $(\mathbf{P}, Y)$. Then $y = \mathrm{im}(y)qx$ for some $q \in \mathbf{P}(\mathbb{R})$, thus $\mathrm{im}(y)^{-1}y = qx \in \mathbf{P}(\mathbb{R})x = Y_{\mathbb{R}}$. Note that the expression $\mathrm{im}(y)^{-1}y$ does not depend on x, and the map $Y \to Y_{\mathbb{R}}$ is well defined.

The $\mathbf{P}(\mathbb{R})$-equivariance is automatic: take $g \in \mathbf{P}(\mathbb{R})$ and $u = \mathrm{im}(y)$ for some $y \in Y$. Then $y = uqx$ for some $q \in \mathbf{P}(\mathbb{R})$, $x \in Y_{\mathbb{R}}$, and $gy = (gug^{-1}g(qx))$ which is the required equivariance.

■

Remark 1.1.19. It is easy to verify that the notions of real part and imaginary part are functorial: if we are given a morphism of Shimura data $(\mathbf{P}, Y) \to (\mathbf{P}', Y')$, then the map $Y \to Y'$ induces $Y_{\mathbb{R}} \to Y'_{\mathbb{R}}$. In particular, for a subdatum $(\mathbf{P}, Y) \subset (\mathbf{P}', Y')$ we have $Y_{\mathbb{R}} \subset Y'_{\mathbb{R}}$.

For any pure section $(\mathbf{G}, X)$ of $(\mathbf{P}, Y)$, the map $X \hookrightarrow Y$ sends X into $Y_{\mathbb{R}}$, and since the latter is homogeneous under $\mathbf{P}(\mathbb{R})$, we have $Y_{\mathbb{R}} = \mathbf{P}(\mathbb{R}) \cdot X$.

Fixing any point $x \in Y_{\mathbb{R}}$, we get $Y_{\mathbb{R}} \cong \mathbf{P}(\mathbb{R})/P_x$ where P_x is a maximal compact Lie subgroup of $\mathbf{P}(\mathbb{R})$, contained in $\mathbf{G}(\mathbb{R})$ for some Levi decomposition $\mathbf{P}_{\mathbb{R}} = \mathbf{W}_{\mathbb{R}} \rtimes \mathbf{G}$ over $\mathbb{R}$. The projection $Y_{\mathbb{R}} \to X = \mathbf{G}(\mathbb{R})/P_x$ is a fibration in $\mathbf{W}(\mathbb{R})$, where $X = \mathbf{G}(\mathbb{R})/P_x$ is a Hermitian symmetric domain. We will come back to this projection later in Chap.3.

1.2 Canonical models and reciprocity maps

Definition 1.2.1. For a Shimura datum $(\mathbf{P}, Y)$, a special subdatum is a subdatum of the form $(\mathbf{H}, y)$ with $\mathbf{H}$ a $\mathbb{Q}$-torus in $\mathbf{P}$ and y a single point in Y.

Definition-Proposition 1.2.2. (cf. [Pink-0] Chap.11, 11.1 Definition, 11.2 Properties) The reflex field of a Shimura datum $(\mathbf{P}, Y)$ is the field of definition of the $\mathbf{P}(\mathbb{C})$-conjugacy class of some (or any) $\mu_y = y \circ \mu$ $(y \in Y)$ in $\mathbf{X}^{\vee}_{\mathbf{P}}(\mathbb{C})$ the group of complex cocharacters of $\mathbf{P}$, namely the subfield E of $\mathbb{C}$ characterized by

$$\mathrm{Aut}(\mathbb{C}/E) = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) | \, \sigma[\mu_y] = [\mu_y]\}$$

where $[\mu_y]$ stands for the class of μ_y in $\mathbf{P}(\mathbb{C}) \backslash \mathbf{X}^{\vee}_{\mathbf{P}}(\mathbb{C})$. Write $E(\mathbf{P}, Y)$ for the reflex field of $(\mathbf{P}, Y)$, then:

(1) Once there is a morphism $(\mathbf{P}_1, Y_1) \to (\mathbf{P}_2, Y_2)$, we then have $E(\mathbf{P}_1, Y_1) \supset E(\mathbf{P}_2, Y_2)$; consequently, if $(\mathbf{G}, X) \subset (\mathbf{P}, Y)$ is any pure section, then $E(\mathbf{G}, X) = E(\mathbf{P}, Y)$;

(2) For $(\mathbf{G}, X)$ a pure Shimura datum, $\mathbf{H} \subset \mathbf{G}$ a maximal $\mathbb{Q}$-torus, we have a $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-equivariant isomorphism $\mathbf{G}(\mathbb{C}) \backslash \mathbf{X}^{\vee}_{\mathbf{G}}(\mathbb{C}) \cong W \backslash \mathbf{X}^{\vee}_{\mathbf{H}}(\mathbb{C})$, W being the Weyl group $W(\mathbf{G}_{\mathbb{C}}, \mathbf{H}_{\mathbb{C}})$. As a corollary, for any special subdatum $(\mathbf{H}, y) \subset (\mathbf{G}, X)$, denote

by $F_{\mathbf{H}}$ the splitting number field of $\mathbf{H}$, then we have $\mathrm{Aut}(\mathbb{C}/E(\mathbf{G},X)) \supset \mathrm{Aut}(\mathbb{C}/F_{\mathbf{H}})$ while the latter acts on $W\backslash \mathbf{X}_{\mathbf{H}}^{\vee}(\mathbb{C})$ trivially. In particular reflex fields are number fields embedded in $\mathbb{C}$.

(3)(cf.[D-1] Thm.5.1) Being given F be a number field (in $\mathbb{C}$) containing the reflex field $E(\mathbf{G},X)$ of some pure Shimura data $(\mathbf{G},X)$, there exists a special subdatum $(\mathbf{H},x) \subset (\mathbf{G},X)$ such that $E(\mathbf{H},x) \cap F = E(\mathbf{G},X)$.

When the reflex fields of subdata vary, their degrees over $\mathbb{Q}$ remain uniformly bounded, shown as follows:

Lemma 1.2.3. *For a given Shimura datum* $(\mathbf{P},Y)$ *of reflex field* $E = E(\mathbf{P},X)$, *there exists a constant* C, *such that for any subdatum* $(\mathbf{P}',Y')$ *of reflex field* $E' = E(\mathbf{P}',Y')$, *we have* $[E':\mathbb{Q}] < C$.

Proof. Let $(\mathbf{G}',X')$ be a pure section of $(\mathbf{P}',Y')$, which extends to a pure section $(\mathbf{G},X)$ of $(\mathbf{P},Y)$. Then $E' = E(\mathbf{G}',X')$ and $E = E(\mathbf{G},X)$, and it suffices to treat the case of pure Shimura data.

Let $\mathbf{T}'$ be a maximal $\mathbb{Q}$-torus of $\mathbf{G}'$, which extends to some maximal $\mathbb{Q}$-torus of $\mathbf{G}$. Let F' be the splitting field of $\mathbf{T}'$, which is the number field characterized by the property that $\mathrm{Gal}_{F'} = \mathrm{Ker}(\mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}_{\mathbb{Z}}(X_{\mathbf{T}'}))$. Then $[F':\mathbb{Q}]$ equals the cardinality of the image of $\mathrm{Gal}_{\mathbb{Q}}$ is $\mathrm{Aut}_{\mathbb{Z}}(X_{\mathbf{T}'})$.

Let N be the rank of $\mathbf{G}'$, then $X_{\mathbf{T}'} \cong \mathrm{GL}_N(\mathbb{Z})$, and the cardinalities of finite subgroups of $\mathrm{GL}_{\mathbb{Z}}(N)$ is uniformly bounded by some constant C_N only dependent on N, cf. [RT] Theorem 1 (actually it suffices to note a weaker form $C_N \leq (2N)!$). Because N is bounded by the rank of $\mathbf{G}$, the C_N's remain bounded by some constant $C_{\mathbf{G}}$ when $\mathbf{T}'$ runs through maximal $\mathbb{Q}$-tori of reductive $\mathbb{Q}$-subgroups of $\mathbf{G}$.

Now that F' splits $\mathbf{T}'$, we have $E \subset E' \subset F'$, and $[E':\mathbb{Q}] \leq [F':\mathbb{Q}] \leq C_{\mathbf{G}}$, with $C_{\mathbf{G}}$ a constant only dependent on $\mathbf{G}$. ∎

Definition-Proposition 1.2.4. (reciprocity map and canonical model in the zero-dimensional case, cf. [Pink-0], Chap.11, 11.3 and 11.4)

Let $(\mathbf{H},x)$ be a pure Shimura datum with $\mathbf{H}$ a $\mathbb{Q}$-torus. Since the $\mathbf{H}(\mathbb{C})$-conjugation on $\mathbf{X}_{\mathbf{H}}(\mathbb{C})$ is trivial, $E = E(\mathbf{H},x)$ is no other than the smallest number field (in $\mathbb{C}$) over which $\mu_x : \mathbb{G}_{\mathrm{m}\mathbb{C}} \to \mathbf{H}_{\mathbb{C}}$ descends. Put $r = r_{(\mathbf{H},x)}$ to be the composition $\mathbb{G}_{\mathrm{m}}^{E} \xrightarrow{\mu_x^E} \mathbf{H}^E \xrightarrow{\mathrm{Nm}_{E\mathbb{Q}}} \mathbf{H}$, and define the reciprocity map $\mathbf{rec}_x$ to be the composition of the following chain:

$$\mathrm{Gal}_E \twoheadrightarrow \bar{\pi}_0\pi(\mathbb{G}_{\mathrm{m}}^{E}) \xrightarrow{r} \bar{\pi}_0\pi(\mathbf{H})$$

where the first arrow is the reciprocity map $\mathbf{rec}_E : \mathrm{Gal}_E \twoheadrightarrow \mathrm{Gal}_E^{\mathrm{ab}} \twoheadrightarrow \bar{\pi}_0\pi(\mathbb{G}_{\mathrm{m}}^{E})$ described in the Notations.

The 0-dimensional scheme $M(\mathbf{H},x)_{\mathbb{C}}$ can be identified with its set of connected components, namely $\bar{\pi}_0\pi(\mathbf{H})$. Let Gal_E acts on it by translation via $\mathbf{rec}_x$, then the action is locally constant, i.e. every point in $M(\mathbf{H},x)_{\mathbb{C}}$ is fixed by an open subgroup of Gal_E, and $M(\mathbf{H},x)_{\mathbb{C}}$ is endowed a structure of a pro-E-scheme, called the canonical model of $M(\mathbf{H},x)$, denoted as $M(\mathbf{H},x)_E$.

More concretely, for every compact open subgroup $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}^{\mathrm{f}})$, $M_{K_{\mathbf{H}}}(\mathbf{H}, x)$ is characterized as the finite $\mathbb{Q}^{\mathrm{ac}}$-scheme with underlying set $\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}^{\mathrm{f}})/K_{\mathbf{H}}$ on which $\mathrm{Gal}_{\mathrm{E}}$ acts as $\sigma[x, aK_{\mathbf{H}}] = [x, \mathbf{rec}_x(\sigma)aK_{\mathbf{H}}]$, $\sigma \in \mathrm{Gal}_{\mathrm{E}}$, $a \in \mathbf{H}(\mathbb{A}^{\mathrm{f}})$. $M(\mathbf{H}, x)$ is equipped with a continuous right action of the locally profinite group $\mathbf{H}(\mathbb{A}^{\mathrm{f}})$: for each $h \in \mathbf{H}(\mathbb{A}^{\mathrm{f}})$ the action

$$h^* : M(\mathbf{H}, x) \cong M(\mathbf{H}, x) \quad [x, g] \mapsto [x, gh]$$

is defined over E because it obviously commutes with the action of $\mathrm{Gal}_{\mathrm{E}}$: only abelian groups intervene. The continuity follows the definition of the action of a locally profinite group on a scheme, see [SGA3] Exposé.VII.

For F a number field (embedded in $\mathbb{C}$) containing $\mathrm{E} = \mathrm{E}(\mathbf{H}, x)$, the quasi-canonical model of $M(\mathbf{H}, x)_{\mathbb{C}}$ over F is nothing but the base change $M(\mathbf{H}, x) \otimes_{\mathrm{E}} \mathrm{F}$, or equivalently, characterized by the action of $\mathrm{Gal}_{\mathrm{F}}$ on the set $\bar{\pi}_0\pi(\mathbf{H})$ by translation through the composition of $\mathrm{Gal}_{\mathrm{F}} \hookrightarrow \mathrm{Gal}_{\mathrm{E}} \xrightarrow{\mathbf{rec}_x} \bar{\pi}_0\pi(\mathbf{H})$, namely the composition of

$$\mathrm{Gal}_{\mathrm{F}} \xrightarrow{\mathbf{rec}_{\mathrm{F}}} \bar{\pi}_0\pi(\mathbb{G}_{\mathrm{m}}^{\mathrm{F}}) \xrightarrow{\mu_{x\mathrm{F}}} \bar{\pi}_0\pi(\mathbf{H}^{\mathrm{F}}) \xrightarrow{\mathrm{Nm}_{\mathrm{F}/\mathbb{Q}}} \bar{\pi}_0\pi(\mathbf{H})$$

justified by the commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}_{\mathrm{F}} \arrow[r, "\mathbf{rec}_{\mathrm{F}}"] \arrow[d, "\mathrm{F}\supset\mathrm{E}"'] & \bar{\pi}_0\pi(\mathbb{G}_{\mathrm{m}}^{\mathrm{F}}) \arrow[r, "\mu_{x\mathrm{F}}"] \arrow[d, "\mathrm{Nm}_{\mathrm{F}/\mathrm{E}}"'] & \bar{\pi}_0\pi(\mathbf{H}^{\mathrm{F}}) \arrow[r, "\mathrm{Nm}_{\mathrm{F}/\mathbb{Q}}"] \arrow[d, "\mathrm{Nm}_{\mathrm{F}/\mathrm{E}}"'] & \bar{\pi}_0\pi(\mathbf{H}) \arrow[d, "="] \\
\mathrm{Gal}_{\mathrm{E}} \arrow[r, "\mathbf{rec}_{\mathrm{E}}"] & \bar{\pi}_0\pi(\mathbb{G}_{\mathrm{m}}^{\mathrm{E}}) \arrow[r, "\mu_x"] & \bar{\pi}_0\pi(\mathbf{H}^{\mathrm{E}}) \arrow[r, "\mathrm{Nm}_{\mathrm{E}/\mathbb{Q}}"] & \bar{\pi}_0\pi(\mathbf{H})
\end{tikzcd}$$

we denote this homomorphism as $\mathbf{rec}_x^{\mathrm{F}}$, or simply $\mathbf{rec}_x$ if F is clear from the context.

Definition-Proposition 1.2.5. (canonical model in the general case, cf.[Pink-0], Chap.11, 11.5 Definition)

Let $(\mathbf{P}, \mathrm{Y})$ be a Shimura datum, and $\mathrm{E} = \mathrm{E}(\mathbf{P}, \mathrm{Y})$ its reflex field. A canonical model of the Shimura variety at some finite level $M_K(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$ is an E-variety such that for any special subdatum $(\mathbf{H}, x) \subset (\mathbf{P}, \mathrm{Y})$, the (0-dimensional) subvariety $M_{K_{\mathbf{H}}}(\mathbf{H}, x)_{\mathbb{C}} \subset M_K(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$ is defined over $\mathrm{E}_x = \mathrm{E}(\mathbf{H}, x)$ on which $\mathrm{Gal}_{\mathrm{E}_x}$ acts as in (1). So we have imposed the rationality conditions on a Zariski dense subset of $M_K(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$. What is less transparent is that these conditions implies the existence of a canonical model of $M_K(\mathbf{P}, \mathrm{Y})_{\mathbb{C}}$ over E, unique up to isomorphism, denoted as $M_K(\mathbf{P}, \mathrm{Y})_{\mathrm{E}}$. We refer the readers to standard references like [Pink-0] and [Milne-0] for further discussions. We also write $M_K(\mathbf{P}, \mathrm{Y})$ for its base change to $\mathbb{Q}^{\mathrm{ac}}$.

Moreover, the notion of canonical model is functorial:

(2-1) Let $M_{K_1}(\mathbf{P}_1, \mathrm{Y}_1)_{\mathbb{C}} \to M_{K_2}(\mathbf{P}_2, \mathrm{Y}_2)_{\mathbb{C}}$ be a morphism of Shimura varieties. Then it is defined over $\mathrm{E}(\mathbf{P}_2, \mathrm{Y}_2)$, namely it is $\mathrm{Aut}(\mathbb{C}/\mathrm{E}(\mathbf{P}_2, \mathrm{Y}_2))$-equivariant.

(2-2) In particular, the transition maps $\mathrm{pr}_{K,K'} : M_{K'}(\mathbf{P},Y)_{\mathbb{C}} \twoheadrightarrow M_K(\mathbf{P},Y)_{\mathbb{C}}$ for compact open subgroups $K' \subset K$, and the translations $g^* : M_{gKg^{-1}}(\mathbf{P},Y)_{\mathbb{C}} \cong M_K(\mathbf{P},Y)_{\mathbb{C}}$ with $g \in \mathbf{P}(\mathbb{A}^f)$, are defined over $E = E(\mathbf{P},Y)$. Hence we have the pro-E-scheme $M(\mathbf{P},Y)_E = \varprojlim_K M_K(\mathbf{P},Y)_E$ equipped with a continuous right action of $\mathbf{P}(\mathbb{A}^f)$.

We have equally the notion of quasi-canonical model of $M(\mathbf{P},Y)_{\mathbb{C}}$ over a number field $F \subset \mathbb{C}$ containing $E = E(\mathbf{P},Y)$. It is the base change $M(\mathbf{P},Y) \otimes_E F$, and is equivalently characterized as the unique F-pro-scheme equipped with a continuous right action of $\mathbf{P}(\mathbb{A}^f)$ such that for any special subdatum $(\mathbf{H},x) \subset (\mathbf{P},Y)$, the subscheme $M(\mathbf{H},x)_{\mathbb{C}}$ is defined over $F_x = F \cdot E(\mathbf{H},x)$, while the $\mathrm{Aut}(\mathbb{C}/F_x)$-action on it is given by the translation via the composition

$$\mathbf{rec}_x^F : \mathrm{Gal}_{F_x} \twoheadrightarrow \bar{\pi}_0\pi(\mathbb{G}_m^{F_x}) \xrightarrow{\mu_x} \bar{\pi}_0\pi(\mathbf{H}^{F_x}) \xrightarrow{\mathrm{Nm}_{F_x/\mathbb{Q}}} \bar{\pi}_0\pi(\mathbf{H}).$$

1.3 Hecke correspondences and special subvarieties

Definition 1.3.1 (Hecke correspondence). Let $M_K = M_K(\mathbf{P},Y)$ be the Shimura variety associated to the datum $(\mathbf{P},Y)$ at level $K \subset \mathbf{P}(\mathbb{A}^f)$. For any $g \in \mathbf{P}(\mathbb{A}^f)$, put $K_g = K \cap gKg^{-1}$, and define the Hecke correspondence $\mathcal{T}_g$ to be diagram

$$M_K \xleftarrow{\mathrm{pr}} M_{K_g} \xrightarrow{g_*\mathrm{pr}} M_K$$

where the first pr is the projection $M_K \longrightarrow M_{K_g}$, $[y, aK_g] \mapsto [y, aK]$, and the second map is the composition $M_{K_g} \xrightarrow{\mathrm{pr}} M_{gKg^{-1}} \xrightarrow{g_*} M_K$, $[x, aK_g] \mapsto [x, agK]$. Note that the two maps are finite, defined over $E(\mathbf{P},Y)$, and are étale if K is torsion free.

The associated Hecke operator $\mathcal{T}_g$ takes an algebraic cycle Z on M_K to the cycle $\mathcal{T}_g(Z) = (g_*\mathrm{pr})_*\mathrm{pr}^*(Z)$, called the Hecke translation of Z by $\mathcal{T}_g$. In particular, it takes a point $z \in M_K(\mathbb{C})$ to a finite subset of $M_K(\mathbb{C})$. Hence for any point $z \in M_K(\mathbb{C})$, we define the Hecke orbit of z to be $\bigcup_{g \in \mathbf{P}(\mathbb{A}^f)} \mathcal{T}_g(z)$ and its rational Hecke orbit to be $\bigcup_{g \in \mathbf{P}(\mathbb{Q})} \mathcal{T}_g(z)$.

Let $(\mathbf{P},Y)$ be a Shimura datum, with $\mathbf{U} \subset \mathbf{P}$ the weight -2 unipotent part. Let $M = M_K(\mathbf{P},Y)$ be a Shimura variety at some neat level $K \subset \mathbf{P}(\mathbb{A}^f)$ (compact open subgroup). Take $z \in M(\mathbb{C})$ contained in a connected component M^+ and write $\mathbb{O}(z)$ to be $M^+ \cap (\bigcup_{g \in \mathbf{P}(\mathbb{Q})} \mathcal{T}_g(z))$

Definition 1.3.2 (Shimura subvariety and special subvariety). A subdatum $(\mathbf{P}_1, Y_1)$ of $(\mathbf{P},Y)$ is characterized by a $\mathbb{Q}$-subgroup $\mathbf{P}_1 \subset \mathbf{P}$ and an inclusion $Y_1 \subset Y$ induced by $\mathfrak{Y}(\mathbf{P}_1) \subset \mathfrak{Y}(\mathbf{P})$ such that $(\mathbf{P}_1, Y_1)$ is a Shimura datum itself. In this case $Y_1 = \mathbf{U}_1(\mathbb{C})\mathbf{P}_1(\mathbb{R})y_1$ for some $y_1 \in Y$ with $\mathbf{U}_1 = \mathbf{P}_1 \cap \mathbf{U}$.

The Shimura subvariety (of finite level) associated to the subdatum $(\mathbf{P}_1, Y_1) \subset (\mathbf{P},Y)$ is the image of the morphism of Shimura varieties $M_{K_1}(\mathbf{P}_1, Y_1) \to M_K(\mathbf{P},Y)$ with $K_1 = \mathbf{P}_1(\mathbb{A}^f) \cap K$, namely $\wp_K(Y_1 \times \mathbf{P}_1(\mathbb{A}^f))$. And we define special subvarieties of $M_K(\mathbf{P},Y)$ to be the geometrically connected components of the Hecke translations of Shimura subvarieties.

Finally, we define weakly special subvarieties in a given (connected) Shimura variety to be a finite union of special subvarieties. For example, if $M_1 \subset M_K(\mathbf{P}, Y)$ is a special subvariety of a Shimura variety whose reflex field is E, then for any number field F containing E, the Gal_F-orbit of M_1 in $M_K(\mathbf{P}, Y)$, written as $\mathrm{Gal}_F \cdot M_1$, is weakly special, because the geometrically irreducible subvariety $M_1 \subset M$ is defined over a number field, and a Galois conjugate of a special subvariety remains special (in fact this is also true for conjugation by $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, cf.[]).

We apologize that this last notion of weakly special subvarieties does not agree with the way it is used in [Pink-1]. Within this thesis no ambiguity is caused because the related results in *loc.cit are not used.*

For example, being given a Shimura subdatum $(\mathbf{P}_1, Y_1) \subset (\mathbf{P}, Y)$ and $g \in \mathbf{P}(\mathbb{A}^f)$, write $K = \coprod_{i=1}^d g_i K'$, then $\mathcal{T}_g$ translate a closed point $[y, aK] \in M_K(\mathbf{P}, Y)(\mathbb{C})$ to the finite set $\{[y, ag_i K] | i = 1, \dots, d\}$. In particular the set of geometrically connected components of $\mathcal{T}_g(M_{K_1}(\mathbf{P}_1, Y_1)_{\mathbb{C}}$ is the finite set

$$\{\wp_K(Y_1^+ \times a_j g_i K) : j \in \mathfrak{R}_{K_1}^{\mathbf{P}_1},\ i = 1, \dots, d\}.$$

So it is equivalent to define special subvarieties to be subvarieties of the form $\wp_K(Y_1^+ \times aK)$ where Y_1^+ comes from some Shimura subdatum $(\mathbf{P}_1, Y_1)$ and $a \in \mathbf{P}(\mathbb{A}^f)$. Since for any $q \in \mathbf{P}(\mathbb{Q})$, qY_1^+ is still deduced from a Shimura subdatum $(q\mathbf{P}_1 q^{-1}, qY_1)$ as long as $(\mathbf{P}_1, Y_1)$ is a subdatum, for a given special subvariety we can choose the expression $M_1 = \wp(Y_1^+ \times aK)$ in such a way that a is taken from the given finite set $\mathfrak{R}_K^{\mathbf{P}}$ representing $\mathbf{P}(\mathbb{Q})_+ \backslash \mathbf{P}(\mathbb{A}^f)/K$. Then via $\gamma = \gamma_K$ we see that $\gamma(M_1) = \Gamma_K(a) \backslash \Gamma_K(a) Y_1^+ \subset \Gamma_K(a) \backslash Y^+$, so special subvarieties are nothing but arithmetically defined locally symmetric subvarieties in $\Gamma \backslash Y^+$ in the prescribed form as above.

Definition 1.3.3 (Mumford-Tate group). (1) For $(\mathbf{P}, Y)$ a Shimura datum, and a subset $Z \subset Y$, the generic Mumford-Tate group of Z is defined to be the smallest $\mathbb{Q}$-subgroup of $\mathbf{P}$ whose complex locus contains $y(\mathbb{S}_{\mathbb{C}})$ for all $y \in Z$, denoted as $\mathbf{MT}(Z)$. When $Z = Y$, it is easy to verify that $(\mathbf{MT}(Y), Y)$ itself is a Shimura datum. Such pairs are called "irreducible" Shimura data by R.Pink in [Pink0] Chap.2.

Note that although the connected components remain of the form $\Gamma \backslash Y^+$ for certain a subgroup $\Gamma \subset \mathbf{P}(\mathbb{Q})$, the total Shimura varieties/schemes associated to $(\mathbf{MT}(Y), Y)$ and $(\mathbf{P}, Y)$ could be different, because the groups of connected components might differ from each other. For example consider a special datum $(\mathbf{H}, x)$ and $\mathbf{T}$ a $\mathbb{Q}$-torus containing $\mathbf{H}$. Then $(\mathbf{T}, x)$ is itself naturally a special datum, and $\pi_0(M(\mathbf{T}, x)) = \bar{\pi}_0\pi(\mathbf{T})$, which is in general different from $\pi_0(M(\mathbf{H}, x)) = \bar{\pi}_0\pi(\mathbf{H})$.

For a subdatum $(\mathbf{P}_1, Y_1) \subset (\mathbf{P}, Y)$, the Mumford-Tate group $\mathbf{MT}(Y_1)$ is independent of the datum $(\mathbf{P}, Y)$ containing it.

For a special subvariety $M_1 = \wp_K(Y_1^+ \times aK) \subset M_K(\mathbf{P}, Y)_{\mathbb{C}}$ with a taken from the fixed finite set $\mathfrak{R}_K^{\mathbf{P}}$, different choices of defining data $(\mathbf{P}_1, Y_1)$ lead to different $\mathbf{MT}(Y_1)$. Since we have fixed a, these choices are permuted transitively by $\Gamma_K(a)$-conjugations, and we define the Mumford-Tate group relative to level K and class $a \in \mathfrak{R}_K^{\mathbf{P}}$ to be the $\Gamma_K(a)$-conjugacy class of the $\mathbb{Q}$-subgroups $\mathbf{P}_1$, denoted as $\mathbf{MT}_{K,a}(M_1) = [\mathbf{P}_1]_{K,a}$, namely we take into consideration various liftings of M_1 in Y^+. Often we simply write $\mathbf{MT}(M_1) = [\mathbf{P}_1]$ if the pair (K, a) is clear in the context, and in most cases we only need a representative $\mathbf{P}_1$ in $[\mathbf{P}_1]$, which is referred to as the Mumford-Tate group by abuse of terminology.

(2) On the other hand, for a variation of polarizable rational mixed Hodge structures $(\mathcal{V}, F^\bullet, W_\bullet)$ over a connected complex analytic variety S we can define the Mumford-Tate group pointwisely, namely for $s \in S$ put $\mathbf{MT}(s) = \mathbf{MT}(\mathcal{V}_s, F_s, W_s)$. Identify $\mathcal{V}_s$ with a fixed fiber V of the pull-back of $\mathcal{V}$ to the universal covering of S, then these $\mathbf{MT}(s)$ are realized as $\mathbb{Q}$-subgroups of $GL_{\mathbb{Q}}(V)$.

Y.André showed that away from a countable union of closed subvarieties the $\mathbf{MT}(s)$'s coincide, and is called the generic Mumford-Tate group of $\mathcal{V}$ over S, denoted as $\mathbf{MT}(S) = \mathbf{MT}(S, \mathcal{V})$. For arbitrary $s \in S$, $\mathbf{MT}(s)$ is a $\mathbb{Q}$-subgroup of $\mathbf{MT}(S)$: actually we can lift $\mathcal{V}$ to the universal covering $\tilde{S}$ of S and get a constant sheaf of fiber V, and all the $\mathbf{MT}(s)$'s are realized as subgroups of $\mathbf{GL}_{\mathbb{Q}}(V)$.

For example, consider a Shimura datum $(\mathbf{P}, Y)$ and the adjoint representation $\mathbf{P} \to GL_{\mathbb{Q}}(\mathbf{Lie}\,\mathbf{P})$. Then we get, due to the universal property of Y, a variation of polarizable mixed Hodge structures $\mathfrak{P}$ on $M_K(\mathbf{P}, Y)_{\mathbb{C}}$, $K \subset \mathbf{P}(\mathbb{A}^f)$ any compact open subgroup. If the adjoint representation is faithful, e.g. $\mathbf{P}$ an adjoint $\mathbb{Q}$-group, then $\mathbf{P}$ is the generic Mumford-Tate group of $\mathfrak{P}$. In general we have the following

Proposition 1.3.4. *(1) (cf.[André4]) Let $\mathcal{V}$ be a variation of polarizable rational mixed Hodge structures on a complex analytic variety* S. *Then there exists countably many closed analytic subvarieties $(S_n)_n$ of* S *such that* $\mathbf{MT}(s)$ *is constant when s varies in* $S - \bigcup_n S_n$.

(2) Let $(\mathbf{P}, Y)$ be a Shimura datum with $\mathbf{P} = \mathbf{MT}(Y)$. Take $\mathbf{P} \to GL_{\mathbb{Q}}(\mathbf{V})$ to be an algebraic representation, and $\mathcal{V}$ the induced variation of mixed Hodge structures on $M = M_K(\mathbf{P}, Y)$ *for some compact open subgroup* $K \subset \mathbf{P}(\mathbb{A}^f)$. *Then the generic Mumford-Tate group of $\mathcal{V}$ equals the image of $\mathbf{P}$ in* $GL_{\mathbb{Q}}(\mathbf{V})$.

Remark 1.3.5. In [Pink0], R.Pink proceeded with a slightly different definition of (mixed) Shimura data: a Shimura datum "à la Pink" is a pair $(\mathbf{P}, \mathbf{U}, \mathbf{Y} \xrightarrow{h} \mathfrak{Y}(\mathbf{P}))$ where $\mathbf{P}$ is a connected linear $\mathbb{Q}$-group, $\mathbf{U} \subset \mathbf{P}$ a unipotent invariant $\mathbb{Q}$-subgroup of $\mathbf{P}$, $\mathbf{Y}$ a homogeneous space under $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$, and h a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-equivariant map from $\mathbf{Y}$ to $\mathfrak{Y}(\mathbf{P}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}_{\mathbb{C}}, \mathbf{P}_{\mathbb{C}})$ such that h is of finite fibers and that $(\mathbf{P}, \mathbf{U}, h(\mathbf{Y}))$ is a Shimura datum in our sense (following P.Deligne). Then in [Pink0] R.Pink developed the theory of mixed Shimura varieties within this framework, including the canonical models of various compactifications.

We present the following lemma describing the difference between these two definitions that would be of interest in the study of special subvarieties:

Lemma 1.3.6. *(1) Let* $(\mathbf{P},\mathrm{Y})$ *be a Shimura datum in the sense of R.Pink, and* $(\mathbf{G},\mathbf{X})$ *a pure section, namely a pure Shimura datum defined by some Levi* $\mathbb{Q}$*-subgroup* $\mathbf{G}$ *of* $\mathbf{P}=\mathbf{W}\rtimes\mathbf{G}$*, and morphisms* $(\mathbf{P},\mathrm{Y})\underset{i}{\overset{\pi_{\mathbf{W}}}{\rightleftarrows}}(\mathbf{G},\mathbf{X})$*. Then for any* $\mathbf{x}\in\mathbf{X}$*, the isotropic subgroup* $\mathrm{G}_{\mathbf{x}}\subset\mathbf{G}(\mathbb{R})$ *of* $\mathbf{x}$ *is a Lie subgroup of* $\mathbf{G}(\mathbb{R})$*, whose reduction modulo* $\mathrm{Z}_{\mathbf{G}}(\mathbb{R})$ *is a compact subgroup of* $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ *of maximal dimension, and* $\mathbf{Y}\cong\mathbf{U}(\mathbb{C})\mathbf{W}(\mathbb{R})\times\mathbf{X}$ *as a real analytic manifold.*

(2) Let $(\mathbf{P},\mathrm{Y})$ *be a Shimura datum in the sense of P.Deligne,* $(\mathbf{P}_1,\mathbf{Y}_1)$ *a subdatum in the sense of R.Pink. Then it is a subdatum in the sense of P.Deligne. As a corollary, let* M *be a Shimura variety in the sense of P.Deligne, and* $\mathrm{M}_1\subset\mathrm{M}$ *a special subvariety in the sense of R.Pink, then* M_1 *is a special subvariety in the sense of P.Deligne.*

Proof. (1) From the datum $(\mathbf{P},h:\mathbf{Y}\to\mathfrak{Y}(\mathbf{P}))$, we see that the pair $(\mathbf{P},\mathrm{Y}=h(\mathbf{Y}))$ is a Shimura datum in our sense, and the map $h:\mathbf{Y}\twoheadrightarrow\mathrm{Y}$ is a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-equivariant finite covering. $\mathbf{G}$ is a Levi $\mathbb{Q}$-subgroup of $\mathbf{P}$. For any $\mathbf{x}\in\mathbf{Y}$, there exists a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-conjugate $\mathbf{x}'$ of $\mathbf{x}$ such that the $\mathbb{C}$-torus $h(\mathbf{x}')$ is contained in $\mathbf{G}_{\mathbb{C}}$, and equals the image under the push-forward by $i\circ\pi_{\mathbf{W}}$. Now that $\pi_{\mathbf{U},*}(h(\mathbf{x}'))$ is defined over $\mathbb{R}$, so it is with $\pi_{\mathbf{W},*}(h(\mathbf{x}'))$, hence $h(\mathbf{x}')=(i\circ\pi_{\mathbf{W}})_*(h(\mathbf{x}'))$ is itself defined over $\mathbb{R}$. To prove the assertion it suffices to treat the case where $\mathbf{x}=\mathbf{x}'$ as they are all conjugated. In this case $x=h(\mathbf{x})\in\mathrm{Y}=h(\mathbf{Y})$ is defined over $\mathbb{R}$ and satisfies the conditions defining a pure Shimura datum in the sense of P.Deligne. Now $\mathrm{Y}=\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})x$ and the pair $(\mathbf{G},\mathrm{X}=\mathbf{G}(\mathbb{R})x)$ is a pure Shimura subdatum of $(\mathbf{P},\mathrm{Y})$. Write G_x to be the isotropic subgroup of $x\in\mathrm{X}$ for the action of $\mathbf{G}(\mathbb{R})$, then $\mathrm{G}_x=\mathbf{G}_x(\mathbb{R})$ where $\mathbf{G}_x$ is the $\mathbb{R}$-subgroup of $\mathbf{G}_{\mathbb{R}}$ fixing $x\in\mathfrak{X}(\mathbf{G})$. By calculating the Hodge types we see that the isotropic subgroup of x in $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$ equals G_x, hence $\mathrm{X}\cong\mathbf{G}(\mathbb{R})/\mathrm{G}_x$ and $\mathrm{Y}\cong\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})/\mathrm{G}_x\cong\mathbf{U}(\mathbb{C})\mathbf{W}(\mathbb{R})\times\mathrm{X}$. $h:\mathbf{Y}\to\mathrm{Y}$ is a $\mathbf{U}(\mathbb{C})\mathbf{P}(\mathbb{R})$-equivariant finite covering, thus the isotropic group of $\mathbf{x}$, denoted as $\mathrm{G}_{\mathbf{x}}$, is a closed subgroup of finite index in G_x, hence an open subgroup of G_x; in particular they are of the same dimension.

By the general theory of non-compact symmetric domains, we know that the reduction of G_x modulo $\mathrm{Z}_{\mathbf{G}}(\mathbb{R})^+$ is a maximal compact subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$. Hence $\mathrm{G}_{\mathbf{x}}$ mod $\mathrm{Z}_{\mathbf{G}}(\mathbb{R})$ is also of maximal dimension among compact subgroups in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$.

Note that the set of connected components of G_x is finite, and there is only finitely many choices of $\mathrm{G}_{\mathbf{x}}$.

It is also easy to verify that $(\mathbf{G},\mathrm{X}=h(\mathbf{X}))$ is a pure section of $(\mathbf{P},\mathrm{Y})$.

(2) Since $(\mathbf{P}_1,h_1:\mathbf{Y}_1\to\mathfrak{Y}(\mathbf{P}_1))$ is a subdatum of $(\mathbf{P},h:\mathrm{Y}\hookrightarrow\mathfrak{Y}(\mathbf{P}))$, h_1 is the restriction of h to $\mathbf{Y}_1\subset\mathrm{Y}$. Thus h_1 is injective and $\mathbf{Y}_1\cong h_1(\mathbf{Y}_1)$ and $(\mathbf{P}_1,\mathbf{Y}_1)$ is a Shimura datum in our sense. The rest is trivial. ■

Lemma 1.3.7. *Let* $(\mathbf{P}_1,\mathrm{Y}_1)\subset(\mathbf{P},\mathrm{Y})$ *be a Shimura subdatum (in the sense of P.Deligne),*

$K \subset \mathbf{P}(\mathbb{A}^f)$ *a neat compact open subgroup, and* $K_1 = K \cap \mathbf{P}_1(\mathbb{A}^f)$. *Then the morphism of Shimura varieties*

$$f : M_{K_1}(\mathbf{P}_1, Y_1) \longrightarrow M_K(\mathbf{P}, Y)$$

is generically injective.

Proof. This is just a word-to-word translation of the Lemma 2.2 in [UY-1] ■

We mention the notion of $\mathbf{C}$-special subdata and $\mathbf{C}$-special subvarieties, which will be widely used in later chapters

Definition 1.3.8. (cf.[UY-1] Definition 3.1, 3.2) We fix $(\mathbf{P}, Y)$ a mixed Shimura datum with a pure section $(\mathbf{G}, X)$, and let $\mathbf{C}$ be a $\mathbb{Q}$-torus in $\mathbf{G}$. Write $\pi : (\mathbf{P}, Y) \to (\mathbf{G}, X)$ for the canonical projection modulo $\mathbf{W}$ the unipotent radical of $\mathbf{P}$.

(1) (the pure case:) a pure subdatum $(\mathbf{G}', X') \subset (\mathbf{G}, X)$ is $\mathbf{C}$-special if $\mathbf{C}$ equals the connected center of $\mathbf{G}'$.

(2) (the mixed case:) a subdatum $(\mathbf{P}', Y')$ of $(\mathbf{P}, Y)$ is $\mathbf{C}$-special if $(\mathbf{G}', X') = (\pi(\mathbf{P}'), \pi(Y'))$ is a $\mathbf{C}$-special subdatum of $(\mathbf{G}, X)$.

We have seen that the unipotent radical $\mathbf{W}'$ of $\mathbf{P}'$ equals $\mathbf{W} \cap \mathbf{P}'$, and if $(\mathbf{G}_1, X_1)$ is a pure section of $(\mathbf{P}', Y')$, then the composition

$$(\mathbf{G}_1, X_1) \hookrightarrow (\mathbf{P}', Y') \hookrightarrow (\mathbf{P}, Y) \twoheadrightarrow (\mathbf{G}, X)$$

induces an isomorphism between $(\mathbf{G}_1, X_1)$ and $(\pi(\mathbf{P}'), \pi(Y'))$.

(3) A Shimura subvariety M' of $M_K(\mathbf{P}, Y)$ is $\mathbf{C}$-special if it is defined by an inclusion of $\mathbf{C}$-special subdatum.

More generally, for a given compact open subgroup $K \subset \mathbf{P}(\mathbb{A}^f)$ and a fixed finite set of representatives $\mathfrak{R}_K^{\mathbf{P}}$ of $\mathbf{P}(\mathbb{Q})_+ \backslash \mathbf{P}(\mathbb{A}^f)/K$, a special subvariety $M' = \wp_K(Y'^+ \times aK)$ with $a \in \mathfrak{R}_K^{\mathbf{P}}$ is $\mathbf{C}$-special if Y'^+ comes from a $\mathbf{C}$-special subdatum $(\mathbf{P}', Y')$.

Remark 1.3.9. In our later applications, we are mainly concerned with the case where K is of the form $K_{\mathbf{W}} \rtimes K_{\mathbf{G}}$ and $\mathfrak{R}_K^{\mathbf{P}} = \mathfrak{R}_{K_{\mathbf{G}}}^{\mathbf{G}}$ is fixed once for all. Very often it suffices to study special subvarieties inside a fixed connected Shimura variety $M^+ = \Gamma \backslash Y^+$ where $\Gamma = \Gamma_K(a)$, namely $M^+ = \wp_K(Y^+ \times aK)$, and the special subvarieties are $M'^+ = \Gamma \backslash \Gamma Y'^+ = \wp(Y'^+ \times aK)$ for $(\mathbf{P}', Y')$ subdatum of $(\mathbf{P}, Y)$. If we change a be a $\mathbf{P}(\mathbb{Q})_+$-conjugation, then all the $(\mathbf{P}', Y')$ are modified coherently.

Finally we present some results concerning the construction of Shimura subdata.

Definition-Proposition 1.3.10. (cf. [CU-3] 4.1 strongly special subvarieties; [E-M-S] Lemma 5.1) Being given $\mathbf{G}$ be a reductive $\mathbb{Q}$-group, a reductive $\mathbb{Q}$-subgroup $\mathbf{H} \subset \mathbf{G}$ is said to be strong, if the following equivalent conditions hold:

(1) For any $\mathbb{Q}$-subgroup $\mathbf{Q} \subset \mathbf{G}$, $\mathbf{Q} \supset \mathbf{H}$ implies that $\mathbf{Q}$ is reductive itself;

(2) For any parabolic $\mathbb{Q}$-subgroup $\mathbf{P} \subset \mathbf{G}$, $\mathbf{P} \supset \mathbf{H}$ implies that $\mathbf{P} = \mathbf{G}$;

(3) The centralizer of $\mathbf{H}$ in $\mathbf{G}$, denoted as $\mathbf{Z}_{\mathbf{G}}\mathbf{H}$, is isogeneous to a product of the form $\mathbf{Z}_{\mathbf{G}}^{\circ} \times \mathbf{C}$ where $\mathbf{C}$ is a $\mathbb{Q}$-anisotropic $\mathbb{Q}$-torus in $\mathbf{G}^{\mathrm{der}}$, where being $\mathbb{Q}$-anisotropic means that there is no split $\mathbb{Q}$-torus contained in $\mathbf{G}$.

Note that the original version in loc.cit only considers the case where **G** and **H** semi-simple; our variant here is obtained by joining a common central $\mathbb{Q}$-torus.

Proof. It suffices to treat the case where **G** is semi-simple, whereby (3) is reformulated as:

(3)' The centralizer $\mathbf{Z_G H}$ of **H** in **G** is F-anisotropic, namely contains no split F-torus.

The proof of the equivalence is essentially the same as in Lemma 5.1 of[E-M-S], where a series of equivalences is established for $\mathbb{Q}$-groups. We can either

(a) replace $\mathbb{Q}$ by F in the proof of $2 \Rightarrow 3$, $3 \Rightarrow 2$, $3 \Rightarrow 5$, $4 \Rightarrow 5$, and $5 \Rightarrow 3$, $5 \Rightarrow 4$; or

(b) in case F is a number field, study the $\mathbb{Q}$-groups $\mathbf{G}^F$, $\mathbf{H}^F$, etc. by Weil restrictions, where the equivalences follow from *loc.cit* directly. ∎

∎

Example 1.3.11. Let $(\mathbf{G},X)$ be a pure Shimura datum, and $(\mathbf{H},X_1)$ a pure subdatum. Then **H** is a strong $\mathbb{Q}$-subgroup of **G**.

It suffices to verify that $\mathbf{Z_G H}$ the centralizer of **H** in **G** is $\mathbb{Q}$-anisotropic modulo $\mathbf{Z_G}$. Take $x \in X_1 \subset X$, then $(\mathbf{Z_G H})_{\mathbb{R}}$ centralizes $x(\mathbb{S}) \subset \mathbf{G}$, hence its reduction modulo $\mathbf{Z_G}$ is compact, namely $\mathbb{R}$-anisotropic, hence $\mathbb{Q}$-anisotropic.

In particular, the center of **H** fixes x, and we deduce that the center of **H** is compact modulo $\mathbf{Z_G}$. As a consequence, when we talk about **C**-special subdata of $(\mathbf{G},X)$ with $\mathbf{G} = \mathbf{MT}(X)$, **C** should be, a priori, a $\mathbb{Q}$-torus containing $\mathbf{Z_G}$ which is also compact modulo $\mathbf{Z_G}$.

Lemma 1.3.12. *(1) Let* $(\mathbf{P},Y)$ *be a mixed Shimura datum with pure section* $(\mathbf{G},X)$*, and* $\mathbf{Q} \subset \mathbf{P}$ *a* $\mathbb{Q}$*-subgroup such that* $\mathbf{Q}_{\mathbb{C}} \supset y(\mathbb{S}_{\mathbb{C}})$ *for some* $y \in Y$*. Then there exists a Shimura subdatum of the form* $(\mathbf{P}_1,Y_1)$ *with* $Y_1 = \mathbf{U}_1(\mathbb{C})\mathbf{Q}(\mathbb{R})y$*, where* $\mathbf{U}_1 = \mathbf{W} \cap \mathbf{Q}$*. If moreover y is defined over* $\mathbb{R}$ *and* $\mathbf{Q}_{\mathbb{R}} \supset y(\mathbb{S})$*, then* $Y_{1\mathbb{R}} = \mathbf{Q}(\mathbb{R})y$

(2) Let $(\mathbf{P},Y)$ *be a Shimura datum with a subdatum* $(\mathbf{P}_1,Y_1)$*. Then the set of Shimura subdata of the form* $(\mathbf{P}_1,Y_1')$ *is finite.*

Proof. (1) Let $x = \pi_{\mathbf{W}*}y \in X$ and $\mathbf{H} = \pi_{\mathbf{W}}(\mathbf{Q}) \subset \mathbf{G}$. We first show that **H** is reductive. It suffices to show that $\mathbf{H}_{\mathbb{R}}$ is reductive. But $\mathbf{T}_x = x(\mathbb{S}) \subset \mathbf{G}_{\mathbb{R}}$ is strong, i.e. $\mathbf{Z}_{\mathbf{G}_{\mathbb{R}}}\mathbf{T}_x$ is compact modulo $\mathbf{Z}_{\mathbf{G}_{\mathbb{R}}}$ which follows from the definition of pure Shimura datum. Hence by the equivalent conditions in the definition above and the chain $\mathbf{T}_x \subset \mathbf{H}_{\mathbb{R}} \subset \mathbf{G}_{\mathbb{R}}$ implies that **H** is reductive, and **Q** admits a Levi-decomposition of the form $\mathbf{W}_1 \rtimes \mathbf{L}$ where $\mathbf{L} = w\mathbf{H}w^{-1}$ for some $w \in \mathbf{W}(\mathbb{Q})$ and $\mathbf{W}_1 = \mathbf{W} \cap \mathbf{Q}$. We put also $\mathbf{U}_1 = \mathbf{U} \cap \mathbf{Q}$, which is central in $\mathbf{W}_1$.

Conjugate x by w we get $z = \mathbf{Int}(w) \circ x : \mathbb{S} \to \mathbf{L}_{\mathbb{R}}$ which satisfies the conditions defining a pure Shimura datum. But **L** is not exactly the $\mathbb{Q}$-group that intervenes in pure Shimura data. **L** admits a decomposition into almost direct product $\mathbf{L} = \mathbf{G}_1 \tilde{\times} \mathbf{L}^c$, where $\mathbf{L}^c$ is the product of compact $\mathbb{Q}$-factors, and $\mathbf{G}_1$ is generated by the center of $\mathbf{L}$ and the non-compact $\mathbb{Q}$-factors of $\mathbf{L}^{\mathrm{der}}$. By definition of pure Shimura data, $\mathbf{G}_{1\mathbb{R}} \supset z(\mathbb{S})$ and $\mathbf{L}^{\mathrm{c}}_{\mathbb{R}}$ fixes z. Hence the $\mathbf{U}_1(\mathbb{R})\mathbf{Q}(\mathbb{R})$-orbit of z is equal to $\mathrm{Y}_1 := \mathbf{U}_1(\mathbb{C})\mathbf{P}_1(\mathbb{R})z$ where $\mathbf{P}_1 = \mathbf{W}_1 \rtimes \mathbf{G}_1$. The pair $(\mathbf{P}_1, \mathrm{Y}_1)$ is a Shimura subdatum of $(\mathbf{P}, \mathrm{Y})$ with a pure section $(\mathbf{G}_1, \mathbf{G}_1(\mathbb{R})z)$.

If moreover $\mathbf{Q}_{\mathbb{R}} \supset y(\mathbb{S})$ for some $y \in \mathrm{Y}_{\mathbb{R}}$. Then from the construction above we see that $y(\mathbb{S})$ is contained in some Levi $\mathbb{R}$-subgroup of $\mathbf{Q}$, which is of the form $w_y \mathbf{L}_{\mathbb{R}} w_y^{-1}$ with $y = \mathbf{Int}(w_y) \circ z$. The compact $\mathbb{R}$-group $w_y \mathbf{L}^{\mathrm{c}}_{\mathbb{R}} w_y^{-1}$ fixes y, and y has image in $w_y \mathbf{G}_{1\mathbb{R}} w_y^{-1}$. Therefore $\mathbf{Q}(\mathbb{R})y = \mathbf{P}_1(\mathbb{R})y = \mathbf{P}_1(\mathbb{R})z = \mathrm{Y}_{1\mathbb{R}}$.

(2) Let $(\mathbf{G}, \mathrm{X})$ be a pure section of $(\mathbf{P}, \mathrm{Y})$. Then $\pi_{\mathbf{W}}(\mathbf{P}_1, \mathrm{Y}_1) = (\mathbf{G}_1, \mathrm{X}_1)$ is a pure subdatum of $(\mathbf{G}, \mathrm{X})$. Lemma 3.7 in [UY] shows the finiteness of pure subdata of the form $(\mathbf{G}_1, \mathrm{X}'_1)$. Back to the total $\mathbb{Q}$-group $\mathbf{P}_1$, we see that the kernel of $\mathbf{P}_1 \to \mathbf{G}_1$ is $\mathbf{W}_1 = \mathbf{P}_1 \cap \mathbf{W}$, and $\mathbf{U}_1(\mathbb{C})\mathbf{W}_1(\mathbb{R})$ remains the same for any $(\mathbf{G}_1, \mathrm{X}_1)$. Hence there is only finitely many Shimura subdata defined by $\mathbf{P}_1$. ■

Lemma 1.3.13. *Let* $\mathrm{M} = \mathrm{M}_{\mathrm{K}}(\mathbf{P}, \mathrm{Y})$ *be a Shimura variety,* $(\mathbf{P}_i, \mathrm{Y}_i)$ *Shimura subdatum of* $(\mathbf{P}, \mathrm{Y})$*, and* M_i *a special subvariety defined by* $(\mathbf{P}_i, \mathrm{Y}_i)$ $(i = 1, 2)$*. If* $\mathrm{M}_1 \cap \mathrm{M}_2 \neq \emptyset$*, then* $\mathrm{M}_1 \cap \mathrm{M}_2$ *is a finite union of special subvarieties.*

Proof. Fix a finite set $\mathfrak{R} = \mathfrak{R}^{\mathbf{P}}_{\mathrm{K}}$ representing $\mathbf{P}(\mathbb{Q})_+ \backslash \mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}$. Suppose $\mathrm{M}_1 \cap \mathrm{M}_2 \neq 0$, then they lie in a common connected component $\wp_{\mathrm{K}}(\mathrm{Y}^+ \times a\mathrm{K}) = \Gamma \backslash \mathrm{Y}^+$, and are of the form $\mathrm{M}_i = \wp_{\mathrm{K}}(\mathrm{Y}_i^+ \times a\mathrm{K}) = \Gamma \backslash \Gamma \mathrm{Y}_i^+$ for some subdatum $(\mathbf{P}_i, \mathrm{Y}_i)$, where $\Gamma = \Gamma_{\mathrm{K}}(a)$ $(i = 1, 2)$.

Take $z \in \mathrm{M}_1 \cap \mathrm{M}_2$, which is lifted to some $y_i \in \mathrm{Y}_i^+$.So y_1 and y_2 lies in the same Γ-orbit in Y^+. Because conjugating $(\mathbf{P}_i, \mathrm{Y}_i)$ by $\gamma \in \Gamma$ does not change the intersection $\mathrm{M}_1 \cap \mathrm{M}_2$, we may suppose, up to conjugating $(\mathbf{P}_2, \mathrm{Y}_2)$ suitably, that $y_1 = y_2 = y$. In particular $\mathrm{Y}_1 \cap \mathrm{Y}_2 \neq \emptyset$ as subsets of Y, and $\mathbf{P}_1 \cap \mathbf{P}_2 = \mathbf{Q} \neq \emptyset$. Now that $\mathbf{Q}$ is a $\mathbb{Q}$-subgroup of $\mathbf{P}$ such that $\mathbf{Q}_{\mathbb{C}} \supset y(\mathbb{S}_{\mathbb{C}})$. By the arguments in Lemma 1.3.13 above, $\mathbf{Q}$ admits a Levi-decomposition $\mathbf{Q} = \mathbf{W}_3 \rtimes \mathbf{L}_3$ with unipotent radical $\mathbf{W}_3 = \mathbf{W} \cap \mathbf{Q}$, and a $\mathbb{Q}$-subgroup $\mathbf{P}_3 = \mathbf{W}_3 \rtimes \mathbf{G}_3$ such that $\mathbf{G}_3 \subset \mathbf{L}_3$ is an invariant $\mathbb{Q}$-subgroup and that $(\mathbf{P}_3, \mathrm{Y}_3)$ is a common Shimura subdatum of $(\mathbf{P}_1, \mathrm{Y}_1)$ and $(\mathbf{P}_2, \mathrm{Y}_2)$, where Y_3 is taken to be the orbit $\mathbf{U}_3(\mathbb{C})\mathbf{Q}(\mathbb{R})y$, with $\mathbf{U}_3 = \mathbf{W}_3 \cap \mathbf{U}$.

We proceed to show that only finitely many subdata are produced in the above way. The subdata constructed in the precedent paragraph are of the form $(\mathbf{P}_3, \mathrm{Y}_3$ with $\mathrm{Y}_3 = \mathbf{U}_3(\mathbb{C})\mathbf{P}_3(\mathbb{R})y = \mathbf{U}_3\mathbf{Q}(\mathbb{R})y)$ for some $y \in \mathrm{Y}_1 \cap \mathrm{Y}_2$. The $\mathbb{Q}$-group $\mathbf{P}_3$ is determined by $\mathbf{Q} = \mathbf{P}_1 \cap \mathbf{P}_2$ independent of $y \in \mathrm{Y}_1 \cap \mathrm{Y}_2$. If $\mathbf{Q} = \mathbf{W}_3 \rtimes \mathbf{L}_3$ is a Levi decomposition which extends to a Levi decomposition $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ with $\mathbf{L}_3 \subset \mathbf{G}$, then $\mathbf{G}_3 \lhd \mathbf{Q}$ is independent of $y \in \mathrm{Y}_1 \cap \mathrm{Y}_2$, because it is the maximal normal $\mathbb{Q}$-subgroup of $\mathbf{L}_3$ such that $\mathbf{L}_3/\mathbf{G}_3$ is either compact semi-simple or trivial. For $\mathbf{P}_3$ fixed, there is at most finitely many subdata of the form $(\mathbf{P}_3, \mathrm{Y}')$, hence only finitely special subvarieties $\mathrm{S}_1, \ldots, \mathrm{S}_{\mathrm{N}}$ are defined by $\mathbf{P}_3$ and contained in $\mathrm{M}_1 \cap \mathrm{M}_2$.

The construction above implies the equality $\mathrm{M}_1 \cap \mathrm{M}_2(\mathbb{C}) = \bigcup_{i=1}^{\mathrm{N}} \mathrm{S}_i(\mathbb{C})$, which forces $\mathrm{M}_1 \cap \mathrm{M}_2 = \bigcup_i \mathrm{S}_i$ weakly special. ■

Chapter 2

Introduction to the André-Oort conjecture

2.1 Statement of the conjecture

Conjecture 2.1.1. *(the André-Oort conjecture, cf. [André-4], [Oort-2]) The following conjecture was raised by Y.André and F.Oort independently around 1990's :*

(0) Let M *be a pure Shimura variety, and* Σ *a set of special points. Then the Zariski closure of* Σ *in* M *is weakly special, namely a finite union of (pure) special subvarieties.*

Note that we have abused the terminology *weakly special*, which was reserved in R.Pink's article [Pink-1] for a more general class of subvarieties and for his generalized André-Oort conjecture. They are not to be treated in this writing and no confusion is caused.

It is easy to show that a special subvariety contains a Zariski dense subset of special points: obviously the rational Hecke orbit of a special point with respect to the Mumford-Tate group of the special subvariety is already dense for the archimedean topology (in the pure case). But in this case the Mumford-Tate groups of these special points are isomorphic to each other as $\mathbb{Q}$-groups: in fact they are conjugate under $\mathbf{G}(\mathbb{Q})$, $\mathbf{G}$ being the Mumford-Tate group of the special subvariety. It is less obvious that there exists sequences of special points dense for the Zariski topology whose Mumford-Tate groups form an infinite set of $\mathbb{Q}$-isomorphism classes. For example consider a sequence of special points $(x_n)_n$ in the modular curve Y(N) with distinct discriminants D_n, then it is not covered by the rational Hecke orbit of a single point. But the set $\{x_n : n \in \mathbb{N}\}$ is Zariski dense: it is an infinite set in a curve. In fact we even know that their $\mathrm{Gal}_{\mathbb{Q}}$-orbits are equidistributed with respect to the canonical measure on the modular curve, proved by W.Duke by explicit estimation via automorphic forms, cf.[Duke-1].

The André-Oort conjecture and its various generalizations are mainly concerned with the higher-dimensional analogue of this phenomenon.

Proposition 2.1.2 (Equivalent formulations of André-Oort). *The following equivalent forms of the conjecture are better adapted for different approaches:*

(1) Let $(\mathrm{S}_n)_n$ *be a sequence of special subvarieties of a pure Shimura variety* M, *then the Zariski closure of* $\bigcup_n \mathrm{S}_n$ *is weakly special.*

(2) Let Z *be a closed subvariety of a pure Shimura variety* M. *Set*

$$\Sigma(\mathrm{Z}) = \{\mathrm{S}_1 \subsetneq \mathrm{Z} : \mathrm{S}_1 \text{ special}\}$$

Then the set $\mathcal{S}(\mathrm{Z})$ *of maximal elements of* $\Sigma(\mathrm{Z})$ *is finite.*

(3) Fix a connected Shimura variety M^+, *identified with a locally symmetric space of the form* $\Gamma\backslash\mathrm{X}^+$ *for some Shimura datum* $(\mathbf{G},\mathrm{X})$ *and congruence subgroup* $\Gamma \subset \mathbf{G}(\mathbb{Q})^+$. *Then every strict sequence of special subvarieties is a generic sequence in the following sense:*

- *A sequence of closed subvarieties* $(\mathrm{Z}_n)_n$ *of* M^+ *is said to be generic if* $(\mathrm{Z}_n)_n$ *converges to the generic point of* M^+, *namely for any closed subvariety* $\mathrm{Z} \subsetneq \mathrm{M}^+$, $\mathrm{Z}_n \not\subseteq \mathrm{Z}$ *for n large enough;*
- *A sequence of special subvarieties* $(\mathrm{S}_n)_n$ *is said to be strict if for any special subvariety* $\mathrm{S} \subsetneq \mathrm{M}^+$ *we have* $\mathrm{S}_n \not\subseteq \mathrm{S}$ *for n large enough.*

The equivalence between the original version (0) and the version (3) is clear. The proof of the remaining equivalences will be given later in the more general context of mixed Shimura varieties, cf.Prop.2.2.5 of this chapter.

Later on we will introduce the notion of (in-)homogeneous sequence of special subvarieties. Then the theorem of L.Clozel and E.Ullmo can be viewed as a a proof of the conjecture (1) in the homogeneous case, and the work of B.Klingler, E.Ullmo, and A.Yafaev covers the inhomogeneous case under the formulation of (2) by assuming the Generalized Riemann Hypothesis for CM fields.

Although the question was raised for pure Shimura varieties, in Y.André's lectures [André-1] is already indicated how this problem could be formulated for mixed Shimura varieties, and this motivates the following:

Conjecture 2.1.3 (André-Oort-Pink). *Let* M *be a mixed Shimura variety, and* Σ *a set of special points. Then the Zariski closure of* Σ *is weakly special, i.e. a finite union of special subvarieties.(cf. [Pink1])*

Of course the André-Oort conjecture bears a strong analogy with the following

Conjecture 2.1.4 (Manin-Mumford). *For* A *an abelian variety over* $\mathbb{C}$, *define special points to be torsion points, and special subvarieties to be torsion subvarieties, namely abelian subvarieties translated by torsion points. Let* Σ *be a set of special points of* A. *Then the Zariski closure of* Σ *is weakly special, namely a finite union of special subvarieties.*

Both of the two conjectures are concerned with the distribution of special sub-objects inside a geometric structure with "a lot of symmetries": in the case of abelian varieties we have translations by torsion points and correspondences by endomorphisms (up to isogenies); in the case of Shimura varieties we have Hecke correspondences.

The Manin-Mumford conjecture was proved first by M.Raynaud (cf.[R-1],[R-2]), then resolved from various approaches by different mathematicians. It is closely related to the following results:

Theorem 2.1.5. *(L.Szpiro, E.Ullmo, S.Zhang; cf.[SUZ], [Zh]) Let* A *be an abelian variety over a number field* K, $\{x_n\}_n$ *a sequence of closed points of* A *such that*

(1) $\lim_{n\to\infty}\mathbf{h}(x_n)=0$, *where* $\mathbf{h}$ *is the Néron-Tate height function, and that*

(2) for any special subvariety $\mathrm{Z}\subsetneq \mathrm{A}$, $\mathrm{Z}\cap\{x_n\}$ *is finite.*

Write $\mathcal{O}(x_n)$ *for the* $\mathrm{Gal}_{\mathrm{K}}$*-orbit of* x_n *inside* A. *Then with respect to any complex embedding* $\sigma:\mathrm{K}\hookrightarrow\mathbb{C}$, *the sequence of measures* $\frac{1}{|\mathcal{O}(x_n)|}\sum_{y\in\sigma(\mathcal{O}(x_n))}\delta_y$ *weakly converges to the normalized Haar measure on the compact Lie group* $\mathrm{A}_\sigma(\mathbb{C})$.

We see that the Galois orbit of a sequence of "small points" (i.e. with height tending to zero) is equidistributed (i.e. the associated Dirac measures tends to the canonical Haar measure of the ambient variety). So it is natural to consider the following statement as a refinement of the André-Oort conjecture:

Conjecture 2.1.6 (the Equidistribution conjecture). *Let* S_n *be a strict sequence of special subvarieties of a pure Shimura variety* S. *Then the sequence of measures* $\frac{1}{|\mathcal{O}(\mathrm{S}_n)|}\sum_{\mathrm{Y}\in\mathcal{O}(\mathrm{S}_n)} i_{\mathrm{Y}*}\mu_{\mathrm{Y}}$ *converges weakly to the canonical measure on* S.

Here $\mathcal{O}(\mathrm{S}_n)$ *is the Galois orbit of the subvariety* S_n *inside* S, *and the canonical measure* μ_{Y} *on a locally symmetric variety* Y *as the complex analytic variety associated to the complex locus of the special variety means the measure deduced from the Haar measure of its universal covering (which is a symmetric domain). All fields encountered here are assumed to be equipped with a fixed embedding into the complex field* $\mathbb{C}$ *and we write simply* $\mathrm{Y}\in\mathcal{O}(\mathrm{S}_n)$ *instead of* $\mathrm{Y}\in\sigma(\mathcal{O}(\mathrm{S}_n))$ *for some fixed embedding* $\sigma:\mathrm{E}\to\mathbb{C}$. *Note that it is known, due to the works of P.Deligne, J.Milne, etc, that the Galois conjugate of a special subvariety remains special, hence* μ_{Y} *is always well defined.*

Some cases of the conjecture are known, deduced from different approaches. L.Clozel and E.Ullmo proved the homogeneous case, namely a sequence of special varieties S_n defined by Shimura data $(\mathbf{G}_n,\mathrm{X}_n)$ with $\mathbf{G}_n=\mathrm{MT}(\mathrm{X}_n)$ such that all the connected center of the $\mathbf{G}_n$'s coincide. On the other hand, various explicit calculations concerning automorphic forms treated the case of certain inhomogeneous sequence of special subvarieties, see for example the paper of D.Jiang, J.Li, and S.Zhang [JLZ]. With our treatment in the next chapter, it will also be reasonable to consider the equidistribution conjecture in the mixed case.

2.2 Reductions and Reformulations

Lemma 2.2.1. *(Reduction 1) Let* $(\mathbf{P}, Y)$ *be a Shimura datum,* $L \subset K$ *a pair of compact open subgroups of* $\mathbf{P}(\mathbb{A}^f)$*. Then the André-Oort conjecture holds for* $M_K = M_K(\mathbf{P}, Y)$ *if and only if it holds for* $M_L = M_L(\mathbf{P}, Y)$.

Proof. We have the finite map $f : M_L \to M_K$; it sends closed subvarieties to closed subvarieties. It suffices to prove:

(1) for a special subvariety $Z \subset M_L$, $f(Z)$ is special;

(2) a closed geometrically irreducible subvariety $Z \subset M_K$ is special if and only if each geometrically irreducible component of $f^{-1}(Z)$ is special in M_L.

A special subvariety Z in M_L is of complex locus $\wp_L(Y_1^+ \times aL)$ for some Shimura subdatum $(\mathbf{P}_1, Y_1)$ and some $a \in \mathbf{P}(\mathbb{A}^f)$. Thus the complex locus of $f(Z)$ is $\wp_K(Y_1^+ \times aK)$, which is special. Hence (1) is true.

Now for Z a special subvariety in M_K of complex locus $\wp_K(Y_1^+ \times aK)$ for some subdatum $(\mathbf{P}_1, Y_1)$ and $a \in \mathbf{P}(\mathbb{A}^f)$. Consider the decomposition into right cosets $K = \coprod_{i=1}^d b_i L$ where $d = [K : L] < \infty$. Then each geometrically irreducible component of $f^{-1}(Z)$ is of complex locus $\wp_L(Y_1^+ \times ab_i L)$ for some i, which special itself. Hence (2) is true. ■

We see that the André-Oort conjecture holds for M_K if and only if it holds for M_L. In particular, it suffices to study special subvarieties in M_K for K torsion free, whose geometrically components are smooth quasi-projective varieties of complex loci $\Gamma \backslash Y^+$ for $\Gamma \subset \mathbf{P}(\mathbb{R})$ some arithmetic lattice that is torsion free. This setting is adopted in the approach of E.Ullmo, A.Yafaev, etc. For the explicit approach via automorphic forms, e.g. as was shown in [JLZ], it is sometimes more convenient to work with certain maximal compact open subgroups for which more specific results become available.

Lemma 2.2.2 (Reduction 2). *Let* $M = M_K(\mathbf{P}, Y)$ *be a Shimura variety,* $(M_n)_n$ *a sequence of special subvarieties, and* Z *the Zariski closure of* $\bigcup_n M_n$. *In order to treat the André-Oort conjecture for* M*, we may assume that* Z *is irreducible and that* Z *is Hodge generic, i.e. the Mumford-Tate group of* $Z \subset M$ *equals* $\mathbf{P}$*, or equivalently, the smallest special subvariety of* M *containing* Z *is a geometrically connected component of* M.

Proof. Consider the intersection of all the special subvarieties of M containing Z. Because M is a noetherian scheme over a field, the intersection is reduced to a finite intersection of special subvarieties, which is weakly special according to the last lemma of Chap.1. Z is irreducible, so the intersection is a single special subvariety M_Z. This is the smallest special subvariety that contains all the M_n's. Suppose M_Z is of the form $\Gamma_K(a) \backslash \Gamma_K(a) Y_Z^+$ for some subdatum $(\mathbf{P}_Z, Y_Z)$, then up to conjugating the datum by elements in $\mathbf{P}(\mathbb{Q})$, we may assume that Z is contained in a geometrically connected component of $M_{K_Z}(\mathbf{P}_Z, Y_Z)$ where $K_Z = K \cap \mathbf{P}_Z(\mathbb{A}^f)$. We may thus assume that $M = M_{K_Z}(\mathbf{P}_Z, Y_Z)$ whence Z is Hodge generic in M. ■

Lemma 2.2.3. *(Reduction 3) Let* $\mathrm{M} = \mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$ *be a mixed Shimura variety, with* $\mathrm{K} = \mathrm{K}_{\mathbf{W}} \rtimes \mathrm{K}_{\mathbf{G}} \subset \mathbf{P}(\mathbb{A}^{\mathrm{f}})$ *a compact open subgroup. To study the André-Oort conjecture, it suffices to study special subvarieties inside a given connected component of* M *of the form* $\Gamma\backslash\mathrm{Y}^+$ *where* Y^+ *is any fixed connected component of* Y *and* $\Gamma = \mathbf{P}(\mathbb{Q})_+ \cap \mathrm{K}$. *Under the equivalent formulations, we may assume that the sequence of special varieties in question is of the form* $\mathrm{M}_n = \Gamma\backslash\Gamma\mathrm{Y}_n^+$ *with* $(\mathbf{P}_n,\mathrm{Y}_n)$ *subdata of* $(\mathbf{P},\mathrm{Y})$ *and* $\mathrm{Y}_n^+ \subset \mathrm{Y}^+$ *a connected component of* Y_n.

Proof. We fix $\mathfrak{R} = \mathfrak{R}_{\mathrm{K}_{\mathbf{G}}}^{\mathbf{G}}$ a set of representatives for $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}_{\mathbf{G}}$ which extends to represent the finite set $\mathbf{P}(\mathbb{Q})_+\backslash\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}$. Write $\wp_{\mathrm{K}}$ for the projection $\mathrm{Y}^+ \times [\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}] \twoheadrightarrow \mathrm{M}(\mathbb{C}) = \mathbf{P}(\mathbb{Q})_+\backslash\mathrm{Y}^+ \times [\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}]$, and $\gamma(\mathfrak{R})$ for the identification $\mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y}) \cong \coprod_{a\in\mathfrak{R}} \Gamma_{\mathrm{K}}(a)\backslash\mathrm{Y}^+$. Then every special subvariety of M is given by some $\wp_{\mathrm{K}}(\mathrm{Y}'^+ \times a\mathrm{K})$ for some $a \in \mathfrak{R}$ and $(\mathbf{P}',\mathrm{Y}')$ some subdatum with Y'^+ a connected component of Y' contained in Y^+, cf. Chap.1 Definition 1.3.3.

Let $(\mathrm{M}_n)_n$ be a sequence of special subvarieties in M, then we may assume that M_n is given by $\wp_{\mathrm{K}}(\mathrm{Y}_n^+ \times a_n\mathrm{K})$ with $a_n \in \mathfrak{R}$ and $(\mathbf{P}_n,\mathrm{Y}_n)$ some subdatum with Y_n^+ a connected component of Y_n contained in Y^+. Note that $\{a_n : n\}$ comes from the finite set $\mathfrak{R}$, we may write $(\mathrm{M}_n)_n$ as a finite union of subsequences $(\mathrm{M}'_n(a))_n$ with $\mathrm{M}'_n(a)$ exhausting the M_n's of the form $\wp_{\mathrm{K}}(\mathrm{Y}_n^+ \times a\mathrm{K})$, i.e. those contained in $\Gamma_{\mathrm{K}}(a)\backslash\mathrm{Y}^+$ under the identification $\gamma(\mathfrak{R})$. To prove that the Zariski closure of $\bigcup_n \mathrm{M}_n$ is weakly special, it suffices to show that the Zariski closure $\mathrm{M}'(a)$ of $\bigcup_n \mathrm{M}'_n(a)$ is weakly special, $a \in \mathfrak{R}$.

For a fixed $a \in \mathfrak{R}$, we put $\mathrm{K}' = a\mathrm{K}a^{-1} = \mathrm{K}'_{\mathbf{W}} \rtimes \mathrm{K}'_{\mathbf{G}}$, with $\mathrm{K}'_{\mathbf{W}} = a\mathrm{K}_{\mathbf{W}}a^{-1}$ and $\mathrm{K}'_{\mathbf{G}} = a\mathrm{K}_{\mathbf{G}}a^{-1}$. Then $\mathbf{P}(\mathbb{Q})_+\backslash\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}'$ (or equivalently $\mathbf{G}(\mathbb{Q})_+\backslash\mathbf{G}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}'_{\mathbf{G}}$) is represented by $\mathfrak{R}a^{-1} = \{ba^{-1} : b \in \mathfrak{R}\}$. And we have an isomorphism $\lambda_a : \mathrm{M}_{a\mathrm{K}a^{-1}}(\mathbf{P},\mathrm{Y}) \to \mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$, $[y, ga\mathrm{K}a^{-1}] \mapsto [y, ga\mathrm{K}]$. And we complete it into a commutative diagram

$$\begin{array}{ccc}
\mathrm{M}_{a\mathrm{K}a^{-1}}(\mathbf{P},\mathrm{Y}) & \xrightarrow{\quad\lambda_a\quad} & \mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y}) \\
{\scriptstyle\gamma(\mathfrak{R}a^{-1})}\Big\downarrow & & \Big\downarrow{\scriptstyle\gamma(\mathfrak{R})} \\
\coprod_{c\in\mathfrak{R}a^{-1}} \Gamma_{a\mathrm{K}a^{-1}}(c)\backslash\mathrm{Y}^+ & \xrightarrow{\quad\iota_a\quad} & \coprod_{c\in\mathfrak{R}} \Gamma_{\mathrm{K}}(c)\backslash\mathrm{Y}^+
\end{array}$$

here ι_a identifies $\Gamma_{a\mathrm{K}a^{-1}}\backslash\mathrm{Y}^+$ with $\Gamma_{\mathrm{K}}(a)\backslash\mathrm{Y}^+$, in other terms, λ_a sends $\wp_{a\mathrm{K}a^{-1}}(\mathrm{Y}^+ \times \mathrm{K})$ isomorphically to $\wp_{\mathrm{K}}(\mathrm{Y}^+ \times a\mathrm{K})$.

Within the same diagram we see that λ_a sends a special subvariety of the form $\wp_{a\mathrm{K}a^{-1}}(\mathrm{Y}'^+ \times \mathrm{K})$ to $\wp_{\mathrm{K}}(\mathrm{Y}'^+ \times a\mathrm{K})$ bijectively. That means in order to study the Zariski closure $\mathrm{M}'(a)$ of the union of the $\mathrm{M}'_n(a) = \wp_{\mathrm{K}}(\mathrm{Y}_n^+ \times a\mathrm{K})$ it suffices to study the Zariski closure of the union of the $\wp_{a\mathrm{K}a^{-1}}(\mathrm{Y}_n^+ \times \mathrm{K})$, or rather, under $\gamma(\mathfrak{R}a^{-1})$, study the closure of $\Gamma'\backslash\Gamma'\mathrm{Y}_n^+$'s inside $\Gamma'\backslash\mathrm{Y}^+$, where $\Gamma' = \Gamma_{a\mathrm{K}a^{-1}}(1) = \Gamma_{\mathrm{K}}(a) = \mathbf{P}(\mathbb{Q})_+ \cap a\mathrm{K}a^{-1}$. Therefore in what follows we are mainly concerned with a sequence of special subvarieties of the form $\mathrm{M}_n = \Gamma\backslash\Gamma\mathrm{Y}_n^+$ inside $\Gamma\backslash\mathrm{Y}^+$, where Γ is some arithmetic subgroup of $\mathbf{P}(\mathbb{R})^+$.

Remark 2.2.4. The advantage to work with special subvarieties of the form $\Gamma\backslash\Gamma\mathrm{Y}'^+$ is that these special subvarieties are actually connected components of Shimura subvarieties defined by $(\mathbf{P}',Y')$, with $\mathbf{P}' = \mathbf{MT}(Y')$, and we can estimate the Galois orbit of such special subvarieties by standard formula of the reciprocity map.

We propose some equivalent reformulations of the André-Oort conjecture in the mixed case.

Proposition 2.2.5. *Let* $(\mathbf{P},Y)$ *be a Shimura datum,* K *a torsion free compact open subgroup of* $\mathbf{P}(\mathbb{A}^f)$*, and* $M = M_K(\mathbf{P},Y)$*. Then the following statement are equivalent:*

(AO-0) For $(z_n)_n$ *any sequence of special points of* M*, the Zariski closure of* $\{z_n : n \in \mathbb{N}\}$ *is weakly special.*

(AO-1) For $(S_n)_n$ *any sequence of pure special subvarieties of* M*, the Zariski closure of* $\bigcup_n S_n$ *is weakly special.*

(AO-2) For $(M_n)_n$ *any sequence of special subvarieties of* M*, the Zariski closure of* $\bigcup_n M_n$ *is weakly special.*

(AO-2)' Let $(M_n)_n$ *a strict sequence of special subvarieties of* M*. Then* $(M_n)_n$ *is generic.*

(AO-3) For $Z \subset M$ *any closed subset, define* $\Sigma(Z) = \{M' \subset Z : M' \text{ special}\}$*, then the set of maximal elements (withe respect to the inclusion order) of* $\Sigma(Z)$*, denoted by* $\mathcal{S}(Z)$*, is finite.*

Proof. The equivalence between (AO-2) and (AO-2)' is evident. Clearly we have (AO-2)$\Rightarrow$(AO-1)$\Rightarrow$(AO-0). On the other hand every special subvariety M' is the Zariski closure of the countable set $\{z \in M'(\mathbb{Q}^{ac}) : z \text{ special}\}$, hence $\bigcup_n M_n$ is of the same Zariski closure with the countable set $\bigcup_n\{z \in M_n(\mathbb{Q}^{ac}) : z \text{ special}\}$, and (AO-0)$\Rightarrow$(AO-2).

(AO-2)$\Rightarrow$(AO-3): Let $Z \subset M$ be a closed subvariety, and $\Sigma = \mathcal{S}(Z)$ as is defined in (AO-3). It is finite when Z is weakly special. Suppose that Z is not weakly special. Then Σ is finite or countable, and it can be written as a sequence of special subvarieties $(M_n)_n$. (AO-2) implies that the Zariski closure Z' of $\bigcup_n M_n$ is weakly special, i.e. $Z' = \bigcup_{i=1}^N S_i$ for finitely many special subvarieties $S_1,\dots,S_N$. In particular $S_i \subset Z$ and every M_n is contained in some S_i, and thus there exists i such that there $M_n \subset S_i$. But we have assumed that the M_n's are maximal, hence $M_n = S_i$ and Σ is finite.

(AO-3)$\Rightarrow$(AO-2): Let $(M_n)_n$ be a sequence of special subvarieties, and Z the Zariski closure of $\bigcup_n M_n$. The set Σ of maximal elements of is finite according to (AO-3), denoted as $\{S_1,\dots,S_N\}$. Then $\bigcup_n M_n \subset \bigcup_{i=1}^N S_i$. Taking Zariski closure of both sides we get $Z = \bigcup_{i=1}^N S_i$, i.e. Z is weakly special. ∎

2.3 An approach under GRH

A notable progress is the proof of the André-Oort conjecture in the pure case by B.Klingler, E.Ullmo, and A.Yafaev, under the Generalized Riemann Hypothesis (abbreviated as **GRH** in what follows). The beginning point is an lower bound of the size of the Galois orbits of special points, estimated under the effective Chebotarev theorem, which relies on the GRH. This estimation is then generalized to higher dimensional case and results in an algorithmic approach to the André-Oort conjecture for pure Shimura varieties. We introduce some terminologies before presenting their approach and our generalization in the mixed case.

2.3.1 Some terminologies

We introduce some terminologies to illustrate a current approach to the André-Oort conjecture assuming the GRH.

We consider a pure Shimura variety $S = M_{K_G}(\mathbf{G}, X)$, with $K_G = \prod_p K_{G,p}$ assumed to be torsion free. Let $\mathscr{L} = \mathscr{L}(K)$ be the canonical line bundle on S defining the Baily-Borel compactification, defined over the same reflex field $E = E(\mathbf{G}, X)$ as S is. Also fix F a number field containing the reflex field E, and a faithful representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$ over some $\mathbb{Q}$-vector space $\mathbf{M}$ with a lattice $\mathbf{M}_{\mathbb{Z}} \subset \mathbf{M}$ such that $\rho(K_G)$ stabilizes $\mathbf{M}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ inside $\mathbf{M} \otimes_{\mathbb{Q}} \mathbb{A}^f$.

• uniform constant : this is nothing but a constant determined by the tuple $(\mathbf{P}, Y, F, \rho)$, independent of the choice of subdata of $(\mathbf{P}, Y)$ and free of the level K.

• test invariant of a pure special subvariety :
We fix an integer $N > 0$.

For a subdatum $(\mathbf{G}', X')$ of $(\mathbf{G}, X)$, it is always assumed that $\mathbf{G}' = \mathbf{MT}(X')$, and the associated Shimura subvariety is $M_{K_{G'}}(\mathbf{G}', X')$, $K_{G'} = K \cap \mathbf{G}'(\mathbb{A}^f)$. Write $\mathbf{C}'$ for the connected center of $\mathbf{G}'$, then for any geometrically connected component S' of the image $M_{K_{G'}}(\mathbf{G}', X')$, define the test invariant of M' to be

$$\tau(S') := c_N (\log D_{C'})^N \max\{1, \prod_{p \in \delta(\mathbf{C}', K)} B | K^{\max}_{C',p} / K_{C',p}\}$$

where c_N and B are uniform constants, $K_{C'} = \prod_p K_{C',p} = K \cap \mathbf{C}'(\mathbb{A}^f)$, $K^{\max}_{C'}$ the unique maximal compact open subgroup of $\mathbf{C}'(\mathbb{A}^f)$, $\delta(\mathbf{C}', K)$ the set of rational primes p such that $K_{C',p} \subsetneq K^{\max}_{C',p}$, and $D_{C'}$ the absolute discriminant of $E_{C'}$ over $\mathbb{Q}$, $E_{C'}$ being the splitting field of $\mathbf{C}'$.

$\tau(S')$ appears in the estimation of the lower bound of the number of Galois conjugates of S': the constants c_N and B can be chosen in such a way that with respect to some fixed number field F containing the reflex field of $(\mathbf{G}, X)$, the inequality

$$\deg_{\mathscr{L}}(\mathrm{Gal}_F S') \geq \tau(S')$$

holds for any special subvariety S'.

• (in-)homogeneous subsequence:

A sequence of pure special varieties $(S_n)_n$ of $M_{K_G}(\mathbf{G}, X)$ is homogeneous with respect to some $\mathbb{Q}$-torus $\mathbf{C} \subset \mathbf{G}$, or simply a $\mathbf{C}$-special sequence, if for each n there exists a subdatum $(\mathbf{G}_n, X_n) \subset (\mathbf{G}, X)$ such that $\mathbf{C}$ is the connected center of $\mathbf{G}_n$ and S_n is a geometrically connected component of the image of $M_{K_{G_n}}(\mathbf{G}_n, X_n) \to M_{K_G}(\mathbf{G}, X)$; it is weakly homogeneous if there exists finitely many $\mathbb{Q}$-tori $\mathbf{C}_i \subset \mathbf{G}$ such that each S_n is $\mathbf{C}_i$-special for some $i = i_n$; it is inhomogeneous if it is not weakly homogeneous.

Note that we can always reduce the problem to a sequence of special subvarieties inside a fixed geometrically connected component $S^+ = \Gamma_{K_G}(a)\backslash X^+$ of $M_{K_G}(\mathbf{G}, X)$, due to Reduction 3.

2.3.2 The approach under GRH

The combination of [CU-0], [UY],and [KY] leads to the proof of the André-Oort conjecture in the pure case under GRH:

We follow the formulation 2.1.2(2). Let Z be a closed subvariety of a given pure Shimura variety S, and we study the finiteness of $\Sigma = \mathcal{S}(Z)$.

If Z is weakly special itself, then Σ is the set of geometrically irreducible component of Z, hence finite.

Therefore we may assume that Z is not weakly special, and we assume for simplicity that Z is geometrically irreducible. We may also assume that the Z is contained in a special subvariety S such that Z is Hodge generic S. Σ is clearly countable, and we may write it as a sequence $(S_n)_n$. We want to show that $(S_n)_n$ is finite. i.e. $\Sigma = \{S_n\}_n$ is a finite set.

(i) If the sequence $(S_n)_n$ is weakly homogeneous, then the theorem of L.Clozel and E.Ullmo implies that the Zariski closure of $\bigcup_n S_n$ in Z is weakly special. Combining with maximality of Σ we deduce that the sequence $(S_n)_n$ is finite.

(ii) If the sequence $(S_n)_n$ is inhomogeneous, then a theorem of E.Ullmo and A.Yafaev affirms that the sequence of test invariants is unbounded. Combining with (i) we see that any subsequence of bounded test invariants can be replaced by finitely many special subvarieties, thus we may, up to rearranging the sequence, assume that $(\tau_n = \tau(S_n))_n$ tends to $+\infty$ as $n \to +\infty$. B.Klingler and A.Yafaev then proved that for τ_n sufficiently large, there exists a special subvariety $S'_n \subset Z$ such that $S_n \subset S'_n$, contradicting the maximality of Σ. Hence Σ is finite in this case. This final step makes essential use of the Effective Chebotarev Theorem, which is established under the Generalized Riemann Hypothesis.

2.4 Main results

In this thesis we follow the strategy of E.Ullmo, A.Yafaev and their collaborators and derive some special cases of the André-Oort-Pink conjecture in the mixed case. However we are not yet ready to treat the most general case even under GRH.

2.4.1 General setting

We work with a mixed Shimura variety $\mathrm{M} = \mathrm{M}(\mathbf{P}, \mathrm{Y})$ equipped with a pure section $\mathrm{S} = \mathrm{M}_{\mathrm{K}_\mathbf{G}}(\mathbf{G}, \mathrm{X})$, where $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ is a Levi decomposition, and $\mathrm{K} = \mathrm{K}_\mathbf{W} \rtimes \mathrm{K}_\mathbf{G}$ a compact open subgroup of $\mathbf{P}(\mathbb{A}^\mathrm{f})$. Denote by $\mathrm{M} \underset{i}{\overset{\pi}{\rightleftarrows}} \mathrm{S}$ the morphisms corresponding to the projection $\pi : \mathbf{P} \to \mathbf{G}$ and the inclusion $i : \mathbf{G} \to \mathbf{P}$. Fix also a finite subset $\mathfrak{R}$ of $\mathbf{G}(\mathbb{A}^\mathrm{f})$ representing $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbb{A}^\mathrm{f})/\mathrm{K}_\mathbf{G}$, which extends to a set of representatives of $\mathbf{P}(\mathbb{Q})_+ \backslash \mathbf{P}(\mathbb{A}^\mathrm{f})/\mathrm{K}$.

As is indicated in the Reductions 1-3, we may assume that K is torsion free, and we study $\mathscr{S}(\mathrm{Z})$ for Z a closed subvariety of a connected component M^+ of M. We assume for simplicity that M^+ is the connected component given by $\Gamma \backslash \mathrm{Y}^+$ for $\Gamma = \mathbf{P}(\mathbb{Q})_+ \cap \mathrm{K}$, and that Z is irreducible, not weakly special. Write $\mathscr{S}(\mathrm{Z})$ as a sequence $(\mathrm{M}_n)_n$, then each M_n is of complex locus $\Gamma \backslash \Gamma \mathrm{Y}_n^+$ for some subdatum $(\mathbf{P}_n, \mathrm{Y}_n)$ with Y_n^+ a connected component of Y_n contained in Y^+.

2.4.2 The (weakly) homogeneous case

For a $\mathbb{Q}$-torus $\mathbf{C}$ in $\mathbf{G}$, we have seen the notion of $\mathbf{C}$-special subvarieties in Chap.1, Definition 1.3.9, with respect to the fixed set of representatives $\mathfrak{R}$. A sequence of special subvarieties $(\mathrm{M}_n)_n$ of M is homogeneous if there exists a $\mathbb{Q}$-torus $\mathbf{C} \subset \mathbf{G}$ such that all the M_n's are $\mathbf{C}$-special. $(\mathrm{M}_n)_n$ is said to be weakly homogeneous if it is a finite union of homogeneous subsequences. It is inhomogeneous if it is not weakly homogeneous.

We see that the notion of $\mathbf{C}$-special subvarieties and (in-)homogeneous sequences is determined by their projections under π into S. The point is that the ergodic approach can be carried over to the mixed case directly such that the map π gives us only a "trivial extension":

(1) The starting point is that the theorem of Mozes-Shah already characterizes the so-called H-measures on spaces of the form $\Omega = \Gamma \backslash \mathbf{Q}(\mathbb{R})^+$, where $\mathbf{Q}$ is a $\mathbb{Q}$-group of type $\mathcal{H}$, namely of the form $\mathbf{R}_u\mathbf{Q} \rtimes \mathbf{H}$ where $\mathbf{R}_u\mathbf{Q}$ is the unipotent $\mathbb{Q}$-radical and $\mathbf{H}$ is a semi-simple $\mathbb{Q}$-group without compact $\mathbb{Q}$-factors, and $\Gamma \subset \mathbf{Q}(\mathbb{R})^+$ is an arithmetic lattice. For a $\mathbb{Q}$-subgroup of type $\mathcal{H}$ $\mathbf{Q}' \subset \mathbf{Q}$ with $\mathbf{R}_u\mathbf{Q}' \subset \mathbf{R}_u\mathbf{Q}$, $\Gamma' = \Gamma \cap \mathbf{Q}'(\mathbb{R})^+$ is an arithmetic lattice, and the canonical measure on $\Omega' = \Gamma' \backslash \mathbf{Q}'(\mathbb{R})^+$, induced by the Haar measure on $\mathbf{Q}'(\mathbb{R})^+$, is pushed-forward to a probability measure ν' on Ω under the inclusion $\Omega' \hookrightarrow \Omega$, whose support is exactly Ω'. Such ν''s are referred to as the H-measures on Ω, denoted as $\mathcal{H}(\Omega)$.

The main results of Mozes-Shah shows that the countable set $\mathcal{H}(\Omega)$ is compact for the weak topology; moreover, if a sequence $(\nu_n)_n$ in $\mathcal{H}(\Omega)$ converges to some ν', then for some $\mathrm{N} > 0$, the union $\bigcup_{n>\mathrm{N}} \mathrm{Supp}\,\nu_n$ is dense in Supp'_ν for the archimedean topology. This establishes "the weakened André-Oort conjecture at the level of lattice spaces": for a sequence of lattice subspaces $(\Omega_n)_n$ in Ω with $\Omega_n = \mathrm{Supp}\,\nu_n$ for some $\nu_n \in \mathcal{H}(\Omega)$, the archimedean closure of $\bigcup_n \Omega_n$ is a finite union of supports of measures in $\mathcal{H}(\Omega)$.

For a mixed Shimura datum $(\mathbf{P},Y)$, we put $\mathbf{P}^\dagger$ to be its maximal $\mathbb{Q}$-subgroup of type $\mathcal{H}$: if $\mathbf{P}=\mathbf{W}\rtimes\mathbf{G}$ is a Levi decomposition over $\mathbb{Q}$, then $\mathbf{P}^\dagger=\mathbf{W}\rtimes\mathbf{G}^{\mathrm{der}}$. This is an invariant $\mathbb{Q}$-subgroup of $\mathbf{P}$. For a congruent subgroup $\Gamma\subset\mathbf{P}(\mathbb{R})$, Γ normalizes $\mathbf{P}^\dagger$, and $\Gamma\cap\mathbf{P}^\dagger(\mathbb{R})^+$ is a congruence lattice of $\mathbf{P}^\dagger(\mathbb{R})^+$. Thus for arithmetic subgroup $\Gamma\subset\mathbf{P}(\mathbb{R})^+$, we write $\Omega=\Gamma\backslash\mathbf{P}^\dagger(\mathbb{R})^+$ for $(\Gamma\cap\mathbf{P}^\dagger(\mathbb{R})^+)\backslash\mathbf{P}^\dagger(\mathbb{R})^+$ and refer to it as the lattice space associated to $(\mathbf{P},Y,\Gamma)$. In the same way we have the notion of lattice subspace defined by a subdatum $(\mathbf{P}',Y')$, namely $\Omega'=\Gamma\backslash\Gamma\mathbf{P}'^\dagger(\mathbb{R})^+$. We then define the notion of $\mathbf{C}$-special H-measures on Ω and $\mathbf{C}$-special lattice subspaces of Ω for a fixed $\mathbb{Q}$-torus $\mathbf{C}$ in $\mathbf{G}$: a lattice subspace is $\mathbf{C}$-special if it is associated to a $\mathbf{C}$-special subdatum of $(\mathbf{P},Y)$, and an H-measure is $\mathbf{C}$-special if it is the canonical measure associated to a $\mathbf{C}$-special lattice subspace.

Write $\mathcal{H}_{\mathbf{C}}(\Omega)$ for the set of $\mathbf{C}$-special H-measures on Ω, and for a closed subspace Z in Ω put $\mathcal{S}_{\mathbf{C}}(Z)$ for the set of maximal $\mathbf{C}$-special lattice subspaces of Ω contained in Z. Then from the theorem of Mozes and Shah we deduce the compactness of $\mathcal{H}_{\mathbf{C}}(\Omega)$ and the finiteness of $\mathcal{S}_{\mathbf{C}}(Z)$: namely we prove the weakened André-Oort conjecture for lattice spaces defined by Shimura data under the assumption of homogeneity.

(2) We then want to project from lattice spaces to Shimura varieties. But the better way proves to be first passing through the "real part" of the Shimura variety. We thus introduce the notion of S-space associated to a Shimura datum $(\mathbf{P},Y)$ at some level K.

With respect to a pure section $(\mathbf{G},X)\subset(\mathbf{P},Y)$, the real part of Y is defined to be the orbit $Y_{\mathbb{R}}=\mathbf{P}(\mathbb{R})x$ of some (or any) $x\in X$, which is actually independent of the choice of pure sections. To each compact open subgroup K of $\mathbf{P}(\mathbb{A}^f)$ we associate the Shimura S-space $\mathscr{S}_K(\mathbf{P},Y):=\mathbf{P}(\mathbb{Q})\backslash[Y_{\mathbb{R}}\times\mathbf{P}(\mathbb{A}^f)/K]\cong\coprod_{g\in\mathfrak{R}}\Gamma_K(g)\backslash Y_{\mathbb{R}}^+$, where $Y_{\mathbb{R}}^+$ is a fixed connected component of $Y_{\mathbb{R}}$, $\mathfrak{R}$ is a set of representatives of $\mathbf{P}(\mathbb{Q})_+\backslash\mathbf{P}(\mathbb{A}^f)/K$, and $\Gamma_K(g)=\mathbf{P}(\mathbb{Q})_+\cap gKg^{-1}$. Parallel to the case of Shimura varieties, we define the notion of Shimura S-subspaces, etc. Let's remark that we regard S-spaces as real analytic spaces, and they coincide with the underlying real spaces of the corresponding Shimura varieties if and only if the defining $\mathbb{Q}$-group $\mathbf{P}$ has no unipotent part of weight -2.

Many constructions for mixed Shimura varieties carry over to S-spaces, such as the notion of $\mathbf{C}$-special S-subspaces, Hecke correspondences, etc. But we do not study the notion of canonical models for S-spaces.

We mainly work with connected S-spaces of the form $S=\Gamma\backslash Y_{\mathbb{R}}^+$ where Γ is some torsion-free arithmetic subgroup of $\mathbf{P}(\mathbb{R})^+$. Let $\Omega=\Gamma\backslash\mathbf{P}^\dagger(\mathbb{R})^+$ be the lattice space associated to $(\mathbf{P},Y,\Gamma)$, and for any $y\in Y_{\mathbb{R}}^+$, we have the projection from lattice space $\kappa_y:\Omega\twoheadrightarrow S$, $\Gamma g\mapsto\Gamma gy$, and the canonical probability H-measure ν on Ω is pushed-forward to the canonical probability measure μ on S.

On S we have the notion of $\mathbf{C}$-special H-measures, which are defined to be the push-forwards of H-measures on Ω whose supports are $\mathbf{C}$-special lattice subspaces, i.e. of the form $\Omega'=\Gamma'\backslash\mathbf{P}'^\dagger(\mathbb{R})^+$, where $\mathbf{P}'$ comes from some Shimura subdatum $(\mathbf{P}',Y')$ such that $\pi(\mathbf{P}')=\mathbf{G}'\subset\mathbf{G}$ is of connected center $\mathbf{C}$, and $\mathbf{P}'^\dagger$ is the maximal $\mathbb{Q}$-subgroup in $\mathbf{P}'$ of type $\mathcal{H}$. Then we have $\nu' \in \mathcal{H}(\Omega)$ of support Ω'. Take $y' \in Y'^{+}_{\mathbb{R}}$ we get the projection $\kappa_{y'} : \Omega' \to S' = \Gamma' \backslash Y'^{+}_{\mathbb{R}}$, which sends ν' to the canonical probability measure on S of support S'. The collection of all these μ' makes the (countable) set of $\mathbf{C}$-special H-measures on S, denoted as $\mathcal{H}_{\mathbf{C}}(S)$.

(3) An important fact, deduced from a proposition of S.Dani and Margulis, is that there exists a compact subset $\mathcal{C}_{\mathbf{C}}$ in $Y^{+}_{\mathbb{R}}$ such that any $\mathbf{C}$-special H-measure on S is of the form $\mu' = \kappa_{y',*}\nu'$ for some $y' \in \mathcal{C}_{\mathbf{C}}$ and $\nu' \in \mathcal{H}(\Omega)$ defined by some Shimura subdatum $(\mathbf{P}', Y')$ and simultaneously $y' \in Y'^{+} \subset Y^{+}$. Therefore $\mathcal{H}_{\mathbf{C}}(S)$ is compact. We thus conclude that for any sequence of $\mathbf{C}$-special S-subspaces $(S_n)_n$, the archimedean closure of $\bigcup_n S_n$ is a finite union of $\mathbf{C}$-special S-subspaces.

(4) We remark that we would first deal with the case where $\mathbf{C}$ equals the connected center of $\mathbf{G}$. And for a general $\mathbb{Q}$-torus $\mathbf{C}'$ inside $\mathbf{G}$, the point is that the set of maximal $\mathbf{C}'$-special lattice subspaces in Ω, and the set of maximal $\mathbf{C}'$-special S-subspaces in S, are finite (possibly empty for some $\mathbf{C}'$). The the study of a sequence of $\mathbf{C}'$-special sub-objects is reduced to the arguments in the case where $\mathbf{C}$ is the connected center of $\mathbf{G}$.

(5) Finally, to treat the original conjecture, it suffices to notice that for any special subvariety $M' \subset M$, the associated S-space S' is dense in M' for the Zariski topology. Hence for any sequence of $\mathbf{C}'$-special subvarieties $(M_n)_n$ in M, with associated special S-subspaces $(S_n)_n$ and Zariski closure $M' = \overline{\bigcup_n M_n}^{\mathrm{Zar}}$, it is easy to show that the $\bigcup_n S_n$ is Zariski dense in M', and therefore M' is a finite union of $\mathbf{C}'$-special subvarieties of M, which terminates the proof of André-Oort conjecture for a homogeneous sequence of special subvarieties.

2.4.3 The estimation of the degree of Galois orbits

(0) The estimation of E.Ullmo and A.Yafaev

One of the main results in [UY-1] is the following: if a sequence of special subvarieties $(S_n)_n$ in a pure Shimura variety S is of bounded test invariants, then the sequence is weakly homogeneous, i.e. there exists finitely many $\mathbb{Q}$-tori $(\mathbf{C}_i)_{i \in I}$ such that every S_n is $\mathbf{C}_i$-special for some i, and consequently the Zariski closure of $\bigcup_n S_n$ is weakly special.

Here we assume that the ambient Shimura variety S is defined at some finite level $K_{\mathbf{G}} = \prod_p K_{\mathbf{G},p}$ which is torsion-free, and the (N-th) test invariant of a special subvariety $S' \subset S$ is defined as

$$\tau(S') = c_{\mathrm{N}}(\log D_{\mathbf{C}'})^{\mathrm{N}} \cdot \max\{1, \prod_{p \in \delta(\mathbf{C}', K_{\mathbf{G}})} B|K^{\max}_{\mathbf{C}',p}/K_{\mathbf{C}'}|\}$$

where $\mathbf{C}'$ is the connected center of some $\mathbb{Q}$-subgroup $\mathbf{G}' \subset \mathbf{G}$ for some Shimura subdatum $(\mathbf{G}', X')$ such that S' is a connected component of the Shimura subvariety defined by $(\mathbf{G}', X')$, $K_{\mathbf{C}'} = K_{\mathbf{G}} \cap \mathbf{C}'(\mathbb{A}^{\mathrm{f}})$, $K^{\max}_{\mathbf{C}'}$ denotes the unique maximal compact open subgroup of $\mathbf{C}'(\mathbb{A}^{\mathrm{f}})$, and $\delta(\mathbf{C}')$ is the finite set of rational primes

p such that $K_{\mathbf{C},p} \subsetneq K^{\max}_{\mathbf{C}',p}$. c_N and B are constants, independent of the choice of special subvarieties.

We remark that N is a positive integer that is prescribed from the very beginning; it is employed in the works of B.Edixhoven, B.Klingler, and A.Yafaev about the comparison between the intersection degrees of Hecke correspondences and the Galois orbits. It is not involved in our applications: what really counts for us in the present writing is whether the sequence of test invariants is bounded or not.

The $\tau(S')$ serves as a lower bound of the intersection degree of the Gal_F-orbit of S' with respect to $\mathscr{L}$, where F is a fixed number field containing the reflex field of S, and $\mathscr{L} = \mathscr{L}(K_{\mathbf{G}})$ is the canonical ample invertible sheaf defining the Baily-Borel compactification of S, namely the sheaf of top degree differential forms allowing at most logarithmic poles along boundary components of codimension one.

(1) By examining the proofs of [UY-1], we find that the constants c_N and B are independent of the torsion free group $K_{\mathbf{G}}$. We thus obtain the following estimation:

Let $\pi : M = M_K(\mathbf{P}, Y) \to S = M_{K_{\mathbf{G}}}(\mathbf{G}, X)$ be a fibration of a mixed Shimura variety M over a pure section S, with Levi decomposition $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ and torsion free compact open subgroup $K = K_{\mathbf{W}} \rtimes K_{\mathbf{G}} \subset \mathbf{P}(\mathbb{A}^f)$. Let M' be a pure special subvariety contained in some $M(w) = M_{K_w}(w\mathbf{G}w^{-1}, w \rtimes X)$ given by a pure subdatum of the form $(w\mathbf{G}'w^{-1}, w \rtimes X')$, where $K_w = K \cap w\mathbf{G}(\mathbb{A}^f)w^{-1} = wK_{\mathbf{G}}(w)w^{-1}$ with $K_{\mathbf{G}}(w) := \{g \in K_{\mathbf{G}} : wgw^{-1}g^{-1} \in K_{\mathbf{W}}\}$, and $(\mathbf{G}', X')$ some subdatum of $(\mathbf{G}, X)$ of connected center $\mathbf{C}'$. Then

$$\deg_{\pi^*\mathscr{L}} \mathrm{Gal}_F M' \geq \tau^w(M') = c_N (\log D_{\mathbf{C}'})^N \cdot \max\{1, \prod_{p \in \Delta_w(\mathbf{C}')} B|K^{\max}_{\mathbf{C}',p}/K_{\mathbf{C}'}(w)_p|\}$$

and $\tau^w(M')$ is referred to as the test invariant of M'.

In the expression of $\tau^w(M')$ the only new thing is the product $\prod_{p\in\Delta_w(\mathbf{C}')} |K^{\max}_{\mathbf{C}',p}/K_{\mathbf{C}'}(w)_p|$, where $K_{\mathbf{C}'}(w) = \prod_p K_{\mathbf{C}'}(w)_p = K_{\mathbf{G}}(w) \cap \mathbf{C}'(\mathbb{A}^f)$, and $\Delta_w(\mathbf{C}')$ is nothing but $\delta(\mathbf{C}', K_{\mathbf{G}}(w))$, i.e. the set of rational primes p such that $K_{\mathbf{C}'}(w)_p \subsetneq K^{\max}_{\mathbf{C}',p}$. The idea behind the formula is that the pure special subvariety $M(w)$ is isomorphic to $S_w := M_{K_{\mathbf{G}}(w)}(\mathbf{G}, X)$ which is equipped with a finite covering map $\pi_w : S_w \to S$. And the degree $\deg_{\pi^*\mathscr{L}}(\mathrm{Gal}_F M')$ is equal to $\deg_{\pi_w^*\mathscr{L}}(\mathrm{Gal}_F S')$, where S' is isomorphic to M' under the isomorphism $S_w \cong M(w)$. By functoriality of $\mathscr{L}$, we know that $\pi_w^*\mathscr{L}$ is isomorphic to the canonical ample invertible sheaf on S_w, namely $\mathrm{pr}_w^*(\mathscr{L}(K_{\mathbf{G}}(w)) \cong \mathscr{L}(K_{\mathbf{G}}(w))$, and the formula of $\tau(S')$ applies in this case, with the same constants c_N and B independent of the levels.

(2) We then proceed to show that the criterion of E.Ullmo and A.Yafaev remains valid in the mixed setting : if $(M_n)_n$ is a sequence of pure special subvarieties in M whose test invariants are uniformly bounded as n ranges over $\mathbb{N}$, then the sequence is weakly homogeneous, and the Zariski closure of $\bigcup_n M_n$ is a finite union of special subvarieties.

The idea is the same as the pure case. We may assume that M_n is a $\mathbf{C}_n$-special pure subvariety of $M(w_n)$ for some $w_n \in \mathbf{W}(\mathbb{Q})$ with respect to some $\mathbf{C}_n \subset \mathbf{G}$, and write $\tau_n(w_n) = \tau^{w_n}(M_n)$. Because $\tau_n(w_n) \geq c_N(\log D_{\mathbf{C}_n})^N$ where $D_{\mathbf{C}_n}$ is the absolute discriminant of $F_{\mathbf{C}_n}$ the splitting field of $\mathbf{C}_n$, the upper bound of $(\tau_n(w_n))_n$ shows that the $\log D_{\mathbf{C}_n}$'s takes only finitely many values. Hence the factors

$$J_n(w_n) = \prod_{p\in\Delta_{w_n}(\mathbf{C}_n)} B|K^{\max}_{\mathbf{C}_n,p}/K_{\mathbf{C}_n}(w)_p|$$

are bounded when n runs over $\mathbb{N}$. Similar to the pure case treated by B.Edixhoven and A.Yafaev, we show that $J_n(w_n) \geq \prod_{p\in\Delta_{w_n}(\mathbf{C}_n)} \frac{1}{2}Bp$, and thus $\sup\{p \in \Delta_{w_n}(\mathbf{C}_n) : n \in \mathbb{N}\}$ is finite. From this we deduce the finiteness of $\bigcup_n \Delta_{w_n}(\mathbf{C}_n)$, and we denote by $m(\in \mathbb{N})$ the cardinality of this union.

On the other hand, we consider the test invariants of $S_n = \pi(M_n) \subset S$. Note that S_n is $\mathbf{C}_n$-special, and the formula in [UY-1] shows that $\tau(S_n) = c_N(\log D_{\mathbf{C}_n})^N \max\{1, J_n(0)\}$ with $J_n(0) = \prod_{\delta(\mathbf{C}_n)} B|K^{\max}_{\mathbf{C}_n,p}/K_{\mathbf{C}_n,p}|$. From the reasoning in [UY-1] we see that the positive constant B is strictly less than 1, and it is easy to calculate the following

$$\frac{J_n(w_n)}{J_n(0)} = \prod_{p\in\delta(\mathbf{C}_n)} |K_{\mathbf{C}_n,p}/K_{\mathbf{C}_n}(w_n)| \cdot \prod_{p\in\Delta_{w_n}(\mathbf{C}_n)-\delta(\mathbf{C}_n)} B|K^{\max}_{\mathbf{C}_n,p}/K_{\mathbf{C}_n}(w)_p|$$

$$\geq B^{|\Delta_{w_n}(\mathbf{C}_n)-\delta(\mathbf{C}_n)|} \geq B^m$$

and thus $J_n(0) \leq B^{-m}J_n(w_n)$. Here $\delta(\mathbf{C}_n) := \delta(\mathbf{C}_n, K_{\mathbf{G}})$. It turns out that the sequence $(S_n)_n$ is of bounded test invariants, and is weakly homogeneous. Because $S_n = \pi(M_n)$, we thus deduce that $(M_n)_n$ is weakly homogeneous, and consequently, the Zariski closure of $\bigcup_n M_n$ is weakly special, consequent to the equidistribution results in the Chapter 3.

(3) Up to now we have seen the case of a sequence of pure special subvarieties, and we extend our approach to the mixed case in a trivial way: for a mixed special subvariety M' in M, we define the test invariant of M' to be the infinum of $\tau(S')$ with S' running through the set of maximal pure special subvarieties of M'. The point is that if S' is a maximal pure special subvariety of M', then M' is $\mathbf{C}$-special in M if and only if S' is $\mathbf{C}$-special, $\mathbf{C}$ being a given $\mathbb{Q}$-torus in $\mathbf{G}$. And the criterion still works in the mixed setting: let $(M_n)_n$ be a sequence of special subvarieties in M, with test invariants $\tau(M_n) = \tau(S_n)$ bounded as n varies, where S_n is some maximal pure special subvariety in M_n, then (S_n) is weakly homogeneous, and so it is with $(M_n)_n$, hence the Zariski closure of $\bigcup_n M_n$ is weakly special.

(4) One might prefer to conjecture that, in the above setting of a seuquence of pure special subvarieties, $(w_n \mod K_{\mathbf{W}})$ should be also be finite as long as $\tau_n(w_n)$ is bounded. But this seems to require more restrictions on the data $(\mathbf{G}_n, X_n)$ and the w_n's. Consider the following trivial example. Take a mixed Shimura datum $(\mathbf{P}, Y) = (\mathbf{H}, X_{\mathbf{H}}) \times (\mathbf{V} \rtimes \mathbf{G}_0, \mathbf{V}(\mathbb{R}) \rtimes X_0)$ with pure section $(\mathbf{H} \times \mathbf{G}_0, X_{\mathbf{H}} \times$

X_0), such that $X_{\mathbf{H}}$ is of dimension > 0, that $\mathbf{G}^0$ is semi-simple, and that the unipotent part $\mathbf{V}$ is non-zero. Consider a sequence of pure subdata $(\mathbf{H}_n, x_n) \times (v_n \mathbf{G}_0 v_n^{-1}, v_n \rtimes X_0)$ with $\mathbf{H}_n$ a $\mathbb{Q}$-torus of $\mathbf{H}$ and $v_n \in \mathbf{V}(\mathbb{Q})$. Fix a torsion-free compact open subgroup $K = K_{\mathbf{V}} \rtimes (K_{\mathbf{H}} \times K_{\mathbf{G}_0})$, we get the corresponding pure special subvarieties M_n. The projection $\pi : M \to S$ induces isomorphisms $M_n \to S_n = \pi(M_n)$, and it is easy to check that $\tau^{v_n}(M_n) = \tau(S_n)$: in fact $\mathbf{H}_n$ serves as the connected center of the $\mathbb{Q}$-group defining S_n, and $\mathbf{H}_n$ fixes v_n because it acts on $\mathbf{V}$ trivially. When $(\tau_n(w_n))_n$ is bounded, we see immediately that $(S_n)_n$ and hence $(M_n)_n$ are both weakly homogeneous. This has nothing to do with the torsion order of $(v_n \mod K_{\mathbf{V}})$.

In general even in the case where $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ is given by a faithful action $\mathbf{G} \to \mathbf{Aut}_{\mathbb{Q}}(\mathbf{W})$, it is possible to choose subdata $(\mathbf{G}_n, X_n)$ of $(\mathbf{G}, X)$ such that the connected center $\mathbf{C}_n$ of $\mathbf{G}_n$ fixes some $w_n \in \mathbf{W}(\mathbb{Q})$, and that the torsion order of $w_n \mod K_{\mathbf{W}}$ tends to infinity. We won't trace further results in this direction.

2.4.4 A generalization of the Manin-Mumford conjecture

Classically the Manin-Mumford was raised for abelian varieties over a field of characteristic zero. We propose two generalized forms of this conjecture, and discuss how they are related to the André-Oort conjecture.

(0) We first recall some standard definitions and properties of abelian S-schemes and the monodromy representation of the fundamental group S on the torsion sections of an abelian S-scheme $f : A \to S$. For simplicity we only work with the case where S is a scheme of characteristic zero, and we assume that S is geometrically integral of generic point η.

For an abelian S-scheme $f : A \to S$ with S geometrically integral of characteristic zero, we fix $\bar{x}$ a generic geometric point of S, $\pi_1(S, x)$ the fundamental group of S, and $\hat{S} \to S$ the Galois covering corresponding to the kernel of the monodromy representation $\mathbf{mon}_{\hat{\mathbb{Z}}}(f, \bar{x}) : \pi_1(S, \bar{x}) \to \mathrm{GL}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$.

(1) We start with the case where $S = \hat{S}$, i.e. $\pi_1(S, \bar{x})$ acts trivially on the torsion sections. Let $(a_n)_n$ be a sequence of torsion sections of $A \to S$. Put $b_n = a_n \times_S \eta$, we get a sequence of torsion points $(b_n)_n$ in the abelian variety A_η over η. The Zariski closure of $\{b_n\}_n$ is a torsion subvariety B of A_η. We take A' to be the Zariski closure of B in A., which is equal to the Zariski closure of $\{a_n\}_n$.

We then show that A' is a finite union of torsion subscheme, and for simplicity we assume that B is irreducible, namely a single torsion subvariety, and so it is with A', which reduces us to show that A' is a torsion subscheme. We may translate B by a torsion point b such that $b + B$ is an abelian subvariety. b can be lifted uniquely to a torsion section a of $A \to S$. Hence $a + A'$ equals the Zariski closure of $b + B$, and it suffices to show that $a + A'$ is an abelian S-subscheme. We may thus assume that $b = 0$ and $a = 0$.

B is then an abelian subvariety. Now that A' is the closure B, it is itself stable under the group law of $A \to S$, hence a group S-subscheme. It remains to show that A' is a smooth S-subscheme. To this end it suffices to show that $A' \to S$ is flat and its fibers are all smooth. The flatness is evident because A' contains sufficiently many torsion sections. The smoothness of the fibers is clear: the fiber of $A' \to S$ at a geometric point $x \hookrightarrow S$ is a group variety over x, and is automatically smooth because S, and henceforth x, is of characteristic zero.

The general case where $S \neq \hat{S}$ is similarly treated: it suffices to take orbits under $\pi_1(S)$. See Chapter 5 for details.

(1)' If S is known to be normal, then from the main theorem in [G] one can show that the following étale sheaves are actually constant sheaves on $S_{\text{ét}}$:

– the torsion sheaves $A[N] = \mathrm{Ker}([N] : A \to A)$, the integral Tate module $\mathbb{T}(A) = \varprojlim_N A[N]$, and the total Tate module $\mathbb{A}^f{}_S \otimes_{\hat{\mathbb{Z}}_S} (\mathbb{T}(A))$;

– the endomorphism sheaves $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$, $\mathbf{End}_{\mathbb{A}^f{}_S}(\mathbb{T}(A))$, and their subsheaves $\mathbf{End}_S(A)$ and $\mathbf{End}^\circ_S(A) := \mathbb{Q}_S \otimes_{\mathbb{Z}_S} \mathbf{End}_S(A)$.

Consequently, the specialization at η $T \mapsto T_\eta$ induces bijections between $\mathbb{T}(A)$ and $\mathbb{T}(A_\eta)$, resp. $\mathrm{End}_S(A)$ and $\mathrm{End}_\eta(A_\eta)$, etc. We thus conclude that when $S = \hat{S}$, the Manin-Mumford conjecture over the generic fiber η extends to the whole abelian S-scheme $f : A \to S$.

We thank Prof.M.Raynaud for communicating to us the theorem of A.Gorthdieck in loc.cit which completes our original approach.

(2) We then consider a more general form of the Manin-Mumford conjecture, which contains the André-Oort conjecture for mixed Shimura varieties as a special case:

Let $f : A \to S$ be an abelian S-scheme, with S geometrically integral of generic point η. Suppose we are given a sequence of closed subschemes $(S_n)_n$ in S, and for each n a special S_n-subscheme T_n of $S_n \times_S A$, viewed as a closed subscheme of A. Assume that $\bigcup_n S_n$ is Zariski dense in S, then under what conditions is the Zariski closure of $\bigcup_n T_n$ a weakly special S-subscheme of S?

The André-Oort conjecture somehow overlaps with the above question: if the abelian S-scheme is given by a morphism of mixed Shimura varieties $\pi : M \to S$ which is a fibration over a pure section, and that the S_n's are special pure subvarieties of S, then the André-Oort conjecture predicts that the Zariski closure of $\bigcup_n T_n$ is a weakly special subvarieties in M. On the other hand there are trivial counter-examples to the question, and much remains to be refined for the formulation.

(4) However, we still provide a simple case where the question becomes meaningful: let T be a closed irreducible S-subscheme of A which is faithfully flat over S such that $T = [N]T$ for some integer $N > 1$, then T is weakly special. This is essentially reduced to the Manin-Mumford conjecture over a general base. The statement also shed some light on the André-Oort conjecture: in order to show the Zariski closure T of a sequence of special subvarieties T_n, it suffices to show that T is stable under some non-trivial homothety.

Chapter 3

Equidistribution of special subvarieties: the homogeneous case

3.1 Introduction

3.1.1 The strategy of L.Clozel and E.Ullmo

In [CU-1], L.Clozel and E.Ullmo studied the equidistribution of strongly special subvarieties in a pure Shimura variety $S = M_{K_G}(\mathbf{G}, X)$ defined by some pure Shimura datum $(\mathbf{G}, X)$ with $\mathbf{G}$ adjoint, where by "strongly special" subvariety is meant a connected component S' of a Shimura subvariety defined by a subdatum $(\mathbf{G}', X')$ such that $\mathbf{G}'$ is semi-simple. Note that this notion is referred to as being 1-special in our terminology, with 1 standing for the trivial $\mathbb{Q}$-torus. The strategy is as follows:

(1) Study the equidistribution of lattice subspaces in $\Omega = \Gamma\backslash\mathbf{G}(\mathbb{R})^+$ for Γ some fixed arithmetic lattice of $\mathbf{G}$. Here the lattice subspaces are defined to be subspaces of the form $\Omega_{\mathbf{H}} = \Gamma\backslash\Gamma\mathbf{H}(\mathbb{R})^+$ for $\mathbf{H} \subset \mathbf{G}$ a $\mathbb{Q}$-subgroup of type $\mathcal{H}$. The advantage of these notions comes from a theorem of Mozes and Shah: consider $\nu_{\mathbf{H}}$ the canonical measure on Ω whose support is $\Omega_{\mathbf{H}}$, then the set $\mathcal{H}(\Omega)$ of such measures, with $\mathbf{H}$ varying over the $\mathcal{H}$-type $\mathbb{Q}$-subgroups of $\mathbf{G}$, is compact for the weak topology; moreover if a sequence $(\nu_n)_n$ in $\mathcal{H}(\Omega)$ converges to some ν, then $\mathrm{Supp}\,\nu$ equals the archimedean closure of $\bigcup_{n>N}\mathrm{Supp}\,\nu_n$ for some $N > 0$.

As a consequence, we see that the archimedean closure of a sequence of lattice subspaces is a finite union of lattice subspaces: this can be regarded as an André-Oort type theorem at the level of lattice subspaces.

(2) The passage from lattice subspaces to strongly special subvarieties is clear: let $(\mathbf{G}', X')$ be a subdatum of $(\mathbf{G}, X)$ with $\mathbf{G}'$ semi-simple, then $\mathbf{G}'$ is of type $\mathcal{H}$, and for any $x \in X' \subset X$ we have the projection $\kappa_x : \Omega' = \Gamma\backslash\Gamma\mathbf{G}'(\mathbb{R})^+ \to S' = \Gamma\backslash\Gamma X'^+$, $\Gamma g \mapsto \Gamma g x$; moreover κ_x pushes forward the canonical measure ν' supported

on Ω' to the canonical measure μ' on S supported on S'. Denote by $\mathcal{H}(S)$ the set of measures μ' obtained this way. Then L.Clozel and E.Ullmo established the analogue of the theorem of Mozes and Shah for $\mathcal{H}(S)$: it is compact for the weak topology, and if a sequence $(\mu_n)_n$ converges to some μ, then $\mathrm{Supp}\,\mu$ is the archimedean closure of $\bigcup_{n>N}\mathrm{Supp}\,\mu_n$ for some $N > 0$.

The proof of the compactness requires a variant of a proposition of S.Dani and G.Margulis: there exists a compact subset C of X such that if $S' \subset S$ is a strongly special subvariety, then we may choose the defining datum $(\mathbf{G}', X')$ such that $X' \cap C \neq \emptyset$. Take $x \in X' \cap C$, then the canonical measure $\mu' \in \mathcal{H}(S)$ associated to S' is $\kappa_{x*}(\nu')$ with $\nu' \in \mathcal{H}(\Omega)$ associated to $\Gamma\backslash\Gamma\mathbf{G}'(\mathbb{R})^+$. We then deduce the compactness of $\mathcal{H}(S)$ from that of C and of $\mathcal{H}(\Omega)$. The remaining part is proved similarly.

(3) Here we encounter the delicate fact that $\mathcal{H}(S)$ is closed for the weak topology: $\mathcal{H}(S)$ is the set of canonical measures supported on strongly special subvarieties, and if a sequence $(\mu_n)_n$ in $\mathcal{H}(S)$ converges to some measure μ', then μ' is automatically associated to some strongly special subvarieties. This is not as difficult as it appears: if we denote by $\mathbf{G}_n$ resp. $\mathbf{G}'$ the $\mathbb{Q}$-groups corresponding to μ_n resp. μ', then the convergence at the level of lattice spaces implies that $\mathbf{G}_n \subset \mathbf{G}'$ for sufficiently many n, and for such n's, take $x_n \in X_n^+$, then using the conditions on Hodge structures one can show that the pair $(\mathbf{G}', X' = \mathbf{G}'(\mathbb{R})x_n)$ is already a Shimura subdatum with $\mathbf{G}'$ semi-simple, independent of the choice of base points x_n, and μ' is the canonical measure associated to S' a connected component of the Shimura subvariety defined by $(\mathbf{G}', X')$.

(4) In [UY], the authors generalized the above results to sequences of $\mathbf{T}$-special subvarieties for some fixed $\mathbb{Q}$-torus $\mathbf{T}$ contained in $\mathbf{G}$. The idea is that in $S = M_{K_G}(\mathbf{G}, X)$ the set of maximal $\mathbf{T}$-special subvarieties is finite, denoted by $(S_i)_{i\in I}$, and when we study a sequence of $\mathbf{T}$-special subvarieties $(S_n)_n$ contained in a single S_i, we can "take quotient modulo $\mathbf{T}$" and reduce to the situation of a sequence of strongly special subvarieties inside a Shimura variety defined by a semi-simple $\mathbb{Q}$-group. Then it suffices to apply the results of [CU].

3.1.2 Our strategy in the mixed case

The basic observation is that the theorem of Mozes-Shah holds for arbitrary $\mathbb{Q}$-groups of type $\mathcal{H}$: namely those $\mathbb{Q}$-groups $\mathbf{Q}$ admitting a Levi decomposition of the form $\mathbf{Q} = \mathbf{W} \rtimes \mathbf{H}$ with $\mathbf{W}$ unipotent and $\mathbf{H}$ semi-simple without compact $\mathbb{Q}$-factors. However:

(i) Such $\mathbf{Q}$ might not readily define some mixed Shimura datum $(\mathbf{Q}, Y_{\mathbf{Q}})$; in general a central $\mathbb{Q}$-torus has to be joined to the reductive part $\mathbf{H}$. Therefore for a mixed Shimura datum $(\mathbf{P}, Y)$ with Levi decomposition $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$, we associate the $\mathbb{Q}$-group $\mathbf{P}^\dagger = \mathbf{W} \rtimes \mathbf{G}^{\mathrm{der}}$ which is of type $\mathcal{H}$, and consider lattice spaces of the form $\Gamma\backslash\mathbf{P}^\dagger(\mathbb{R})^+$. The theorem of Mozes-Shah works in this situation.

(ii) But the projection $\kappa_x : \Gamma\backslash\mathbf{P}^\dagger(\mathbb{R})^+ \to \Gamma\backslash Y^+$ is not onto: the image is only dense for the Zariski topology instead of the archimedean one, and the canonical measure on $\Gamma\backslash Y^+$ is not of finite volume.

In order to work coherently with the ergodic arguments, we introduce the notion of S-spaces associated to mixed Shimura varieties. The basic idea is more transparent when we consider the example of a sequence of torsion points $(t_n)_n$ in the complex multiplicative group $\mathbb{C}^\times$: if $(t_n)_n$ is stable under Galois conjugation with torsion order tending to infinity, then the sequence is equidistributed in the real circle of modulus one, and is dense in $\mathbf{C}^\times$ for the complex Zariski topology. Similarly, the S-spaces serve as real parts of the corresponding mixed Shimura varieties. They are dense for the complex Zariski topology, and they carry canonical probability measures. Moreover, for any x in the real part $Y_\mathbb{R}$ of the mixed Shimura datum $(\mathbf{P}, Y)$, we have a projection $\kappa_x : \Gamma\backslash\mathbf{P}^\dagger(\mathbb{R})^+ \to \Gamma\backslash Y_\mathbb{R}^+$, $\Gamma g \mapsto \Gamma g x$, where $Y_\mathbb{R}^+$ is the connected component of $Y_\mathbb{R}$ containing x. This projection is surjective and quasi-compact, and it sends the canonical measure on the lattice space to the canonical measure on the S-space.

The notion of S-spaces is well adapted for our strategy in the mixed case:

Fix a mixed Shimura datum $(\mathbf{P}, Y)$ with pure section $(\mathbf{G}, X)$, we denote by $\mathbf{C}$ the connected center of $\mathbf{G}$. We fix also a torsion free compact open subgroup $K = K_\mathbf{W} \rtimes K_\mathbf{G}$, and associate to it the mixed Shimura variety fibred over the pure section $\pi : M = M_K(\mathbf{P}, Y) \to S = M_{K_\mathbf{G}}(\mathbf{G}, X)$. We denote by $Y_\mathbb{R}$ for the real part of Y and $\mathcal{M}$ for the S-space associated to M. Note that $\mathcal{M} = M(\mathbb{C})_{an}$ as topological spaces if and only if there is no unipotent part of weight -2 in $\mathbf{P}$. For example, for a pure Shimura variety $S = M_{K_\mathbf{G}}(\mathbf{G}, X)$, the associated S-space $\mathcal{S}$ is no other than the real analytic space underlying $S(\mathbb{C})_{an}$.

To study the closure of a sequence of special subvarieties, it suffices to restrict to a connected component of M, say M^+. We assume that M^+ is given by the connected component Y^+ of Y, and the corresponding connected S-spaces is $\mathcal{M}^+$ given by $Y_\mathbb{R}^+$. Then $\mathcal{M}^+ = \Gamma\backslash Y_\mathbb{R}^+$ and $M^+ = \Gamma\backslash Y^+$, for some fixed arithmetic lattice $\Gamma \subset \mathbf{P}(\mathbb{Q})^+$.

(1) We define the set of H-measures on the lattice space $\Omega = \Gamma\backslash\mathbf{P}^\dagger(\mathbb{R})^+$ to be the set $\mathcal{H}(\Omega)$ of probability measures of the form $i_*\nu_0$, where for $\mathbf{P}_0^\dagger \subset \mathbf{P}^\dagger$, a $\mathbb{Q}$-subgroup of type $\mathcal{H}$, we associate ν_0 the canonical measure on $\Omega_0 = \Gamma\backslash\Gamma\mathbf{P}_0^\dagger(\mathbb{R})^+$, $i : \Omega_0 \hookrightarrow \Omega$ being the closed immersion of the lattice subspace. The theorem of Mozes and Shah implies the compactness of $\mathcal{H}(\Omega)$, and that the archimedean closure of a sequence of lattice subspaces remains a finite union of lattice subspaces: this is the "coarse" André-Oort conjecture for lattice spaces.

(2) Then we consider the subset $\mathcal{H}_\mathbf{C}(\Omega)$ of $\mathcal{H}(\Omega)$ consisting of the measures given by $\mathcal{H}$-type $\mathbb{Q}$-subgroup $\mathbf{P}_0^\dagger$ such that there exists a $\mathbf{C}$-special subdatum $(\mathbf{P}_0, Y_0)$ with $\mathbf{P}_0^\dagger$ equal to the maximal $\mathbb{Q}$-subgroup of $\mathbf{P}_0$ of type $\mathcal{H}$, where by $\mathbf{C}$-special subdata we mean subdata of the form $(\mathbf{P}', Y') \subset (\mathbf{P}, Y)$ such that the reduction modulo the unipotent radical $\pi : \mathbf{P} \to \mathbf{G}$ maps $\mathbf{P}'$ onto a reductive $\mathbb{Q}$-

subgroup $\mathbf{G}'$ of $\mathbf{G}$ whose connected center is $\mathbf{C}$. It turns out that $\mathcal{H}_{\mathbf{C}}(\Omega)$ is closed in $\mathcal{H}(\Omega)$ and hence is compact itself. The idea is that we can "remove" $\mathbf{C}$ simultaneously and then add it back: thus we get the André-Oort conjecture for $\mathbf{C}$-special lattice subspaces in Ω.

(3) We then pass from the lattice space Ω to the S-space $\mathcal{M} = \Gamma\backslash Y^+_{\mathbb{R}}$. For any $x \in Y^+_{\mathbb{R}}$, the projection $\kappa_x : \Omega \to \mathcal{M}$, $\Gamma g \mapsto \Gamma g x$ pushes the canonical measure on Ω forward to the canonical probability measure on $\mathcal{M}$. Similarly, for a $\mathbf{C}$-special subdatum $(\mathbf{P}', Y')$, we have $\mathbf{P}'^{\dagger} \subset \mathbf{P}'$ the maximal $\mathbb{Q}$-subgroup of type $\mathcal{H}$, the $\mathbf{C}$-special lattice subspace $\Omega' = \Gamma\backslash\Gamma\mathbf{P}'^{\dagger}(\mathbb{R})^+$, and the $\mathbf{C}$-special S-subspace $\mathcal{M}' = \Gamma\backslash\Gamma Y'^+_{\mathbb{R}}$. Take $x \in Y'^+_{\mathbb{R}}$, the projection κ_x sends Ω' onto $\mathcal{M}'$, and it pushes the H-measure ν' associated to Ω' to the canonical measure on $\mathcal{M}$ supported on $\mathcal{M}'$. Such measures makes up the set $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$, referred to as the $\mathbf{C}$-special H-measures on $\mathcal{M}$.

(4) To show that $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ is compact, it suffices to show the existence of a compact subset $C \subset Y^+_{\mathbb{R}}$ such that any $\mathbf{C}$-special S-subspace $\mathcal{M}' \subset \mathcal{M}$ is obtained from some $\mathbf{C}$-special subdatum $(\mathbf{P}', Y')$ with $Y'^+_{\mathbb{R}} \cap C \neq \emptyset$. This is essentially the Dani-Margulis argument in the pure case. In fact we have the fibration of $\mathcal{M}$ over the pure Shimura variety $\mathcal{S} = \Gamma_{\mathbf{G}}\backslash X^+$ by compact real tori, and the compact subset chosen for $\mathcal{S}$ can be lifted to a compact subset $C \subset \mathcal{M}$ that meets the requirements.

(5) Pay attention to the compactness of $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$. For a sequence $(\mu_n)_n$ in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$, the results in (3) and (4) confirm that $(\mu_n)_n$ has a subsequence converges to some probability measure on $\mathcal{M}$. Say $(\mu_n)_n$ converges to some probability measure μ' on $\mathcal{M}$. To see that it lies in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$, we need to show that $\mathrm{Supp}\,\mu'$ is of the form $\Gamma\backslash\mathbf{P}'^{\dagger}(\mathbb{R})^+ x'$ drawn from some Shimura subdatum $(\mathbf{P}', Y')$ and that $x' \in Y'^+_{\mathbb{R}}$. The proof is similar to the case treated by L.Clozel and E.Ullmo.

(6) Finally, by taking Zariski closure of the special S-subspaces, we get the André-Oort conjecture for homogeneous sequences of special subvarieties in a mixed Shimura variety.

Remark 3.1.3. Let $(\mathbf{G}, X)$ be a pure section of a mixed Shimura datum $(\mathbf{P}, Y)$, and $\mathbf{C}$ a $\mathbb{Q}$-torus in $\mathbf{G}$. Write $\pi : (\mathbf{P}, Y) \to (\mathbf{G}, X)$ for the canonical projection, and $\mathbf{W}$ the unipotent radical of $\mathbf{P}$.. We have seen the notion of $\mathbf{C}$-special subdata and $\mathbf{C}$-special subvarieties. The following two cases should mentioned explicitly:

(1) The case where $\mathbf{C}$ is of CM type, in the sense that for some $x \in X$ we have $x(\mathbb{S}) \subset \mathbf{C}_{\mathbb{R}}$. In this case $(\mathbf{C}, x)$ is the unique $\mathbf{C}$-special pure subdatum of $(\mathbf{G}, X)$, and $(\mathbf{W} \rtimes \mathbf{C}, \pi^{-1}(x))$ is the unique maximal $\mathbf{C}$-special subdatum of $(\mathbf{P}, Y)$. A $\mathbf{C}$-special subvariety defined by $(\mathbf{W} \rtimes \mathbf{C}, \pi^{-1}(x))$ is a $(\mathbb{C}^\times)^r$-torsor $B \to A$ over some CM abelian variety A, and the $\mathbf{C}$-special subvarieties are torsion subvarieties in B. Note that if $\mathbf{W} = \mathbf{U} \oplus \mathbf{V}$ is a trivial extension (including the case where $\mathbf{U} = 0$ and the case where $\mathbf{V} = 0$), then B is the product of a torus and an abelian variety, and

the notion of torsion subvarieties is the evident one; on the other hand, if **W** is a non-trivial extension of **V** by **U**, then the notion of torsion subvarieties involved here is defined in an ad.hoc way so as to coincide with our definition of special subvarieties in general.

(2) The case where **C** is not of CM type, i.e. for any $x \in X$, we have $x(\mathbb{S}) \not\subset \mathbf{C}_{\mathbb{R}}$. We only consider the case where there exists non-trivial **C**-special subdata, namely we assume the existence of a subdatum $(\mathbf{G}', X')$ of $\mathbf{G}, X)$ such that **C** is the connected center of $\mathbf{G}'$. In this case $(\mathbf{W} \rtimes \mathbf{G}', \pi^{-1}(X'))$ is the maximal **C**-special subdatum, and one can verify easily that each **C**-special subvariety is of dimension > 0.

The ergodic approach towards the equidistribution of **C**-special subvarieties (of dimensions > 0) works well in the case (2), which is to be developed in the following sections. On the other hand it is less effective in the case (1): here one might encounter infinitely many special points. Instead the Manin-Mumford conjecture already implies positive answers in this case.

We postpone the detailed study of the case (1) to a forthcoming preprint [Ch-2]. And in this chapter, we mainly consider the non-CM case (2).

It is also necessary to simplify the intermediate object $\mathbf{P}^{\dagger}$ introduced in the illustration above, as is in the following:

Lemma 3.1.4. *Let* $(\mathbf{P}, Y)$ *be a mixed Shimura datum. Then* $\mathbf{P}^{\dagger} = \mathbf{P}^{\mathrm{der}}$.

Proof. Let $(\mathbf{G}, X)$ be a pure section of $(\mathbf{P}, Y)$ and denote by ρ the conjugation action of **G** on **W**. As has been assumed since Chapter 1, we have $\mathbf{G} = \mathbf{MT}(X)$ and $\mathbf{P} = \mathbf{MT}(Y)$. We also have $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ and $\mathbf{P}^{\dagger} = \mathbf{W} \rtimes \mathbf{G}^{\mathrm{der}}$.

By definition, $\mathbf{P}^{\mathrm{der}}$ is the $\mathbb{Q}$-subgroup generated by $(a, b) = aba^{-1}b^{-1}$ for $a, b \in \mathbf{P}$, and $\mathbf{P}^{\mathrm{ab}} = \mathbf{P}/\mathbf{P}^{\mathrm{der}}$ is the maximal commutative quotient of **P**. In particular, $\mathbf{P}^{\mathrm{der}}$ contains $(w\mathbf{G}w^{-1})^{\mathrm{der}} = w\mathbf{G}^{\mathrm{der}}w^{-1}$ for all $w \in \mathbf{W}(\mathbb{Q})$, and it also contains $\mathbf{W}'$ by which we mean the $\mathbb{Q}$-subgroup generated by all the elements of the form $(w, g) = wgw^{-1}g^{-1} = w \cdot (g(w))^{-1}$ for $w \in \mathbf{W}(\mathbb{Q})$ and $g \in \mathbf{G}(\mathbb{Q})$, where $g(w) := gwg^{-1}$. $\mathbf{W}' \subset \mathbf{W}$ is clearly stable under the conjugation action by **G**, and $\mathbf{W}'$ is the smallest **G**-stable invariant $\mathbb{Q}$-subgroup such that **G** acts trivially on $\mathbf{W}/\mathbf{W}'$. From the conditions on Hodge types in the definition of mixed Shimura data, we deduce that $\mathbf{W} = \mathbf{W}'$.

Consequently, $\mathbf{P}^{\mathrm{der}} \supset \mathbf{W} \rtimes \mathbf{G}^{\mathrm{der}}$. Since $\mathbf{P}/(\mathbf{W} \rtimes \mathbf{G}^{\mathrm{der}}) = \mathbf{G}/\mathbf{G}^{\mathrm{der}}$ is commutative, we deduce that $\mathbf{P}^{\mathrm{der}}$ coincides with $\mathbf{P}^{\dagger} = \mathbf{W} \rtimes \mathbf{G}^{\mathrm{der}}$.

■

3.2 Preliminaries on groups and ergodic theory

Definition 3.2.1. (1) (cf. [Spr] Chap.16) Let F be a field, **G** a linear F-group, and $E \supset F$ an extension of fields. A linear F-group **G** is said to be E-isotropic if $\mathbf{G}_E$ is of E-rank at least one, namely contains a split E-torus, and E-anisotropic otherwise. In particular **G** being E-anisotropic implies that it is F-anisotropic. Note that here the E-rank is meant to be rank of a maximal split E-torus, and the group $\mathbf{G}$ is not assumed to be reductive.

This notion is trovial for unipotent linear groups, which contains no tori. Interesting examples come from real reductive groups: a reductive $\mathbb{R}$-group $\mathbf{G}$ is $\mathbb{R}$-anisotropic if and only if $\mathbf{G}(\mathbb{R})$ is compact as a Lie group. Consequently, if F is a subfield of $\mathbb{R}$, then $\mathbf{G}$ is said to be compact if it is $\mathbb{R}$-anisotropic, or equivalently, if $\mathbf{G}(\mathbb{R})$ is a compact Lie group. The definition of non-compact linear F-group is similar.

(2) (cf.[Milne-2] Chap.28) For $\mathbf{G}$ a linear $\mathbb{Q}$-group, a congruence subgroup of $\mathbf{G}(\mathbb{R})$ is a subgroup of the form $\Gamma = K \cap \mathbf{G}(\mathbb{Q})$ where $K \subset \mathbf{G}(\mathbb{A}^f)$ is a compact open subgroup; an arithmetic subgroup of $\mathbf{G}(\mathbb{R})$ is a subgroup Λ commensurable with some congruence subgroup, i.e. there exists some congruence subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ such that the two indices $[\Gamma : \Gamma \cap \Lambda]$ and $[\Lambda : \Lambda \cap \Gamma]$ are both finite.

An arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})$ is a lattice if the quotient space $\Gamma\backslash\mathbf{G}(\mathbb{R})$ is of finite volume with respect to the measure μ induced from the left-invariant Haar measure on $\mathbf{G}(\mathbb{R})$. When this is the case, we normalize μ to be of total mass 1, and refer to it as the canonical measure, or simply the Haar measure, on $\Gamma\backslash\mathbf{G}(\mathbb{R})$. If moreover the arithmetic lattice Γ is contained in $\mathbf{G}(\mathbb{R})^+$, we call the normalized restriction of μ to the quotient $\Omega = \Gamma\backslash\mathbf{G}(\mathbb{R})^+$ as the canonical, or, again by abuse of terminologies, the Haar measure on Ω.

Definition 3.2.2. For $\mathbf{G}$ a linear $\mathbb{Q}$-group, $\mathbf{G}$ is said to be of type $\mathcal{H}$, written as $\mathbf{G} \in \mathcal{H}$, if the maximal reductive quotient of $\mathbf{G}$ is semi-simple without compact $\mathbb{Q}$-factors. This is equivalent to the condition that $\mathbf{G} = \mathbf{W} \rtimes \mathbf{H}$ for $\mathbf{W}$ unipotent and $\mathbf{H}$ semi-simple without compact $\mathbb{Q}$-factors. In particular, its radical is unipotent.

This implies that some (hence each) arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})$ is a lattice, i.e. is of finite co-volume. In general, being given a reductive $\mathbb{Q}$-group $\mathbf{G}$, an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})$ is a lattice, i.e. $\Gamma\backslash\mathbf{G}(\mathbb{R})^+$ is of finite volume with respect to the measure induced from the left-invariant Haar measure on $\mathbf{G}(\mathbb{R})$, if and only if $\mathbf{G}$ admits only trivial $\mathbb{Q}$-characters, namely $\mathbf{X}_{\mathbf{G}}(\mathbb{Q}) = 1$; cf.[B-HC] Theorem.3 On the other hand, an arithmetic subgroup of a unipotent $\mathbb{Q}$-group is always a lattice, cf.[Ragh] Chap.2 Theorem.2.1. Hence by Levi-decomposition, a linear $\mathbb{Q}$-group $\mathbf{G}$ admits arithmetic lattices if and only if $\mathbf{X}_{\mathbf{G}}(\mathbb{Q}) = 1$. In particular this is the case for $\mathbb{Q}$-groups of type $\mathcal{H}$.

Moreover a reductive $\mathbb{Q}$-group $\mathbf{G}$ is said to be of Hermitian type if its derived group is non-compact of type $\mathcal{H}$ and the associated symmetric space is hermitian, namely carries an Hermitian metric invariant under $\mathbf{G}(\mathbb{R})^+$. For such a $\mathbf{G}$ the adjoint quotient $\mathbf{G}^{\mathrm{ad}}$ is of type $\mathcal{H}$. For example in a pure Shimura datum $(\mathbf{G}, X)$ the $\mathbb{Q}$-group $\mathbf{G}$ is of Hermitian type.

Recall some basic definitions from ergodic theory, cf. [Zim], Chap.2, 2.1.1 Definition :

Definition 3.2.3. For a measure space (Ω, μ), a map $T : \Omega \to \Omega$ is measure-preserving if for any measurable subset $S \subset \Omega$, $T^{-1}S$ remains measurable and $\mu(S) = \mu(T^{-1}S)$. T is said to be ergodic if for any T-invariant measurable subset $S \subset \Omega$, we have either $\mu(S) = 0$ or $\mu(\Omega - S) = 0$.

Let (Ω, μ) be a probability space defined over some Hausdorff space Ω, and H a countable Hausdorff topological group that acts on Ω from the right. The action is (weakly) continuous if the map $\Omega \times H \to \Omega$, $(x, g) \mapsto xg$ is continuous with respect to the product topology on $\Omega \times H$. The action is said to be ergodic if any element $g \in H$ is a measure-preserving map $g : \Omega \to H$ and any H-invariant measurable subset of Ω is of measure either 0 or 1.

We are particularly interested in the case where Ω is a homogeneous space of the form $\Gamma\backslash\mathbf{G}(\mathbb{R})^+$ and H is some unipotent subgroup of $\mathbf{G}(\mathbb{R})^+$, where $\mathbf{G}$ is some linear $\mathbb{Q}$-group of type $\mathcal{H}$ and $\Gamma \subset \mathbf{G}(\mathbb{R})^+$ is an arithmetic lattice. The existence of a unipotent subgroup $H \subset \mathbf{G}(\mathbb{R})^+$ that acts on Ω ergodically plays an essential role in the study of certain classes of measures on Ω, as is presented later in the theorem of M.Ratner, S.Mozes and N.Shah.

Definition 3.2.4. (cf. [CU-3] Section 2) Let $\mathbf{G}$ be a $\mathbb{Q}$-group of type $\mathcal{H}$, $\Gamma \subset \mathbf{G}(\mathbb{R})^+$ an arithmetic lattice, μ_G the normalized Haar measure on $\Omega = \Gamma\backslash\mathbf{G}(\mathbb{R})^+$. A connected closed Lie subgroup F of $\mathbf{G}(\mathbb{R})^+$ is said to be of type $\mathcal{K}$, written as $F \in \mathcal{K}$, if the following hold:

(1) $\Gamma_F = \Gamma \cap F$ is a lattice of F, and the inclusion $i_L : \Omega_F = \Gamma_F\backslash F \hookrightarrow \Omega$ is a closed immersion of real analytic spaces;

(2) write μ_F the canonical probability measure on Ω_F deduced from the Haar measure of $\mathbf{G}(\mathbb{R})^+$ and L(F) the subgroup of F generated by one-parameter unipotent subgroups of F, then L(F) acts on the probability space (Ω_F, μ_F) ergodically.

Lemma 3.2.5. *(cf. [CU-3] Lemme 2.1 and Lemme 2.2; [U-3], Lemme 2.3 and Lemme 2.4; [S] Lemma 2.9, Prop.3.2, Remark 3.7) Let* $\mathbf{G}$ *be a linear* $\mathbb{Q}$*-group of type* $\mathscr{H}$*, and* $\Gamma \subset \mathbf{G}(\mathbb{R})^+$ *an arithmetic lattice, then*

(1) for an arbitrary $\mathbb{Q}$*-subgroup* $\mathbf{H} \subset \mathbf{G}$, $\mathbf{H} \in \mathcal{H}$ *implies that* $H^+ \in \mathcal{K}$ *where* $H^+ = \mathbf{H}(\mathbb{R})^+$*;*

(2) if $F \in \mathcal{K}$ *is a connected closed Lie subgroup of* $\mathbf{G}(\mathbb{R})^+$*, then there exists a* $\mathbb{Q}$*-subgroup* $\mathbf{H} \subset \mathbf{G}$ *of type* $\mathscr{H}$ *such that* $F = \mathbf{H}(\mathbb{R})^+$*. In this case* $\mathbf{H}$ *is the Mumford-Tate group of* F *in* $\mathbf{G}$*, i.e. the minimal* $\mathbb{Q}$*-subgroup of* $\mathbf{G}$ *whose real locus contains* F.

Recall that for a locally compact Hausdorff topological space Ω, the set $\mathscr{R}(\Omega)$ of continuous linear functionals on the set of compactly supported continuous functions on $\mathscr{C}_c^0(\Omega)$ is endowed with the weak topology, in the sense that a sequence $(\mu_n)_n$ in $\mathscr{R}(\Omega)$ converges to some $\mu \in \mathscr{R}(\Omega)$ if and only if for any $f \in \mathscr{C}_c^0(\Omega)$ we have $\lim_{n\to\infty} \mu_n(f) = \mu(f)$. We realize the set $\mathcal{P}(\Omega)$ of Borelian probability measures on Ω as a closed subset of $\mathscr{R}(\Omega)$, endowed with the induced weak topology.

Definition-Proposition 3.2.6. (cf. [CU-3] Thm.2.4, Prop.2.5; [MS] Thm.1.1, Cor.1.4) For $\mathbf{G}$, Γ, Ω as above, consider $\mathcal{H}(\Omega)$ the set of probability measure on Ω of the form $i_{\mathbf{H}*}\nu_{\mathbf{H}}$ where $\mathbf{H}\subset\mathbf{G}$ is some $\mathbb{Q}$-subgroup of type $\mathcal{H}$, $\nu_{\mathbf{H}}$ the canonical probability measure on $\Omega_{\mathbf{H}}=\Gamma\backslash\Gamma\mathbf{H}(\mathbb{R})^+\cong(\Gamma\cap\mathbf{H}(\mathbb{R})^+)\backslash\mathbf{H}(\mathbb{R})^+$, and $i_{\mathbf{H}}:\Omega_{\mathbf{H}}\hookrightarrow\Omega$ the closed immersion of real analytic spaces. The elements in $\mathcal{H}(\Omega)$ are referred to as the H-measures on Ω, and we regard $\mathcal{H}(\Omega)$ as a subset of $\mathcal{P}(\Omega)$ the set of Borelian probability measures on Ω, equipped with the weak topology.

S.Mozes and N.Shah showed that $\mathcal{H}(\Omega)$ is compact. Moreover if a sequence $(\nu_n)_n$ in $\mathcal{H}(\Omega)$ converges to some $\nu\in\mathcal{H}(\Omega)$, then for some $\mathrm{N}\in\mathbb{N}$ the set $\bigcup_{n>\mathrm{N}}\mathrm{Supp}\,\nu_n$ is dense in $\mathrm{Supp}\,\nu$ for the archimedean topology.

Remark 3.2.7. The results in [Rat-1], [Rat-2], and in [MS] provide us with a delicate description of certain measures on the lattice space Ω. We briefly list their works as follows:

For each $\mu\in\mathcal{P}(\Omega)$, define $\Lambda(\mu):=\{g\in\mathbf{G}(\mathbb{R})^+:\mu\cdot g=\mu\}$, which is a closed Lie subgroup of $\mathbf{G}(\mathbb{R})^+$, and we set $\mathrm{L}(\mu)=\mathrm{L}(\Lambda(\mu))$ to be the subgroup of $\Lambda(\mu)$ generated by one-parameter ad-unipotent subgroups, and $\overline{\mathrm{L}(\Lambda(\mu))}$ the closure of $\mathrm{L}(\Lambda(\mu))$ in $\Lambda(\mu)$ for the archimedean topology. Consider the following sets of probability measures

$$\text{algebraic measures } \mathcal{A}(\Omega)=\{\mu\in\mathcal{P}(\Omega):\mathrm{Supp}\,\mu=x\cdot\Lambda(\mu)\ \text{ for some } x\in\Omega\}$$

$$\text{ergodic measures } \mathcal{E}(\Omega)=\{\mu\in\mathcal{P}(\Omega):\mathrm{L}(\mu)\text{ acts ergodically on }(\Omega,\mu)\}$$

$$\text{and its subset } \mathcal{E}^+(\Omega)=\{\mu\in\mathcal{E}(\Omega):\Gamma\cdot e_{\mathbf{G}^+}\in\mathrm{Supp}\,\mu\}$$

and $\mathcal{H}(\Omega)$ the set of H-measures on Ω. Then

(1) (M.Ratner, [Rat-1]) $\mathcal{E}(\Omega)\subset\mathcal{A}(\Omega)$ i.e. every ergodic measure is algebraic.

(2) (Mozes-Shah, [MS]) $\mathcal{E}^+(\Omega)$ and $\mathcal{E}(\Omega)$ are closed in $\mathcal{P}(\Omega)$ and $\mathcal{E}^+(\Omega)$ is compact.

(3) (Mozes-Shah, [MS]) $\mathcal{H}(\Omega)=\mathcal{E}^+(\Omega)$ i.e. an H-measure is the same as an algebraic measure whose support contains the point Γe, where e is the neutral element of $\mathbf{G}(\mathbb{R})^+$. For a convergent sequence $(\nu_n)_n$ in $\mathcal{H}(\Omega)$ of limit ν, we have the description of $\mathrm{Supp}\,\nu$ through $\mathrm{Supp}\,\nu_n$ as is stated in the above Definition-Proposition.

Recall the notions of $\mathbf{C}$-special sub-objects:

Definition 3.2.8. Fix a $\mathbb{Q}$-torus $\mathbf{C}\subset\mathbf{G}$, with $(\mathbf{G},X)$ a pure section of a mixed Shimura datum $(\mathbf{P},Y)$.

(1) A Shimura subdatum $(\mathbf{P}_1,Y_1)\subset(\mathbf{P},Y)$ is said to be $\mathbf{C}$-special if $\mathbf{C}$ equals the connected center of $\pi_{\mathbf{W}}(\mathbf{P}_1)=\mathbf{G}_1$. Note that by calculating the Hodge types we see that $\mathbf{W}_1=\mathbf{W}\cap\mathbf{P}_1$ equals the unipotent radical of $\mathbf{P}_1$, and the image of $\mathbf{P}_1$ in $\mathbf{G}$, namely $\pi_{\mathbf{W}}(\mathbf{P}_1)$, is isomorphic to a Levi $\mathbb{Q}$-subgroup of $\mathbf{P}_1$.

In particular 1-special subdata are given by $\mathbb{Q}$-groups of the form $\mathbf{W}_1\rtimes w\mathbf{G}_1w^{-1}$ with $\mathbf{W}_1\subset\mathbf{W}$, $w\in\mathbf{W}(\mathbb{Q})$ and $\mathbf{G}_1\subset\mathbf{G}$ semi-simple. These are the mixed versions of the "strongly special" subdata considered in [CU]. However in order to include mixed Shimura data we could not assume à priori that $\mathbf{G}$ is adjoint: were it adjoint, then from the associated representations $\mathbf{G} \to \mathrm{GL}_{\mathbb{Q}}(\mathbf{V_P})$ and $\mathbf{G} \to \mathrm{GL}_{\mathbb{Q}}(\mathbf{U})$ we get Hodge structures of weight zero with respect to any $x : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ in X, and $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ would not produce mixed Shimura data in the prescribed way.

(2) For a fixed set $\mathfrak{R}_{\mathrm{K}}^{\mathbf{P}}$ of representatives of $\mathbf{P}(\mathbb{Q})_{+}\backslash\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}$, a special subvariety $\mathrm{S} \subset \mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$ is $\mathbf{C}$-special if it is of the form $\wp(\mathrm{Y}_1^+ \times g\mathrm{K})$ for some $\mathbf{C}$-special subdatum $(\mathbf{P}_1, \mathrm{Y}_1)$ and some $g \in \mathfrak{R}_{\mathrm{K}}^{\mathbf{P}}$.

Remark 3.2.9. Note that, unless in the case $\mathbf{C} = 1$, different $\mathbf{C}$-special data might produce the same special subvariety: for example, for any $q \in \Gamma_{\mathbf{G}}$ with $\Gamma_{\mathbf{G}} = \mathbf{G}(\mathbb{Q})^+ \cap \mathrm{K}$, being $\mathbf{C}$-special in $(\Gamma_{\mathbf{W}} \rtimes \Gamma_{\mathbf{G}})\backslash\mathrm{Y}^+$ is the same as being $q\mathbf{C}q^{-1}$-special. In fact it is known from the construction that conjugating $\mathbf{C}$ by $\Gamma_{\mathbf{G}}$ does not change the underlying space of a $\mathbf{C}$-special variety. Besides the definition of $\mathbf{C}$-special variety is only concerned with the image under the projection π, hence the invariance under conjugation by $\mathbf{W}(\mathbb{Q})$.

In this chapter we are only concerned with sequences of special subvarieties with prescribed defining data.

3.3 Equidistribution of lattice subspaces

Definition 3.3.1. Let $\mathbf{Q}$ be a $\mathbb{Q}$-group of type $\mathcal{H}$, and $\Gamma \subset \mathbf{Q}(\mathbb{R})^+$ an arithmetic lattice. The quotient $\Omega = \Gamma\backslash\mathbf{Q}(\mathbb{R})^+$ is referred to as the lattice space (associated to $\mathbf{Q}$ and Γ), endowed with the probability measure ν induced from the left invariant Haar measure on $\mathbf{Q}(\mathbb{R})^+$.

A lattice subspace of Ω is a subset of the form $\Omega' = \Gamma\backslash\Gamma\mathbf{Q}'(\mathbb{R})^+$ where $\mathbf{Q}' \subset \mathbf{Q}$ is a $\mathbb{Q}$-subgroup of type $\mathcal{H}$. Note that Ω' is a real analytic subspace.

The Haar measure on $\mathbf{Q}(\mathbb{R})^+$ induces a probability measure on Ω'. Under the inclusion map $i' : \Omega' \hookrightarrow \Omega$ it is pushed-forward to a probability measure ν' on Ω with support Ω'. ν' is referred to as the H-measure on Ω of support Ω'.

The set of H-measures on Ω is denoted as $\mathcal{H}(\Omega)$. This is a countable subset of the set of probability measures on Ω. We endow it with the weak topology.

Proposition 3.3.2. *Let $\mathbf{Q}$, Γ, and Ω be as above. Then:*

(1) $\mathcal{H}(\Omega)$ is compact.

(2) For any closed subset $\mathrm{Z} \subset \Omega$, the set $\mathscr{S}(\mathrm{Z})$ of maximal lattice subspaces contained in Z is finite. Equivalently, for $(\Omega_n)_n$ a sequence of lattice subspaces of Ω, the archimedean closure of $\bigcup_n \Omega_n$ is always a finite union of lattice subspaces.

Proof. (1) is part of the theorem of S.Mozes and N.Shah.

(2) First we note that the set of lattice subspaces in Ω is countable, just as it is with $\mathcal{H}(\Omega)$. In particular $\mathscr{S}(Z)$ is countable for Z closed in Ω. If it is infinite, we write it as a sequence (Ω_n) and write (ν_n) for the corresponding infinite sequence of H-measures on Ω. Then $(\nu_n)_n$ contains a convergent subsequence $(\nu'_n)_n$ of limit $\nu' \in \mathcal{H}(\Omega)$, and we may suppose that $\bigcup_n \mathrm{Supp}_n \nu'_n$ is dense in $\mathrm{Supp}\,\nu'$ for the archimedean topology. Therefore we have $\mathrm{Supp}\,\nu'_n \subsetneq \mathrm{Supp}\,\nu' \subset Z$ for n large, which contradicts the maximality of the $\mathrm{Supp}\,\nu'_n$'s.

Thus we conclude the finiteness of $\mathscr{S}(Z)$, and the equivalent formulation is derived similarly. ■

We then proceed to the notion of **C**-special H-measures.

We fix $(\mathbf{P}, Y)$ a mixed Shimura datum with pure section $(\mathbf{G}, X)$, given by a Levi decomposition $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$. $\mathbf{C} = \mathbf{C}_\mathbf{G}$ denotes the connected center of $\mathbf{G}$, and by Lemma 3.1.4 we have $\mathbf{P}^{\mathrm{der}} = \mathbf{W} \rtimes \mathbf{G}^{\mathrm{der}}$, which is the unique maximal $\mathbb{Q}$-subgroup of type $\mathcal{H}$ in $\mathbf{P}$. Take torsion-free arithmetic subgroup $\Gamma_\mathbf{W} \subset \mathbf{W}(\mathbb{R})$, $\Gamma_\mathbf{G} \subset \mathbf{G}(\mathbb{R})^+$, and $\Gamma = \Gamma_\mathbf{W} \rtimes \Gamma_\mathbf{G} \subset \mathbf{P}(\mathbb{R})^+$ stabilizing $\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$. Take $\Gamma^\dagger = \Gamma \cap \mathbf{P}^{\mathrm{der}}(\mathbb{Q})^+$, then the quotient $\Omega = \Gamma^\dagger \backslash \mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ is referred to as the lattice space associated to $(\mathbf{P}, Y)$ and Γ. But for most of this section we only need the arithmetic subgroup $\Gamma^\dagger$ of $\mathbf{P}^{\mathrm{der}}(\mathbb{R})$, therefore in this section the discrete subgroups $\Gamma \subset \mathbf{P}(\mathbb{R})^+$ are understood to be arithmetic subgroups of $\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$.

We note that the quotient $\Gamma_\mathbf{G} \backslash \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ is of finite volume with respect to the quotient measure $\nu_\mathbf{G}$ induced from the Haar measure of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$. Similarly, the Haar measure on $\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ induces a probability measure ν on $\Omega = \Gamma \backslash \mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$, which equals the H-measure of Ω.

The **C**-special subvarieties play the role of strongly special subvarieties in the work of L.Clozel and E.Ullmo. We introduce analog notions for lattice subspaces as follows:

Definition 3.3.3. (1) A subspace $\Omega' \subset \Omega$ is a **C**-special lattice subspace if it is given by a closed immersion of lattice subspaces $\Omega' = \Gamma' \backslash \mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+ \xrightarrow{i'} \Omega$, where $\mathbf{P}'^{\mathrm{der}}$ is the derived $\mathbb{Q}$-group (of type $\mathcal{H}$) of the $\mathbb{Q}$-group $\mathbf{P}'$ which comes from a **C**-special subdatum $(\mathbf{P}', Y') \subset (\mathbf{P}, Y)$. Here $\Gamma' = \Gamma \cap \mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ is an arithmetic subgroup of $\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$.

For $Z \subset \Omega$ a closed subset, write $\mathscr{S}_\mathbf{C}(Z)$ for the set of maximal **C**-special lattice subspaces contained in Z.

(2) An H-measure $\nu' \in \mathcal{H}(\Omega)$ is said to be **C**-special, if it is the canonical H-measure associated to a **C**-special lattice subspace $\Omega' = \Gamma' \backslash \mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ for some **C**-specail subdatum $(\mathbf{P}', Y')$.

We also write $\mathcal{H}_\mathbf{C}(\Omega)$ for the set of **C**-special H-measures on Ω, which is of course a countable subset of $\mathcal{H}(\Omega)$.

(3) A sequence of **C**-special lattice subspaces $(\Omega_n)_n$ is said to be **C**-strict if for any non-maximal **C**-special lattice subspace $\Omega' \subsetneq \Omega$, we have $\Omega_n \not\subseteq \Omega'$ for n large enough. Similarly, a sequence $(\nu_n)_n$ is said to be **C**-strict if so it is with the sequence $(\mathrm{Supp}\,\nu_n)_n$.

Proposition 3.3.4. *Let $(\Omega_n)_n$ be a* **C**-*strict sequence of* **C**-*special lattice subspaces of* Ω *contained in a maximal* **C**-*special lattice subspace* Ω', *and denote by* ν_n *resp.* ν' *the corresponding* **C**-*special measures. Then* $(\nu_n)_n$ *converges to* ν' *for the weak topology, and* $\operatorname{Supp}\nu'$ *equals the closure of* $\bigcup_n \operatorname{Supp}\nu_n$.

Proof. We first note that for $\mathbf{C}\subset\mathbf{G}$ fixed, there exists only finitely many maximal **C**-special subdata of $(\mathbf{P}, Y)$, produced from the $\mathbb{Q}$-subgroup $\mathbf{W}\rtimes\mathbf{Z_G C}$ following Lemma 1.3.13 of Chapter 1.

Since it suffices to work within Ω', we may assume for simplicity that **C** equals the connected center of **G**, hence $\Omega=\Omega'$ is the unique maximal **C**-special lattice subspace. It remains to show that $(\nu_n)_n$ converges to the canonical measure ν of Ω.

We first show that $(\Omega_n)_n$ is generic, in the sense that if $\Omega'\subsetneq\Omega$ is an arbitrary lattice subspace associated to a $\mathbb{Q}$-subgroup $\mathbf{Q}'$, not necessarily **C**-special, then $\Omega_n\not\subseteq\Omega'$ for n-large enough. Assume that the contrary holds, namely there exists a lattice subspace Ω' associated to some $\mathbb{Q}$-subgroup $\mathbf{Q}'\subsetneq\mathbf{P}^{\mathrm{der}}$, such that $\Omega_n\subset\Omega'$ for infinitely many n's, and we may assume by restricting to a subsequence that $\Omega_n\subset\Omega'$. Consider the projection $\pi:\Omega=\Gamma\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+\to\Omega_{\mathbf{G}}=\Gamma_{\mathbf{G}}\backslash\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$. Then $\pi(\Omega_n)$ is a **C**-strict sequence of **C**-special lattice subspaces of $\Omega_{\mathbf{G}}$, whose supports are of the form $\Gamma_{\mathbf{G}}\backslash\Gamma_{\mathbf{G}}\mathbf{G}_n^{\mathrm{der}}(\mathbb{R})^+$. $\Omega_n\subset\Omega'=\Gamma\backslash\Gamma\mathbf{Q}'(\mathbb{R})^+$ implies that

$$\pi(\Omega_n)=\Gamma_{\mathbf{G}}\backslash\Gamma_{\mathbf{G}}\mathbf{G}_n^{\mathrm{der}}(\mathbb{R})^+\subset\pi(\Omega')=\Gamma_{\mathbf{G}}\backslash\Gamma_{\mathbf{G}}\mathbf{G}'(\mathbb{R})^+$$

where $\mathbf{G}':=\pi(\mathbf{Q}')$. Computing the tangent space of $\pi(\Omega_n)$ and $\pi(\Omega')$ at the origin we find that $\mathbf{G}_n^{\mathrm{der}}\subset\mathbf{G}'$ for all n. $\mathbf{G}_n^{\mathrm{der}}$ are strong $\mathbb{Q}$-subgroups of $\mathbf{G}^{\mathrm{der}}$, it turns out that $\mathbf{G}'$ is reductive, according to Definition-Proposition 1.3.11 of Chapter 1 (cf.[CU-3] 4.1). The theorem of Mozes-Shah implies that $\mathbf{G}'$ is of type $\mathcal{H}$, hence semi-simple without compact $\mathbb{Q}$-factors. $\mathbf{CG}'\supset\mathbf{CG}_n^{\mathrm{der}}=\mathbf{G}_n$, and thus $\mathbf{CG}'$ is the Mumford-Tate group of some **C**-special subdatum $(\mathbf{CG}', X')$. Because the sequence $(\Omega_n)_n$ is **C**-strict in Ω, we deduce that the images $\pi(\Omega_n)$ is **C**-strict in $\pi(\Omega)$, and this implies the equalities $\mathbf{CG}'=\mathbf{G}$ and $\mathbf{G}'=\mathbf{G}^{\mathrm{der}}$.

We thus have show that $\pi(\mathbf{Q}')=\mathbf{G}^{\mathrm{der}}$. Then $\mathbf{W}':=\mathbf{W}\cap\mathbf{Q}'$ equals the unipotent radical of $\mathbf{Q}'$, and $\mathbf{U}'=\mathbf{U}\cap\mathbf{Q}'=\mathbf{U}\cap\mathbf{W}'$ is a central $\mathbb{Q}$-subgroup of $\mathbf{W}'$. Because $\mathbf{Q}'\supset\mathbf{P}_n^{\mathrm{der}}$ for all n, we have $\mathbf{W}'\supset\mathbf{W}_n$ and $\mathbf{U}'\supset\mathbf{U}_n$, where $\mathbf{W}_n$ resp. $\mathbf{U}_n$ is the unipotent radical resp. the weight 2 unipotent part of $\mathbf{P}_n$. $\mathbf{W}'$ is a $\mathbb{Q}$-subgroup of **W** stable under the action of $\mathbf{G}^{\mathrm{der}}$. Because the action of $\mathbf{G}^{\mathrm{der}}$ on **W** and that of **C** commute, we deduce that $\mathbf{W}'$ is also stable under **C**, hence stable under **G**. It is also easy to check that $\mathbf{U}':=\mathbf{U}\cap\mathbf{W}$ is central in $\mathbf{W}'$, and for any $y\in Y$, $\mathbf{Lie}(\mathbf{W}'\rtimes\mathbf{G})$ satisfies the conditions of Hodge structures in the definition of mixed Shimura datum. By putting $\mathbf{P}'=\mathbf{W}'\rtimes\mathbf{G}$, we see that $(\mathbf{P}',\mathbf{U}'(\mathbb{C})\mathbf{P}'(\mathbb{R})y_n)$ is a **C**-special subdatum containing $(\mathbf{P}_n, Y_n)$, $\forall y_n\in Y_n$. We have assumed that Ω_n is **C**-strict, thus we must have $\mathbf{P}'=\mathbf{P}$.

In particular, the limit of ν_n exists and it is the **C**-special measure of the total **C**-special lattice space Ω. The density of $\bigcup_n\Omega_n$ in Ω is clear.

■

Corollary 3.3.5. *For $Z \subset \Omega$ a closed subset, and $\mathcal{H}_C(Z)$ the subset of $\mathcal{H}_C(\Omega)$ consisting of* **C***-special measures with supports contained in* Z*. Then $\mathcal{H}_C(Z)$ is a compact closed subset of $\mathcal{H}_C(\Omega)$, and there exists only finitely maximal* **C***-special lattice subspaces contained in Z: $\#\mathscr{S}_C(Z) < \infty$.*

Proof. Let $(\nu_n)_n$ be a sequence in $\mathcal{H}_C(Z)$. We need to show that $(\nu_n)_n$ admits convergent subsequence, and every convergent sequence in $\mathcal{H}_C(Z)$ has its limit in $\mathcal{H}_C(Z)$.

(1) We first consider the case where $Z = \Omega$. Suppose $\mathcal{H}_C(\Omega)$ is not closed in $\mathcal{H}(\Omega)$. Then there exists some measure $\nu \in \mathcal{H}(\Omega) - \mathcal{H}_C(\Omega)$ which is the limit of a sequence $(\nu_n)_n$ in $\mathcal{H}_C(\Omega)$. We assume for simplicity that the ν_n's are mutually distinct. Write $\Omega_n = \mathrm{Supp}\,\nu_n = \Gamma\backslash\Gamma\mathbf{P}_n^{\mathrm{der}}(\mathbb{R})^+$ for some subdatum $(\mathbf{P}_n, Y_n)$.

We claim that there exists a smallest **C**-special lattice subspace in Ω containing all the Ω_n's. In fact it suffices to take $\mathbf{P}'$ to be the $\mathbb{Q}$-subgroup generated by $\bigcup_n \mathbf{P}_n$. By the same arguments in the last proposition, $\mathbf{P}'$ is a $\mathbb{Q}$-subgroup coming from some **C**-special subdatum, and $\Omega' = \Gamma\backslash\Gamma\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ is the smallest **C**-special lattice subspace containing all the Ω_n's.

We may regard $(\nu_n)_n$ as a sequence in $\mathcal{H}_C(\Omega') \subset \mathcal{H}(\Omega')$. This sequence is **C**-strict in $\mathcal{H}_C(\Omega')$: if for some **C**-special subspace $\Omega'' \subsetneq \Omega'$ we have infinitely many Ω_n's contained in Ω''. By induction on the dimension of Ω', we have a closed subset $\mathcal{H}_C(\Omega')$ inside $\mathcal{H}(\Omega)$, and $(\nu_n)_n$ has a convergent subsequence whose limit ν' lies in $\mathcal{H}_C(\Omega')$. The convergence of $(\nu_n)_n$ in $\mathcal{H}(\Omega)$ shows that nu' is also the limit of $(\nu_n)_n$ in $\mathcal{H}_C(\Omega') \subset \mathcal{H}_C(\Omega)$.

We thus conclude that $\mathcal{H}_C(\Omega)$ is closed in $\mathcal{H}(\Omega)$, and is in particular compact for the weak topology.

(2) We then pass to a general closed subset $Z \subset \Omega$. Let $(\nu_n)_n$ be a sequence in $\mathcal{H}_C(\Omega)$ that converges to some ν in $\mathcal{H}_C(\Omega)$, such that $\mathrm{Supp}\,\nu_n \subset Z$ for all n. Then $\mathrm{Supp}\,\nu$ is the closure of $\bigcup_{n>N} \mathrm{Supp}\,\nu_n$ for some $N > 0$, and is thus contained in Z, which confirms that $\mathcal{H}_C(Z)$ is closed and compact.

(3) Finally we show the finiteness of $\mathscr{S}_C(Z)$ for an arbitrary closed subset $Z \subset \Omega$. If $\mathscr{S}_C(Z)$ is not finite, then it contains an infinite sequence $(\Omega_n)_n$. The elements of $\mathscr{S}_C(Z)$ are maximal among those **C**-special lattice subspaces in Z. If we put Ω' to be the minimal lattice subspace containing Z, which certainly exists by dimensional induction, then $(\Omega_n)_n$ makes up a **C**-strict sequence in Ω', and so it is with the corresponding sequence of **C**-special measures $(\nu_n)_n$ in $\mathcal{H}_C(\Omega')$. Hence $(\nu_n)_n$ converges to some **C**-special measure ν' on Ω. Because $\nu' = \lim \nu_n$ we have $\mathrm{Supp}\,\nu \subset Z$ and meantime $\mathrm{Supp}\,\nu_n \subsetneq \mathrm{Supp}\,\nu'$ for n large enough, which contradicts the maximality of $\mathscr{S}_C(Z)$, hence the finiteness of $\mathscr{S}_C(Z)$.

■

3.4 S-spaces and special S-subspaces

With the André-Oort property established at the level of **C**-special lattice subspaces, we then want to transfer it to **C**-special subvarieties. As we have mentioned in the introduction of this chapter, we'll first study the projections from lattice space to an intermediate class of objects called "S-subspaces". In this section we develop the formalism of S-subspaces: they can be viewed as the real part of Shimura varieties equipped with canonical probability measures, which enables the usage of ergodic arguments.

Recall that for a mixed Shimura datum $(\mathbf{P},\mathrm{Y})$, we have defined the real part of Y to be the orbit $\mathbf{P}(\mathbb{R})x$ of some (or equivalently, any) $x \in \mathrm{X}$, $(\mathbf{G},\mathrm{X})$ being any pure section of $(\mathbf{P},\mathrm{Y})$. This notion is independent of the choice of pure sections.

Definition 3.4.1 (The formalism of S-spaces). Fix a mixed Shimura datum $(\mathbf{P},\mathrm{Y})$ and a compact open subgroup $\mathrm{K} \subset \mathbf{P}(\mathbb{A}^{\mathrm{f}})$.

(1) The S-space associated to $(\mathbf{P},\mathrm{Y})$ at level K is the real analytic space defined as

$$\mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y}) := \mathbf{P}(\mathbb{Q})\backslash[\mathrm{Y}_{\mathbb{R}} \times \mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}]$$

where $\mathrm{Y}_{\mathbb{R}}$ is the real part of Y.

Fix $\mathfrak{R}$ a set of representatives of $\mathbf{P}(\mathbb{Q})_+\backslash\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}$. Note that $\mathrm{Y} \cong \mathbb{I}(\mathbf{U}) \times \mathrm{Y}_{\mathbb{R}}$ as real analytic spaces, and we have a bijection between $\pi_0(\mathrm{Y})$ and $\pi_0(\mathrm{Y}_{\mathbb{R}})$. We thus get an isomorphism

$$\mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y}) \cong \mathbf{P}(\mathbb{Q})_+\backslash[\mathrm{Y}_{\mathbb{R}}^+ \times \mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}] \cong \coprod_{g\in\mathfrak{R}} \Gamma_{\mathrm{K}}(g)\backslash\mathrm{Y}_{\mathbb{R}}^+$$

where $\mathrm{Y}_{\mathbb{R}}^+$ is any fixed connected component of $\mathrm{Y}_{\mathbb{R}}$, $\Gamma_{\mathrm{K}}(g) = \mathbf{P}(\mathbb{Q})_+ \cap g\mathrm{K}g^{-1}$.

$\mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$ is identified with a real analytic subspace of $M_{\mathrm{K}}(\mathbf{P},\mathrm{Y})_{\mathbb{C}}$. With respect to the projection $\wp_{\mathrm{K}} : \mathrm{Y} \times \mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K} \to M_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$, $\Gamma_{\mathrm{K}}(g)^{\dagger}\backslash\mathrm{Y}_{\mathbb{R}}^+$ is identified with $\wp_{\mathrm{K}}(\mathrm{Y}_{\mathbb{R}}^+ \times g\mathrm{K})$.

For a general arithmetic subgroup $\Gamma \subset \mathbf{P}(\mathbb{R})^+$, the quotient $\Gamma\backslash\mathrm{Y}_{\mathbb{R}}^+$ is referred to as the connected S-space associated to $(\mathbf{P},\mathrm{Y})$ at level Γ, $\mathrm{Y}_{\mathbb{R}}^+$ being a fixed connected component of $\mathrm{Y}_{\mathbb{R}}$.

(2) The morphisms between S-spaces and Hecke correspondences are defined in the obvious way. In particular we have the notions of Shimura S-subspaces and special S-subspaces of $\mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$: a Shimura S-subspace is the image of a morphism $\mathcal{M}_{\mathrm{K}_1}(\mathbf{P}_1,\mathrm{Y}_1) \to \mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$ for some Shimura subdatum $(\mathbf{P}_1,\mathrm{Y}_1)$;and a special S-subspace is a connected component of a Hecke translate of some Shimura S-subspace, or equivalently, a subvariety of the form $\wp_{\mathrm{K}}(\mathrm{Y}_{1\mathbb{R}}^+ \times a\mathrm{K})$ in $\mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$ for some Shimura subdatum $(\mathbf{P}_1,\mathrm{Y}_1)$.

Note that if $(\mathbf{P}_1,\mathrm{Y}_1)$ is a Shimura subdatum of $(\mathbf{P},\mathrm{Y})$, and $(\mathbf{G}_1,\mathrm{X}_1)$ a pure section of $(\mathbf{P}_1,\mathrm{Y}_1)$, then it extends to a pure section $(\mathbf{G},\mathrm{X}) \supset (\mathbf{G}_1,\mathrm{X}_1)$: $\mathbf{G}_1$ is a maximal reductive $\mathbb{Q}$-subgroup of $\mathbf{P}_1$, and extends to a maximal reductive $\mathbb{Q}$-subgroup of $\mathbf{P} \supset \mathbf{P}_1$. Two pure sections of $(\mathbf{P},\mathrm{Y})$ differ by a $\mathbf{W}(\mathbb{Q})$-conjugation.

We define weakly special S-subspaces to be finite unions of special S-subspaces.

(3) In the same way as we have seen for Shimura varieties, for a $\mathbb{Q}$-torus $\mathbf{C}' \subset \mathbf{G}$, a special S-subspace is $\mathbf{C}'$-special if it of the form $\wp(\mathrm{Y}_{1\mathbb{R}}^+ \times g\mathrm{K})$ for some

$\mathbf{C}'$-special subdatum $(\mathbf{P}_1, Y_1)$ and $g \in \mathfrak{R}_K^{\mathbf{P}}$. Being $\mathbf{C}'$-special is the same as being $q\mathbf{C}'q^{-1}$-special, for any $q \in \mathbf{W}(\mathbb{Q}) \rtimes \Gamma_K(g)$, g the representative corresponding to the special S-subspace.

(Here we write $\mathbf{C}'$ and reserve $\mathbf{C}$ for the connected center of $\mathbf{G}$.)

(4) The volume form on $Y_{\mathbb{R}} = \mathbf{P}(\mathbb{R})y$ leads to a Borel measure on $Y_{\mathbb{R}}$ invariant under the left action of $\mathbf{P}(\mathbb{R})$, which is equivalently induced by the Haar measure on $\mathbf{P}(\mathbb{R})$, and it gives rise to a probability measure on the S-space $\mathbf{P}(\mathbb{Q})\backslash((Y_{\mathbb{R}}) \times \mathbf{P}(\mathbb{A}^f)/K)$. The construction is obvious, as can be seen from the canonical probability measure μ_S on the connected component of the form $\mathcal{M}_K(g) = \Gamma_K(g)\backslash Y_{\mathbb{R}}^+$. The finiteness of the canonical measure is justified in Lemma 3.4.2 below, where we introduce a useful map

$$\kappa_y : \Omega = \Gamma_K(g)^\dagger \backslash \mathbf{P}^{\mathrm{der}}(\mathbb{R})^+ \to \mathcal{M} = \Gamma_K(g)\backslash Y_{\mathbb{R}}^+,\ \Gamma_K(g)^\dagger q \mapsto \Gamma_K(g) q y$$

(5) $\mathbf{C}'$ being a $\mathbb{Q}$-torus in $\mathbf{G}$, we define the set of $\mathbf{C}'$-special H-measures on $\mathcal{M} = \mathcal{M}_K(\mathbf{P}, Y)$, denoted by $\mathcal{H}_{C'}(\mathcal{M})$, to be the set of measures of the form $\mu' = \iota_* \mu_{\mathcal{M}'}$, where $i : \mathcal{M}' \hookrightarrow \mathcal{M}$ is the inclusion of a $\mathbf{C}'$-special S-subspace, and $\mu_{\mathcal{M}'}$ is the canonical probability measure on $\mathcal{M}'$ deduced from the volume form. This is a priori a countable subset of the set of Borelian probability measures on $\mathcal{M}$.

(6) We can similarly define the notion of generic Mumford-Tate group for a special S-subspace $\mathcal{M}' = \wp_K(Y_{1,\mathbb{R}}^+ \times gK) \subset \mathcal{M} = \mathcal{M}_K(\mathbf{P}, Y)$ for some Shimura subdatum $(\mathbf{P}_1, Y_1)$, to be the $\Gamma_K(g)$-conjugacy class of $\mathbf{P}_1$ in $\mathbf{P}$. It is equal to the generic Mumford-Tate group of the Zariski closure of $\mathcal{M}'$ in $M_K(\mathbf{P}, Y)$, namely $[\mathbf{P}_1] = \mathbf{MT}(\wp_K(Y_1^+ \times gK))$.

(7) Let $(\mathcal{M}_n)_n$ be a sequence of special S-subspaces of $\mathcal{M} = \mathcal{M}_K(\mathbf{P}, Y)$. The sequence is generic resp. strict if so it is with the corresponding sequence $(\overline{M_n}^{\mathrm{Zar}})_n$ of special subvarieties of $M = M_K(\mathbf{P}, Y)_{\mathbb{C}}$, namely:

generic: for every real analytic subspace $Z \subsetneq S$, $\mathcal{M}_n \not\subseteq Z$ for n large enough;

strict: for every subdatum $(\mathbf{P}', Y') \subsetneq (\mathbf{P}, Y)$ we have $\mathcal{M}_n \not\subseteq \mathcal{M}'$ for n large enough, where $\mathcal{M}'$ is the Shimura S-subspace in $\mathcal{M}$ defined by $(\mathbf{P}', Y')$;

$\mathbf{C}$-strict: for every $\mathbf{C}$-special S-subspace $\mathcal{M}' \subset \mathcal{M}$, we have $\mathcal{M}_n \not\subseteq \mathcal{M}'$ for n large enough.

One can define in a parallel way the notions of generic resp. strict resp. $\mathbf{C}$-strict sequence of canonical measures on $\mathcal{M}$, just as we have seen in the setting of lattice spaces.

In the following of this section we fix $(\mathbf{G}, X)$ a pure section of $(\mathbf{P}, Y)$, and $\mathbf{C}$ denotes the connected center of $\mathbf{G}$.

Lemma 3.4.2. *(1) Let Ω be the lattice space associated to $(\mathbf{P}, Y)$ and an arithmetic subgroup $\Gamma \subset \mathbf{P}(\mathbb{R})^+$, namely $\Omega = \Gamma^\dagger \backslash \mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$, where $\Gamma^\dagger := \Gamma \cap \mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$. Then for any $y \in Y_{\mathbb{R}}^+$, the map*

$$\kappa_y : \Omega \to \mathcal{M} = \Gamma\backslash Y_{\mathbb{R}}^+,\ \Gamma^\dagger q \mapsto \Gamma q y$$

is surjective and quasi-compact. If ν *denotes the canonical measure on* Ω*, then* $\mu = \kappa_{y*}\nu$ *equals the canonical measure on* $\mathcal{M}$ *induced by the left* $\mathbf{P}(\mathbb{R})^+$*-invariant volume form on* $Y^+_{\mathbb{R}}$*, and* μ *is independent of the choice of* y*.*

(2) Fix $\mathcal{M}$ *a connected component of the S-space* $\mathcal{M}_K(\mathbf{P},Y)$ *of the form* $\Gamma\backslash Y^+_{\mathbb{R}}$ *for some arithmetic subgroup* $\Gamma \subset \mathbf{P}(\mathbb{R})^+$*, and* $\mathcal{M}' = \Gamma\backslash\Gamma Y'^+_{\mathbb{R}}$ *a* $\mathbf{C}$*-special S-subspace defined by some* $\mathbf{C}$*-special subdatum* $(\mathbf{P}',Y')$*. Let* $\Omega = \Gamma^\dagger\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ *be the lattice space corresponding to* $\mathcal{M}$*, and* $\Omega' = \Gamma^\dagger\backslash\Gamma^\dagger\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ *the* $\mathbf{C}$*-special lattice subspace corresponding to* $\mathcal{M}'$*. Then for any* $y \in Y'^+_{\mathbb{R}} \subset Y^+_{\mathbb{R}}$*, the projection* $\kappa_y : \Omega \to \mathcal{M}, \Gamma g \mapsto \Gamma g y$ *maps* Ω' *onto* $\mathcal{M}'$*, and it pushes* ν *resp.* ν' *forward to* μ *resp.* μ'*, where* ν *resp.* μ *denotes the canonical probability measure on* Ω *resp.* $\mathcal{M}'$*, and* ν' *resp.* μ' *denotes the* $\mathbf{C}$*-special H-measure of support* Ω' *resp.* $\mathcal{M}'$*.*

Proof. We take $(\mathbf{G},X)$ a pure section of $(\mathbf{P},Y)$, and assume that $\Gamma = \Gamma_{\mathbf{W}} \rtimes \Gamma_{\mathbf{G}}$ for arithmetic subgroups $\Gamma_{\mathbf{W}} \subset \mathbf{W}(\mathbb{R})^+$ resp. $\Gamma_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R})^+$. $\Omega_{\mathbf{G}} := \Gamma^\dagger_{\mathbf{G}}\backslash\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$, and $\mathcal{S}$ denotes the S-space associated to $(\mathbf{G},X)$ at level $\Gamma_{\mathbf{G}}$: it is no other than the real analytic space underlying $\Gamma_{\mathbf{G}}\backslash X^+$. Denote by $\pi_{\mathbf{W}} : (\mathbf{P},Y) \to (\mathbf{G},X)$ the projection modulo $\mathbf{W}$, and the corresponding maps for lattice spaces and

It is clear that (1) implies (2) by functoriality.

For (1), we first consider the following commutative diagram

$$\begin{tikzcd} \Omega \arrow[r, "\kappa_y"] \arrow[d, "\pi_{\mathbf{W}}"'] & \mathcal{M} \arrow[d, "\pi_{\mathbf{W}}"] \\ \Omega_{\mathbf{G}} \arrow[r, "\kappa_x"] & \mathcal{S} \end{tikzcd}$$

where $x = \pi_{\mathbf{W}}(y) \in X^+$, $\kappa_x : \Gamma_{\mathbf{G}} g \mapsto \Gamma_{\mathbf{G}} g x$. Both of the vertical maps are fibration by spaces isomorphic to $\Gamma_{\mathbf{W}}\backslash\mathbf{W}(\mathbb{R})$, and κ_y induces isomorphisms of the fibers.

$\kappa_x : \Omega_{\mathbf{G}} \to \Gamma\backslash X^+$ is surjective: $\Gamma^\dagger_{\mathbf{G}} = \Gamma \cap \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ \subset \Gamma$ and $X^+ = \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ x$ because the center of $\mathbf{G}(\mathbb{R})$ acts on X^+ trivially. The projection $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ \to X^+$ $q \mapsto qx$ is quasi-compact and $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$-equivariant, whose fiber over x is a maximal compact subgroup of $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ (because X^+ is an Hermitian symmetric domain). By commutativity of the diagram above, we deduce that κ_y is surjective and quasi-compact.

$\mathbf{P}^{\mathrm{der}}$ is of type $\mathcal{H}$ and $\Gamma^\dagger$ is an arithmetic subgroup, thus the left Haar measure on $\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ induces a probability measure ν on Ω. Hence $\mu = \kappa_{y*}\nu$ is a probability measure on $\mathcal{M}$. To see that μ is deduced from the volume form, consider the commutative diagram below:

$$\begin{tikzcd} \mathbf{P}^{\mathrm{der}}(\mathbb{R})^+ \arrow[r, "\kappa_y"] \arrow[d, "\wp^\dagger_\Gamma"'] & Y^+_{\mathbb{R}} \arrow[d, "\wp_\Gamma"] \\ \Omega \arrow[r, "\kappa_y"] & \mathcal{M} \end{tikzcd}$$

where the vertical map $\wp^\dagger_\Gamma$ resp. $\wp_\Gamma$ is taking quotient modulo $\Gamma^\dagger$ resp. Γ, and the upper horizontal map κ_y sends q to qy. Let ν' be the normalized left Haar measure on $\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ such that $\wp^{\dagger}_{\Gamma*}\nu' = \nu$ is of total mass 1. Then ν' is given by the left invariant volume form on $\mathbf{P}^{\mathrm{der}}(\mathbb{R})$, $\kappa_{y*}\nu'$ is given by left invariant volume form on $Y^+_{\mathbb{R}}$. We then see that

$$\mu = (\kappa_y \circ \wp^{\dagger}_{\Gamma})_* \nu' = (\wp_{\Gamma} \circ \kappa_y)_* \nu'$$

is the required probability measure. ■

Remark 3.4.3. We would also like to point out that the canonical map $\psi_{\mathbf{P}} : \Gamma^{\dagger}\backslash Y^+_{\mathbb{R}} \to \Gamma\backslash Y^+_{\mathbb{R}}$ is an isomorphism. The verification of this fact is reduced to the pure case via the following commutative diagram

$$\begin{tikzcd} \Gamma^{\dagger}\backslash Y^+_{\mathbb{R}} \arrow[r, "\psi_{\mathbf{P}}"] \arrow[d, "\pi"'] & \Gamma\backslash Y^+_{\mathbb{R}} \arrow[d, "\pi"] \\ \Gamma^{\dagger}_{\mathbf{G}}\backslash X^+ \arrow[r, "\psi_{\mathbf{G}}"'] & \Gamma_{\mathbf{G}}\backslash X^+ \end{tikzcd}$$

The lower horizontal map $\psi_{\mathbf{G}}$ is actually the identity: $\Gamma_{\mathbf{G}} \subset \mathbf{G}(\mathbb{R})^+$ acts on X^+ by conjugation, and $\Gamma_{\mathbf{G}} \cap Z_{\mathbf{G}}(\mathbb{R})$ acts trivially, hence the quotient is the same as $\Gamma^{\dagger}_{\mathbf{G}}\backslash X^+_{\mathbb{R}}$. Check the fibers of the two vertical maps we see that $\psi_{\mathbf{P}}$ is also the identity.

Note that this is not true for lattice spaces: the lattice space $\Gamma^{\dagger}\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ is the quotient of $\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$ by the left translation of $\Gamma^{\dagger}$; replacing $\mathbf{P}^{\mathrm{der}}$ by $\mathbf{P}$ and $\Gamma^{\dagger}$ by Γ produces a different quotient space, to which we cannot apply directly the theorem of Mozes and Shah.

Remark 3.4.4. From the lemma above, we see that $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ is countable, and every element μ in it is a push-forward of some $\nu \in \mathcal{H}_{\mathbf{C}}(\Omega)$. However the operation of push-forward requires a base point from the real part $Y'_{\mathbb{R}}$ of the corresponding Shimura subdatum $(\mathbf{P}', Y')$: different ν''s might fail to be pushed forward through a common base point.

But the compactness of $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ is crucial to us: once the compactness is established, we can immediately deduce the André-Oort conjecture for a sequence of $\mathbf{C}$-special S-subspaces, namely: for any closed subspace $Z \subset \mathcal{M}$, the set of maximal $\mathbf{C}$-special S-subspaces in Z is finite. Taking Zariski closure gives us the André-Oort conjecture for a sequences of $\mathbf{C}$-special subvarieties.

In the next section we'll show the compactness of $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ via an argument of S.Dani-Margulis, as has been applied in [CU-3].

3.5 The Dani-Margulis argument

To prove the compactness of $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$, we'll show that there exists a compact subset of $Y^+_{\mathbb{R}}$ such that every $\mu' \in \mathcal{H}_{\mathbf{C}}(\mathcal{M})$ is given by some subdatum $(\mathbf{P}', Y')$ with $Y'_{\mathbb{R}} \cap C \neq \emptyset$, hence the elements in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ are of the form $\kappa_{x*}\nu'$ for $x \in C$ and

$\nu' \in \mathcal{H}_{\mathbf{C}}(\Omega)$. This will result in the existence of convergent subsequences for arbitrary sequences in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$.

The starting point is the following argument by S.Dani and G.Margulis which we quote without proof:

Proposition 3.5.1. *(S.Dani and G.Margulis, cf. [DM] Theorem 2) Let* $\mathbf{H}$ *be a semi-simple* $\mathbb{Q}$*-group without compact* $\mathbb{Q}$*-factors, and* $\Gamma_{\mathbf{H}} \subset \mathbf{H}(\mathbb{R})^+$ *an arithmetic lattice and* $\Omega_{\mathbf{H}} = \Gamma_{\mathbf{H}}\backslash\mathbf{H}(\mathbb{R})^+$ *the lattice space associated to* $\mathbf{H}$ *and* $\Gamma_{\mathbf{H}}$*. Then there exists a compact subset* C *of* $\Omega_{\mathbf{H}}$ *such that for any one-parameter subgroup* $\mathcal{U} = (u_t)_{t\in\mathbb{R}}$ *of* $\mathbf{H}(\mathbb{R})$ *and* $g \in \mathbf{H}(\mathbb{R})^+$*, if* $\mathrm{C} \cap (\Gamma_{\mathbf{H}}\backslash\Gamma_{\mathbf{H}}g\mathcal{U}) \neq \emptyset$*, then there exists a parabolic* $\mathbb{Q}$*-subgroup* $\mathbf{Q} \subsetneq \mathbf{H}$ *such that* $g\mathcal{U}g^{-1} \subset \mathbf{Q}(\mathbb{R})$*.*

Corollary 3.5.2. *(cf. [CU-3] Section 4, Lemma 4.4, Lemma 4.5) Let* $(\mathbf{P},\mathrm{Y})$, Γ, $\Omega = \Gamma^\dagger\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$, $\mathcal{M} = \Gamma\backslash\mathrm{Y}^+_{\mathbb{R}}$*, etc. be as before.*

(1) There exists a compact subset Δ *of* Ω *such that if* $\mathbf{L} \subset \mathbf{P}^{\mathrm{der}}$ *is a strong* $\mathbb{Q}$*-subgroup of type* $\mathcal{H}$*, then we have* $\Delta \cap \Gamma^\dagger\backslash\Gamma^\dagger g\mathbf{L}(\mathbb{R})^+ \neq \emptyset$ *for some* $g \in \mathbf{P}^{\mathrm{der}}(\mathbb{Q})^+$

(2) There exists a compact subset D *of* $\mathcal{M}$ *such that if* $\mathcal{M}' \subset \mathcal{M}$ *is a subspace of the form* $\Gamma\backslash\Gamma a\mathbf{L}(\mathbb{R})^+ y$ *for some* $y \in \mathrm{Y}^+_{\mathbb{R}}$*,* $a \in \mathbf{P}^{\mathrm{der}}(\mathbb{Q})^+$*, and* $\mathbf{L} \subset \mathbf{P}^{\mathrm{der}}$ *a strong* $\mathbb{Q}$*-subgroup of type* $\mathcal{H}$*, then* $\mathcal{M}' \cap \mathrm{D} \neq \emptyset$*. In particular,* D *meets every* $\mathbf{C}$*-special S-subspace of* $\mathcal{M}$ *non-trivially.*

(3) There exists a compact subset $\mathcal{C}$ *in* $\mathrm{Y}^+_{\mathbb{R}}$ *such that if* $\mathcal{M}' \subset \mathcal{M}$ *is a* $\mathbf{C}$*-special S-subspace, then we can find some* $\mathbf{C}$*-special subdatum* $(\mathbf{P}',\mathrm{Y}')$ *such that* $\mathcal{M}' = \Gamma\backslash\Gamma\mathrm{Y}'^+_{\mathbb{R}}$ *with* $\mathrm{Y}'^+_{\mathbb{R}} \cap \mathcal{C} \neq \emptyset$*.*

Proof. We have fixed a pure section $(\mathbf{G},\mathrm{X})$ of $(\mathbf{P},\mathrm{Y})$, and $\Gamma = \Gamma_{\mathbf{W}} \rtimes \Gamma_{\mathbf{G}}$ is torsion-free. In order to simplify the superscripts and subscripts, we put $\mathbf{H} = \mathbf{G}^{\mathrm{der}}$ and $\Gamma_{\mathbf{H}}$ to be $\Gamma^\dagger_{\mathbf{G}} = \Gamma \cap \mathbf{H}(\mathbb{R})^+$. Then we have$\Omega_{\mathbf{G}} = \Gamma^\dagger_{\mathbf{G}}\backslash\mathbf{H}(\mathbb{R})^+ = \Gamma_{\mathbf{H}}\backslash\mathbf{H}(\mathbb{R})^+$. Similarly, by putting $\mathbf{Q} = \mathbf{W} \rtimes \mathbf{H} = \mathbf{P}^{\mathrm{der}}$ and $\Gamma_{\mathbf{Q}} = \Gamma \cap \mathbf{Q}(\mathbb{R})^+$ we have $\Gamma_{\mathbf{Q}}\backslash\mathbf{Q}(\mathbb{R})^+ = \Omega$. We only consider $\Gamma_{\mathbf{Q}}\backslash\mathbf{Q}(\mathbb{R})^+$, and for simplicity of notations we write $\Gamma = \Gamma_{\mathbf{Q}}$. Note that $\Gamma = \Gamma_{\mathbf{W}} \rtimes \Gamma_{\mathbf{H}}$ is a torsion free lattice in $\mathbf{Q}(\mathbb{R})^+$.

(1) The projection $\pi_{\mathbf{W}} : \Omega = \Gamma\backslash\mathbf{Q}(\mathbb{R})^+ \to \Gamma_{\mathbf{H}}\backslash\mathbf{H}(\mathbb{R})^+$ is quasi-compact, whose fibers are compact sets of the form $\Gamma_{\mathbf{W}}\backslash\mathbf{W}(\mathbb{R})$. The proposition above provides us with the compact subset $\mathrm{C} \subset \Gamma_{\mathbf{H}}\backslash\mathbf{H}(\mathbb{R})^+$, and we set Δ to be $\pi^{-1}_{\mathbf{W}}(\mathrm{C})$.

Let $\mathbf{L} \subset \mathbf{Q} = \mathbf{P}^{\mathrm{der}}$ be a strong $\mathbb{Q}$-subgroup of type $\mathcal{H}$. We want to show that $\Delta \cap (\Gamma\backslash\Gamma g\mathbf{L}(\mathbb{R})^+) \neq \emptyset$.

Suppose that for any $g \in \mathbf{Q}(\mathbb{Q})^+$, $\Delta \cap \Gamma\backslash\Gamma g\mathbf{L}(\mathbb{E})^+ = \emptyset$. Write Λ to be the subgroup of $\mathbf{L}(\mathbb{R})^+$ generated by one-parameter unipotent subgroups of $\mathbf{L}(\mathbb{R})^+$. We have $\mathbf{L} = \mathbf{MT}(\Lambda)$. Take a one-parameter unipotent subgroup $\mathcal{U} = \{u_t : t \in \mathbb{R}\} \subset \Lambda$. The decomposition of $\mathbf{L}_{\mathbb{R}}$ into almost direct products of $\mathbb{R}$-factors gives a similar description of Λ, and we may suppose that $\mathcal{U}$ is not contained in any invariant subgroup $\Lambda' \subsetneq \Lambda$, up to replacing $\mathcal{U}$ by a product of factors intersecting each minimal invariant subgroup of Λ non-trivially.

Then for each $h \in \mathbf{L}(\mathbb{R})^+$, $\Delta \cap \Gamma\backslash\Gamma gh\mathcal{U} \subset \Delta \cap \Gamma\backslash\Gamma g\mathbf{L}(\mathbb{R})^+ = \emptyset$. Apply the projection $\pi_{\mathbf{W}}$, we get $\mathrm{C} \cap \Gamma_{\mathbf{H}}\backslash\Gamma_{\mathbf{H}}\overline{gh\mathcal{U}} \subset \mathrm{C} \cap \Gamma_{\mathbf{H}}\backslash\Gamma_{\mathbf{H}}\overline{g\mathbf{L}(\mathbb{R})^+} = \emptyset$, with $\overline{*}$ standing for the image under $\pi_{\mathbf{W}}$. The proposition of S.Dani-Margulis confirms the existence of a parabolic $\mathbb{Q}$-subgroup $\overline{\mathbf{Q}_\beta} \subsetneq \mathbf{H}$ such that $\overline{gh\mathcal{U}h^{-1}g^{-1}} \subset \overline{\mathbf{Q}_\beta}(\mathbb{R})$. Since there is only countably many such parabolic $\mathbb{Q}$-subgroups in $\mathbf{H}$, there is a single $\mathbb{Q}$-parabolic $\overline{\mathbf{Q}_\beta} \subsetneq \mathbf{H}$ such that $\overline{gh\mathcal{U}h^{-1}g^{-1}} \subset \overline{\mathbf{Q}_\beta}(\mathbb{R})$ for h runs over some subset $\mathrm{A} \subset \mathbf{L}(\mathbb{R})^+$ of positive mass with respect to the Haar measure on $\mathbf{L}(\mathbb{R})^+$.

Put $\mathbf{Q}_\beta \subsetneq \mathbf{P}$ to be the parabolic $\mathbb{Q}$-subgroup of $\mathbf{Q}$ whose reduction modulo $\mathbf{W}$ is $\overline{g^{-1}\mathbf{Q}_\beta g}$, well-defined over $\mathbb{Q}$ since $g \in \mathbf{Q}(\mathbb{Q})^+$. Then $h\mathcal{U}h^{-1} \subset \mathbf{Q}_\beta(\mathbb{R})^+$ for h runs over A. Note that $\mathrm{A} \subset \mathbf{L}(\mathbb{R})^+$ is of positive mass, and that $\mathcal{U}$ is not contained in any invariant subgroup of Λ, we see that $\Lambda \subset \mathbf{Q}_\beta(\mathbb{R})$, and $\mathbf{L} \subset \mathbf{Q}_\beta \subsetneq \mathbf{Q}$, contradicting the fact that $\mathbf{L}$ is strong, hence the conclusion.

(2) Let $\mathcal{V} \subset \mathbf{Q}(\mathbb{R})^+$ be a connected compact neighborhood of the neutral element. Put $\mathrm{D}_1 = \Delta\cdot\mathcal{V} = \{ag : a \in \Delta, g \in \mathcal{V}\}$, which is a compact subset of Ω.

For $x \in \mathrm{Y}_{\mathbb{R}}$ we have the projection $\kappa_x : \Omega \twoheadrightarrow \mathrm{S} = \Gamma\backslash\mathrm{Y}_{\mathbb{R}}^+$, $\Gamma g \mapsto \Gamma gx$. Fix a point $y \in \mathrm{Y}_{\mathbb{R}}^+$, set $\mathrm{D} = \kappa_y(\mathrm{D}_1)$. Then for $g \in \mathcal{V}$, $\kappa_{gy}(\Delta) \subset (\mathrm{D})$.

We have $\mathrm{S}' = \Gamma\backslash\Gamma a\mathbf{L}(\mathbb{R})^+ y \subset \mathrm{S}$, $y \in \mathrm{Y}_{\mathbb{R}}^+$, $\mathbf{L} \subset \mathbf{Q}$ strong of type $\mathcal{H}$. Since $\mathbf{Q}(\mathbb{Q})\cdot\mathcal{V} = \mathbf{Q}(\mathbb{R})^+$, we have $y = qgx$ for some $q \in \mathbf{Q}(\mathbb{Q})^+$ and $g \in \mathcal{V}$. Hence $\mathrm{S}' = \kappa_{gx}(\Gamma\backslash\Gamma aq\mathbf{L}_1(\mathbb{R})^+)$, where $\mathbf{H}_1 = (aq)^{-1}\mathbf{L}aq$.

We show that $\mathrm{D}\cap\mathrm{S}' \neq \emptyset$. If $\mathrm{D}\cap\mathrm{S}' = \emptyset$, then $\kappa_{gx}(\mathrm{C})\cap\mathrm{S}' = \emptyset$, and $\mathrm{C}\cap\Gamma\backslash\Gamma aq\mathbf{L}_1(\mathbb{R})^+ = \emptyset$. The proposition of S.Dani and G.Margulis confirms that $\mathbf{L}_1 \subset \mathbf{Q}_\beta$ for some $\mathbb{Q}_\beta$-parabolic $\mathbf{Q}_\beta \subsetneq \mathbf{Q}$, contradicting the fact that $\mathbf{H} = aq\mathbf{H}_1q^{-1}a^{-1}$ is strong.

To see that D meets every $\mathbf{C}$-special S-subspace non-trivially, it suffices to notice that, by the Lemma 1.4.2, every $\mathbf{C}$-special S-subspace is of the form $\mathcal{M}' = \Gamma\backslash\Gamma\mathrm{Y}_{\mathbb{R}}'^+ = \Gamma\backslash\Gamma\mathbf{Q}'(\mathbb{R})^+y'$, where $(\mathbf{P}', \mathrm{Y}')$ is some $\mathbf{C}$-special subdatum, $\mathbf{Q}' = \mathbf{P}'^{\mathrm{der}}$, and $y' \in \mathrm{Y}_{\mathbb{R}}'^+ \subset \mathrm{Y}_{\mathbb{R}}^+$: namely this is the case where $a = 1$, $\mathbf{L} = \mathbf{Q}'$, and $y = y'$.

(3) Let $\mathrm{D} \subset \mathcal{M}$ be the compact subset in (2). We want to lift D back to some subset in $\mathrm{Y}_{\mathbb{R}}^+$.

Keep the notations as above, the map $\pi_{\mathbf{W}} : \Gamma\backslash\mathrm{Y}_{\mathbb{R}}^+ \to \Gamma_{\mathbf{H}}\backslash\mathrm{X}^+$ is a quasi-compact fibration by compact sets of the form $\Gamma_{\mathbf{W}}\backslash\mathbf{W}(\mathbb{R})$ (here we have followed the identification pointed out in Remark 3.4.3). The quotient $\Gamma_{\mathbf{W}}\backslash\mathbf{W}(\mathbb{R})$ admits an open fundamental set $\mathcal{D}_{\mathbf{W}}$ whose closure in $\mathbf{W}(\mathbb{R})$ is a compact.

On the other hand, the locally symmetric Hermitian space $\Gamma_{\mathbf{H}}\backslash\mathrm{X}^+$ has a fundamental domain in X^+ of the form

$$\mathcal{D}(\xi) := \{\, x \in \mathrm{X}^+ : \mathbf{d}(\xi, x) < \mathbf{d}(\xi, qx), \forall q \in \Gamma_{\mathbf{H}} \,\}$$

where $\xi \in \mathrm{X}^+$ is an arbitrary base point. $\mathcal{D}(\xi)$ is connected, open, and its closure is compact in X^+.

Now we take

$$\mathcal{D} = \mathcal{D}_{\mathbf{W}} \rtimes \mathcal{D}(\xi) = \{\, wx \in \mathbf{W}(\mathbb{R}) \rtimes \mathrm{X}^+ : w \in \mathcal{D}_{\mathbf{W}},\ x \in \mathcal{D}(\xi) \,\}$$

which is an open fundamental domain in $\mathrm{Y}_{\mathbb{R}}^+$ whose closure $\bar{\mathcal{D}}$ is compact in $\mathrm{Y}_{\mathbb{R}}^+$. The subset D in $\mathcal{M}$ is compact, thus the intersection of $\mathcal{D}$ with the pre-image of D under the covering map $Y_{\mathbb{R}}^{+} \twoheadrightarrow \mathcal{M}$ contains a compact subset $\mathcal{C}$ such that $\Gamma\backslash\Gamma\mathcal{C} \supset \mathcal{D}$, i.e. the image of $\mathcal{C}$ in $\mathcal{M}$ is a compact subset containing D.

We proceed to show that the $\mathcal{C}$ obtained this way satisfies the requirements of (3). Let $\mathcal{M}'$ be a **C**-special S-subspace in $\mathcal{M}$. In (2) we have seen that $\mathcal{M}' \cap \mathrm{D} \neq \emptyset$. Say $\mathcal{M}' = \Gamma\backslash\Gamma\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^{+}y'$ where $(\mathbf{P}', \mathrm{Y}')$ is some **C**-special subdatum of $(\mathbf{P}, \mathrm{Y})$, $y' \in Y_{\mathbb{R}}^{+}$ such that $Y_{\mathbb{R}}'^{+} = \mathbf{P}'^{\mathrm{der}}(\mathbb{R})^{+}y'$ is a connected component of $Y'_{\mathbb{R}}$. Now that $\mathcal{M}' \cap \mathrm{D} \neq \emptyset$, there is some $y' \in Y_{\mathbb{R}}'^{+}$ such that $\Gamma y' \in \mathrm{D}$, and hence for some $\gamma \in \Gamma$ we have $\gamma y' \in \mathcal{D} \subset \mathcal{C}$. The subdatum $(\gamma\mathbf{P}'\gamma^{-1}, \gamma Y'_{\mathbb{R}})$ is **C**-special, and it defines the same **C**-special S-subspace $\mathcal{M}' = \Gamma\backslash\Gamma\gamma Y_{\mathbb{R}}'^{+} = \Gamma\backslash\Gamma Y_{\mathbb{R}}'^{+}$ in $\mathcal{M}$, which ends the proof. ■

Theorem 3.5.3. *Let* $(\mathbf{P}, \mathrm{Y})$, $\mathbf{C}$, Γ, $\mathcal{M} = \Gamma\backslash Y_{\mathbb{R}}^{+}$ *be as above. Then* $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ *is compact for the weak topology. Moreover, if a sequence* $(\mu_n)_n$ *in* $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ *converges to some* $\mu' \in \mathcal{H}_{\mathbf{C}}(\mathcal{M})$, *then there exists* $\mathrm{N} > 0$ *such that* $\operatorname{Supp}\mu_n \subset \operatorname{Supp}\mu'$ *for any* $n > \mathrm{N}$, *and that* $\operatorname{Supp}\mu'$ *equals the archimedean closure of* $\bigcup_{n>\mathrm{N}}\operatorname{Supp}\mu_n$.

Proof. (1) The compactness:

Let $(\mu_n)_n$ be an arbitrary sequence in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$. We need to show that $(\mu_n)_n$ admits a convergent subsequence, whose limit again lies in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$.

Assume that μ_n is the **C**-special H-measure on $\mathcal{M}$ associated to the **C**-special S-subspace $\mathcal{M}_n$ defined by a **C**-special subdatum $(\mathbf{P}_n, \mathrm{Y}_n)$: $\mathcal{M}_n = \Gamma\backslash\Gamma Y_{n\mathbb{R}}^{+}$. According to Lemma 3.5.2 (3), we may assume that $Y_{n\mathbb{R}}^{+} \cap \mathcal{C} \neq \emptyset$, and $\mu_n = \kappa_{y_n *}\nu_n$ where ν_n is the H-measure on $\Omega = \Gamma^{\dagger}\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^{+}$ of support $\Omega_n = \Gamma^{\dagger}\backslash\Gamma^{\dagger}\mathbf{P}_n^{\mathrm{der}}(\mathbb{R})^{+}$, and $y_n \in Y_{n\mathbb{R}}^{+} \cap \mathcal{C}$. We have shown the compactness of $\mathcal{H}_{\mathbf{C}}(\Omega)$ and of $\mathcal{C}$, thus up to replacing $(\nu_n)_n$ resp. $(y_n)_n$ by a convergent subsequence, we may assume that both $(\nu_n)_n$ and $(y_n)_n$ converge. Say $y = \lim_n y_n \in \mathcal{C}$ and $\nu = \lim_n \nu_n \in \mathcal{H}_{\mathbf{C}}(\Omega)$. We want to show that $\mu = \kappa_{y*}\nu$ is the limit of $(\mu_n)_n$ and it lies in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$.

Firstly the convergence:

Write $\langle\ ,\ \rangle$ for the pairing between the set $\mathscr{C}_c^{\infty}(\Omega)$ of compactly supported smooth functions and $\mathscr{D}(\Omega)$ the set of distributions on Ω. Take an arbitrary $f \in \mathscr{C}_c^{\infty}(\mathcal{M})$ and put $\mathrm{F} = f \circ \kappa_y$, $\mathrm{F}_n = f \circ \kappa_{y_n}$, which lie in $\mathscr{C}_c^{\infty}(\Omega)$. Then

$$\int_{\mathcal{M}} f\,d\mu_n = \langle \mathrm{F}_n, \nu_n\rangle \text{ and } \int_{\mathcal{M}} f\,d\mu = \langle \mathrm{F}, \nu\rangle.$$

Consider the following estimation

$$|\langle \mathrm{F}, \nu\rangle - \langle \mathrm{F}_n, \nu_n\rangle| \leq |\langle \mathrm{F} - \mathrm{F}_n, \nu\rangle| + |\langle \mathrm{F}_n, \nu - \nu_n\rangle| \leq \|\mathrm{F} - \mathrm{F}_n\| \cdot \|\nu\| + \|\mathrm{F}_n\| \cdot \|\nu - \nu_n\|$$

where $\|\ \|$ stands for either the sup norm of functions or the induced norm on continuous functionals. It suffices to show that $\|\mathrm{F} - \mathrm{F}_n\|$ tends to zero, whence $\|\mathrm{F}_n\|$ remains bounded, and that $\|\nu - \nu + n\|$ tends to zero, as n tends to ∞.

Since $\lim y_n = y$ for the archimedean topology, we have $(\mathrm{F}_n)_n$ converges to F uniformly on Ω (as compactly supported functions). $\lim_n \|\mathrm{F} - \mathrm{F}_n\| = 0$ implies that $(\|\mathrm{F}_n\|)_n$ is bounded, therefore the term $\|\mathrm{F}_n\|\|\nu_n - \nu\|$ and hence the total difference above tends to zero as $n \to \infty$. f being arbitrary, we conclude that $\lim_n \mu_n = \mu$ for the weak topology.

Now that the limit μ exists as a probability measure on $\mathcal{M}$, we proceed to show that μ lies in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$.

$(\nu_n)_n$ converges to ν in $\mathcal{H}_{\mathbf{C}}(\Omega)$. Up to restricting to a subsequence without changing the limit, we may assume that $\operatorname{Supp}\nu_n \subset \operatorname{Supp}\nu$ for all n. Write $\operatorname{Supp}\nu_n = \Gamma^\dagger\backslash\Gamma^\dagger\mathbf{P}_n^{\mathrm{der}}(\mathbb{R})^+$, then $\bigcup_n \operatorname{Supp}\nu_n$ is dense in $\operatorname{Supp}\nu$. Because ν lies in $\mathcal{H}_{\mathbf{C}}(\Omega)$, we have $\operatorname{Supp}\nu = \Gamma^\dagger\backslash\Gamma^\dagger\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ with $\mathbf{P}'$ given by some Shimura subdatum $(\mathbf{P}',\mathrm{Y}') \subset (\mathbf{P},\mathrm{Y})$. By the proof of Prop.3.3.4, it is known that $\bigcup_n \mathbf{P}_n$ generates $\mathbf{P}'$.

Note that $\mathbf{P}'$ arises as the defining $\mathbb{Q}$-group for only finitely many subdata of $(\mathbf{P},\mathrm{Y})$. Since $\mathbf{MT}(y_n) \subset \mathbf{P}_n \subset \mathbf{P}$, we may, again by the convergent subsequence argument, assume that the y_n's all lie in a common $\mathrm{Y}'^+_{\mathbb{R}}$ for some subdatum of the form $(\mathbf{P}',\mathrm{Y}')$. Then the limit $y = \lim y_n$ also lies in the closed subspace $\mathrm{Y}'^+_{\mathbb{R}} \subset \mathrm{Y}_{\mathbb{R}}$. And $\mu = \kappa_{y*}\nu$ does lie in $\mathcal{H})\mathbf{C}(\mathcal{M})$.

(2) The inclusion $\operatorname{Supp}\mu_n \subset \operatorname{Supp}\mu$ for n large enough and the density of $\bigcup \operatorname{Supp}\mu_n$ in $\operatorname{Supp}\mu$:

We already have $\Omega_n = \operatorname{Supp}\nu_n \subset \operatorname{Supp}\nu = \Omega'$ for all n. Thus $\kappa_{y_n}(\Omega_n) \subset \kappa_{y_n}(\Omega)$ for all n. Now that $y_n \in \mathrm{Y}'^+_{\mathbb{R}}$, we have $\kappa_{y_n}(\Omega') = \mathcal{M}' = \kappa_y(\Omega')$, namely $\operatorname{Supp}\mu_n \subset \operatorname{Supp}\mu$ for all μ.

The density of $\bigcup_n \operatorname{Supp}\mu_n$ in $\operatorname{Supp}\mu$ is clear from the convergence $\lim\mu_n = \mu$. ■

Corollary 3.5.4. *Keep the notions as in the theorem.*

(1) For any closed subset $\mathrm{Z} \subset \mathcal{M}$, *the set* $\mathscr{S}_{\mathbf{C}}(\mathrm{Z})$ *of maximal* $\mathbf{C}$*-special S-subspaces in* Z *is finite. Equivalently, for any sequence* $(\mathcal{M}_n)_n$ *of* $\mathbf{C}$*-special S-subspaces in* $\mathcal{M}$, *the archimedean closure of* $\bigcup_n \mathcal{M}_n$ *is a finite union of* $\mathbf{C}$*-special S-subspaces.*

(2) Let M *be a connected Shimura variety associated to* $(\mathbf{P},\mathrm{Y})$.

Then for any closed subvariety $\mathrm{Z} \subset \mathrm{M}$, *the set* $\mathscr{S}_{\mathbf{C}}(\mathrm{Z})$ *of maximal* $\mathbf{C}$*-special subvarieties contained in* Z *is finite. Equivalently, if* $(\mathrm{M}_n)_n$ *is a sequence of* $\mathbf{C}$*-special subvarieties, then the Zariski closure of* $\bigcup_n \mathrm{M}_n$ *is a finite union of* $\mathbf{C}$*-special subvarieties.*

Proof. (1) It is clear that for any closed subspace $\mathrm{Z} \subset \mathcal{M}$, $\mathscr{S}_{\mathbf{C}}(\mathrm{Z})$ is always a countable set. If it is not finite, then we write it as an infinite sequence $(\mathcal{M}_n)_n$. The corresponding sequence of $\mathbf{C}$-special H-measures $(\mu_n)_n$ is also infinite, and thus contains a convergent subsequence $(\mu'_n)_n$. Write μ' for the limit of $(\mu'_n)_n$ in $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$, and $\mathcal{M}'_n$ for $\operatorname{Supp}\mu'_n$. Because $\operatorname{Supp}\mu_n = \mathcal{M}_n \subset \mathrm{Z}$ for all n, we have $\operatorname{Supp}\mu' \subset \mathrm{Z}$ which is also a $\mathbf{C}$-special S-subspace. But the convergence of $(\mu'_n)_n$ implies, according to the theorem above, that $\operatorname{Supp}\mu'_n \subset \operatorname{Supp}\mu'$ for n large enough, and in particular the $\mathcal{M}'_n$ is not maximal, which contradicts the assumption.

The equivalence of the two formulations has been shown in Chapter 2.

(2) It suffices to derive (2) from (1) under the second formulation: the closure of a homogeneous sequence is weakly special.

Let $\mathcal{M}$ resp. $\mathcal{M}_n$ be the S-space corresponding to M resp. $\mathcal{M}_n$. Now that $\mathcal{M}_n$ is $\mathbf{C}$-special for all n, the archimedean closure of $\bigcup_n \mathcal{M}_n$ is a finite union of $\mathbf{C}$-special S-subspaces. Taking Zariski closure we see that the Zariski closure of $\bigcup_n \mathrm{M}_n$ is a finite union of $\mathbf{C}$-special subvarieties.

3.6 The general case of a homogeneous sequence of special subvarieties

Let $(\mathbf{P},\mathrm{Y})$ be a mixed Shimura datum, K a compact open subgroup of $\mathbf{P}(\mathbb{A}^{\mathrm{f}})$, and M a connected component of $\mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$. Assume that $(\mathbf{P},\mathrm{Y})$ has a pure section $(\mathbf{G},\mathrm{X})$ and denote by $\mathbf{C}$ the connected center of $\mathbf{G}$. Write $\Omega=\Gamma^{\dagger}\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^{+}$ and $\mathcal{M}=\Gamma\backslash\mathrm{Y}_{\mathbb{R}}^{+}$ the connected lattice space resp. S-space corresponding to M, where Γ is an arithmetic subgroup of $\mathbf{P}(\mathbb{R})_{+}$ of the form $\mathbf{P}(\mathbb{Q})_{+}\cap g\mathrm{K}g^{-1}$, g coming from a fixed set $\mathfrak{R}$ of representatives of $\mathbf{P}(\mathbb{Q})_{+}\backslash\mathbf{P}(\mathbb{A}^{\mathrm{f}})/\mathrm{K}$.

From the equidistribution of $\mathbf{C}$-special S-subspaces we have deduced that the André-Oort conjecture for a sequence of $\mathbf{C}$-special subvarieties.

We then consider families of $\mathbf{C}'$-special subvarieties, with $\mathbf{C}'$ a $\mathbb{Q}$-torus in $\mathbf{G}$ containing $\mathbf{C}$.

Lemma 3.6.1. *Let $\mathbf{C}'$ be a $\mathbb{Q}$-torus in $\mathbf{G}$ containing $\mathbf{C}$. Then*

(1) The set of maximal $\mathbf{C}'$-special subdata of $(\mathbf{P},\mathrm{Y})$ is finite.

(2) The set of maximal $\mathbf{C}'$-special lattice subspaces of Ω is finite.

(3) The set of maximal $\mathbf{C}'$-special S-subspaces of $\mathcal{M}$ is finite.

(4) The set of maximal $\mathbf{C}'$-special subvarieties of M is finite.

Note that in the statement we allow the empty set as a finite set.

Proof. (1) We first consider the pure case, then the mixed case.

(i) The pure case:

We assume the existence of a non-trivial $\mathbf{C}'$-special subdatum $(\mathbf{G}_1,\mathrm{X}_1)\subset(\mathbf{G},\mathrm{X})$. Then $\mathbf{C}'$ equals the connected center of $\mathbf{G}_1$.

Put $\mathbf{L}=\mathbf{Z}_{\mathbf{G}}\mathbf{C}'$, and write $\mathbf{L}'=\mathbf{C}_{\mathbf{L}}\mathbf{L}_1\mathbf{L}_2$ where $\mathbf{C}_{\mathbf{L}}$ is the connected center of $\mathbf{L}$, $\mathbf{L}_1$ the product of non-compact $\mathbb{Q}$-factors of $\mathbf{L}^{\mathrm{der}}$, and $\mathbf{L}_2$ is the product of compact $\mathbb{Q}$-factors. Note that $\mathbf{C}'\subset\mathbf{C}_{\mathbf{L}}$.

We set $\mathbf{G}'=\mathbf{C}'\mathbf{L}_1$. If $(\mathbf{G}_2,\mathrm{X}_2)$ is a second $\mathbf{C}'$-special subdatum of $(\mathbf{G},\mathrm{X})$, then $\mathbf{G}_2^{\mathrm{der}}\subset\mathbf{L}_1\mathbf{L}_2$. $\mathbf{L}_2\cap\mathbf{G}_2^{\mathrm{der}}$ is a compact $\mathbb{Q}$-factor of $\mathbf{G}_2^{\mathrm{der}}$, and it has to be trivial. It turns out that $\mathbf{G}_2^{\mathrm{der}}\subset\mathbf{L}_1$ and $\mathbf{G}_2\subset\mathbf{G}'$. We conclude that for any maximal $\mathbf{C}'$-special subdatum, its generic Mumford-Tate group has to be $\mathbf{G}'$. With $\mathbf{G}'$ fixed, there could be only finitely many subdata of the form $(\mathbf{G}',\mathrm{X}')\subset(\mathbf{G},\mathrm{X})$, which are exactly the maximal $\mathbf{C}'$-special subdata.

(ii) The mixed case:

$(\mathbf{P},\mathrm{Y})$ has a pure section $(\mathbf{G},\mathrm{X})$ given by a Levi decomposition $(\mathbf{P}=\mathbf{W}\rtimes\mathbf{G}$. Say $(\mathbf{G}',\mathrm{X}')$ is a maximal $\mathbf{C}'$-special subdatum of $(\mathbf{G},\mathrm{X})$, then obviously $(\mathbf{W}\rtimes\mathbf{G}',\mathbf{U}(\mathbb{C})\mathbf{W}(\mathbb{R})\rtimes$

X') is a maximal $\mathbf{C}'$-special subdatum of $(\mathbf{P},Y)$. The finiteness is clear from the result in the pure case.

(2) Note that the lattice subspace corresponding to a subdatum $(\mathbf{P}',Y')$ only depends on the $\mathbb{Q}$-group $\mathbf{P}'$. Thus from (1) we see that there exists a unique maximal $\mathbf{C}'$-special lattice subspace $\Omega' = \Gamma\backslash\Gamma\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ in $\Omega = \Gamma\backslash\mathbf{P}^{\mathrm{der}}(\mathbb{R})^+$, where $\mathbf{P}'$ comes from some (or any) maximal $\mathbf{C}'$-special subdatum $(\mathbf{W}\rtimes\mathbf{G}',\mathbf{U}(\mathbb{C})\mathbf{W}(\mathbb{R})\rtimes X')$.

(3) and (4):

The set of maximal $\mathbf{C}'$-special subdata if finite, whose elements are of the form $(\mathbf{P}',Y'_i)$ for i varying in I some finite index set. For each i, Y'_i has only finitely many connected components, and they give rise to only finitely many $\mathbf{C}'$-special S-subspaces resp. $\mathbf{C}'$-special subvarieties in an arbitrarily fixed connected component $\Gamma\backslash Y^+_{\mathbb{R}}$ resp. $\Gamma\backslash Y^+$. ■

We can directly apply the results in the above section to the case of $\mathbf{C}'$-special subvarieties. Nevertheless we prefer to start with the more precise results on H-measures.

Proposition 3.6.2. *Keep the notations* $(\mathbf{P},Y)$, $(\mathbf{G},X)$, K, Γ, Ω, $\mathcal{M}$, *etc. Fix a* $\mathbb{Q}$*-torus* $\mathbf{C}'$ *in* $\mathbf{G}$ *containing* $\mathbf{C}$ *as above. Then*

(1) The set $\mathcal{H}_{\mathbf{C}'}(\Omega)$ *of* $\mathbf{C}'$*-special H-measures on* Ω *is compact for the weak topology; i.e. it is closed in* $\mathcal{H}(\Omega)$.

(2) There exists a compact subset $D' = D(\mathbf{C}')$ *of* $\mathcal{M}$ *which meets every* $\mathbf{C}'$*-special S-subspace non-trivially.*

(3) There exists a compact subset $\mathcal{C}' = \mathcal{C}(\mathbf{C}')$ *of* $Y^+_{\mathbb{R}}$ *such that if* $\mathcal{M}'' \subset \mathcal{M}$ *is a* $\mathbf{C}'$*-special S-subspace, then there exists a* $\mathbf{C}'$*-special subdatum* $(\mathbf{P}'',Y'')$ *such that* $\mathcal{M}'' = \Gamma\backslash\Gamma Y''^+_{\mathbb{R}}$ *with* $Y''^+_{\mathbb{R}} \cap \mathcal{C}' \neq \emptyset$.

(4) The set $\mathcal{H}_{\mathbf{C}'}(\mathcal{M})$ *of* $\mathbf{C}'$*-special H-measures on* $\mathcal{M}$ *is compact for the weak topology. If a sequence* $(\mu_n)_n$ *in* $\mathcal{H}'_{\mathbf{C}}(\mathcal{M})$ *converges to some* $\mu' \in \mathcal{H}_{\mathbf{C}'}(\mathcal{M})$, *then for some* $N > 0$, *we have* $\mathrm{Supp}\,\mu_n \subset \mathrm{Supp}\,\mu'$ $\forall n > N$, *and* $\bigcup_{n>N}\mathrm{Supp}\,\mu_n$ *is dense in* $\mathrm{Supp}\,\mu'$ *for the archimedean topology.*

Proof. As is justified by the (3) and (4)of the lemma above, we write $\{(\mathbf{P}',Y'_i) : i \in I\}$ for the finite set of maximal $\mathbf{C}'$-special subdata of $(\mathbf{P},Y)$, with I some fixed finite index set, and $\{\mathcal{M}_j : j \in J\}$ for the finite set of maximal $\mathbf{C}'$-special S-subspaces in $\mathcal{M}$, J some finite index set. We may suppose that $\mathcal{M}_j$ is defined by $(\mathbf{P}',Y'_{i(j)})$ for some $i(j) \in I$.

(1) $\Omega' = \Gamma^\dagger\backslash\Gamma^\dagger\mathbf{P}'^{\mathrm{der}}(\mathbb{R})^+$ is the unique maximal $\mathbf{C}'$-special lattice subspace of Ω, and it is clear that the inclusion $\lambda : \Omega' \hookrightarrow \Omega$ induces a bijection $\lambda_* : \mathcal{H}_{\mathbf{C}'}(\Omega) \to \mathcal{H}_{\mathbf{C}'}(\Omega')$: every $\mathbf{C}'$-special lattice subspace is contained in Ω', and the corresponding $\mathbf{C}'$-special H-measure on Ω is the image under λ of the corresponding H-measure on Ω'. $\lambda_* : \mathcal{H}(\Omega') \to \mathcal{H}(\Omega)$ is clearly continuous. And $\mathcal{H}_{\mathbf{C}'}(\Omega')$ is compact for the weak topology, because $(\mathbf{P}',Y'_i)$ is $\mathbf{C}'$-special itself for any i. Therefore the image of $\mathcal{H}_{\mathbf{C}'}(\Omega')$ under λ_* is also compact.

(2) For each $j \in J$, $\mathcal{M}_j$ contains a compact subset D_j that meets every $\mathbf{C}'$-special S-subspace non-trivially: this is just an application of Corollary 3.5.2 (2). Now we take $D' = \bigcup_{j\in J} D_j$. J is finite, hence D' is compact itself. The maximality of the $\mathcal{M}_j$'s shows that D' meets every $\mathbf{C}'$-special S-subspace.

(3) Similar to (2), this is an application of Corollary 3.5.2 (3). In fact there are only finitely many connected components of the form $Y'^{+}_{j\mathbb{R}}$ that give rise to the maximal $\mathbf{C}'$-special S-subspaces $\mathcal{M}_j$'s, each contains a compact subset $\mathcal{C}_j \subset Y'^{+}_{j\mathbb{R}}$ as is described in 3.5.2 (3), and it suffices to take $\mathcal{C}' = \bigcup_j \mathcal{C}_j$.

(4) Write $\lambda_j : \mathcal{M}_j \hookrightarrow \mathcal{M}$ for the inclusion of the maximal $\mathbf{C}'$-special S-subspaces, $j \in J$. Then we have continuous inclusions $\lambda_{j*} : \mathcal{H}_{\mathbf{C}'}(\mathcal{M}_j) \hookrightarrow \mathcal{H}_{\mathbf{C}'}(\mathcal{M})$, and it is evident that $\mathcal{H}_{\mathbf{C}'}(\mathcal{M}) = \bigcup_j \lambda j*(\mathcal{H}_{\mathbf{C}'}(\mathcal{M}_j))$. Note that each $\mathcal{H}_{\mathbf{C}'}(\mathcal{M}_j)$ is compact for the weak topology according to Theorem 3.5.3. Hence the finite union $\bigcup_j(\lambda_{j*}(\mathcal{H}_{\mathbf{C}})) = \mathcal{H}_{\mathbf{C}'}(\mathcal{M})$ is compact.

Now $\mathcal{H}_{\mathbf{C}'}(\mathcal{M})$ is a finite union of compact subsets $\mathcal{H}_j := \lambda_{j*}\mathcal{H}_{\mathbf{C}'}(\mathcal{M}_j)$. Let $(\mu_n)_n$ be a sequence in $\mathcal{H}_{\mathbf{C}'}(\mathcal{M})$ that converges to some $\mu \in \mathcal{H}_{\mathbf{C}'}(\mathcal{M})$. We may suppose that $\mu \in \mathcal{H}_j$ for some fixed $j \in J$.

If for some $N > 0$ we have $\mu_n \in \mathcal{H}_j$ for all $n > N$, then it suffices to apply Theorem 3.5.3 to $(\lambda_j^*\mu_n)_n$ which is a convergent sequence in $\mathcal{H}_{\mathbf{C}'}(\mathcal{M}_j)$ of limit $\lambda_j^*(\mu)$.

In general, we may decompose μ_n into a union of sequences $(\mu^j_{(n)})_n$ with j running over J. $(\mu^j_{(n)})_n$ could finite for only finitely many j. Let N be the largest index m that appears in these $\mu_m \in \{\mu^j_{(n)} : n \in \mathbb{N}\}$. Then $(\mu_n)_{n>N}$ is decomposed into a finite union of convergent subsequences in $\mathcal{H}_j$. Apply the argument in the last paragraph to each of them, we get the required results on the supports. ■

Corollary 3.6.3. *Let $\mathcal{M}$ be a connected S-space defined by some Shimura datum $(\mathbf{P}, Y)$ with pure section $(\mathbf{G}, X)$, $\mathbf{C}$ the connected center of $\mathbf{G}$, and $\mathbf{C}'$ a $\mathbb{Q}$-torus of $\mathbf{G}$ containing $\mathbf{C}$. Write M for the corresponding connected mixed Shimura variety. Then*

(1) If $(\mathcal{M}_n)_n$ is a sequence of $\mathbf{C}'$-special S-subspaces, then the archimedean closure of $\bigcup_n \mathcal{M}_n$ in $\mathcal{M}$ is a finite union of $\mathbf{C}'$-special S-subspaces. Equivalently, for any closed subset $Z \subset \mathcal{M}$, the set $\mathscr{S}_{\mathbf{C}'}(Z)$ of maximal $\mathbf{C}'$-special S-subspaces contained in Z is finite.

(2) If $(M_n)_n$ is a sequence of $\mathbf{C}'$-special subvarieties of M, then the Zariski closure of $\bigcup_n M_n$ in M is a finite union of $\mathbf{C}'$-special subvarieties. Equivalently, for any closed subvariety $Z \subset M$, the set $\mathscr{S}_{\mathbf{C}'}(Z)$ of maximal $\mathbf{C}'$-special subvarieties contained in Z is finite.

Proof. (1) Let $\{\mathcal{M}_j : j \in J\}$ be the finite set of maximal $\mathbf{C}'$-special S-subspaces in $\mathcal{M}$, J being some finite index set. Then it is clear that for each $j \in J$, $\mathscr{S}_{\mathbf{C}'}(Z \cap \mathcal{M}_j)$ is finite, by the same arguments in Corollary 3.5.4 applied to $Z \cap \mathcal{M}_j$ in $\mathcal{M}_j$. It remains to check that $\mathscr{S}_{C'}(Z)$ equals the finite union of finite subsets $\mathscr{S}_{C'}(Z \cap \mathcal{M}_j)$, which is clear because of the maximality of the $\mathcal{M}_j$'s.

(2) It suffices to take Zariski closure in the conclusions of (1), following the arguments we have used in Corollary 3.5.4 (2). ■

Chapter 4

The degree of the Galois orbit of a special subvariety

In [UY-1], E.Ullmo and A.Yafaev studied the lower bound of the intersection degree of the Galois orbit of a special subvariety in a given pure Shimura variety with respect to the canonical sheaf defining the Baily-Borel compactification. They also deduce the following criterion: if a sequence of special subvarieties whose Galois orbits are of uniformly bounded degree with respect to the canonical sheaf, then the sequence is a finite union of homogeneous subsequences. These results play a main role in the work of B.Klingler and A.Yafaev, cf. [KY].

We would like to study the analogue of these results in the framework of mixed Shimura varieties. We work with special subvarieties in a mixed Shimura variety $M = M_K(\mathbf{P}, Y)$ with a pure section $S = M_{K_G}(\mathbf{G}, X)$, where $K = K_W \rtimes K_G \subset \mathbf{P}(\mathbb{A}^f)$ is a compact open subgroup. According to the general theory developed by R.Pink, there exists a canonical ample invertible sheaf $\mathcal{L}$ on M of the form $\mathcal{L} = \pi^*\mathscr{L} \otimes_{\mathcal{O}_M} \mathcal{T}$ where $\pi : M \to S$ is the canonical fibration over the pure section S, $\mathscr{L} = \mathscr{L}(K)$ is the canonical ample sheaf on S, and $\mathcal{T}$ is a π-ample invertible sheaf on M (ample along each fiber of π). Note that $\mathscr{L}$ depends only on S, while $\mathcal{T}$ depends on the choice of "compactification data" namely a complete admissible cone decomposition involving all the rational boundary components of $(\mathbf{P}, Y)$.

In this chapter we first concentrate on the degrees of Galois orbits of pure subvarieties in M with respect to $\pi^*\mathscr{L}$. We will see that these degrees are subject to similar lower bounds as in the pure case treated in [UY-1]. Then we adapt the estimation to the notion of a test invariant of a general special subvariety of M, not necessarily pure.

The results in [UY-1] deal with special subvarieties in a Shimura variety $M_K(\mathbf{G}, X)$ with $\mathbf{G}$ adjoint. But it was already indicated in their treatment that this assumption on $\mathbf{G}$ can be dropped. Moreover the estimation they arrived is level-free: the constants involved only depend on the given representation $\mathbf{G} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$. It allows immediately an interpretation in the case of mixed Shimura varieties, and in this chapter we draw some consequences to the estimation in some special cases.

4.1 Outline of the estimation of E.Ullmo and A.Yafaev

In this section, we fix a pure Shimura datum $(\mathbf{G},X)$ with $\mathbf{G} = \mathbf{MT}(X)$, $K = K_{\mathbf{G}} = \prod_p K_{\mathbf{G},p} \subset \mathbf{G}(\mathbb{A}^f)$ a torsion-free compact open subgroup, and $E = E(\mathbf{G},X)$ the reflex field.

We first study the intersection degree of the Galois orbit of a geometrically connected component S of $M_K(\mathbf{G},X)$, where the degree is computed against $\mathscr{L}$ the canonical ample line bundle of the Baily-Borel compactification of M. It is also known that $\mathscr{L}$ equals the sheaf of top degree differential forms allowing at most logarithmic singularities along boundary components of codimension one. Write $\mathbf{C}$ for the connected center of $\mathbf{G}$, $\mathbf{T}$ the quotient $\mathbf{G}/\mathbf{G}^{der}$, and $\mathrm{pr}: \mathbf{G} \to \mathbf{T}$ inducing an isogeny $\mathrm{pr}: \mathbf{C} \to \mathbf{T}$. Put $K_{\mathbf{C}} = K_{\mathbf{G}} \cap \mathbf{C}(\mathbb{A}^f) = \prod_p K_{\mathbf{C},p}$ and $K_{\mathbf{C}}^{\max} = \prod_p K_{\mathbf{C},p}^{\max}$ the maximal compact open subgroup of $\mathbf{C}(\mathbb{A}^f)$, and denote by $\delta(\mathbf{C})$ the finite set of rational primes p such that $K_{\mathbf{C},p} \subsetneq K_{\mathbf{C},p}^{\max}$.

In the case where S is a geometrically connected component, the estimation in [UY-1] can be formulated as follows:

Proposition 4.1.1. *(cf.[UY-1] 2.3-2.10) Assume* $(\mathbf{G},X)$ *to be a subdatum of some pure Shimura datum* $(\mathbf{H},Y)$, *and that* $\mathbf{H}$ *carries a faithful representation* $\rho: \mathbf{H} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$ *on some finite dimensional* $\mathbb{Q}$*-vector space* $\mathbf{M}$*. We also assume that* $\mathbf{M}$ *contains a lattice* $\Gamma_{\mathbf{M}}$ *whose profinite completion is a finite free* $\hat{\mathbb{Z}}$*-module* $K_{\mathbf{M}}$ *such that* ρ *takes* $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}^f)$ *into* $\mathrm{GL}_{\hat{\mathbb{Z}}}(K_{\mathbf{M}})$*. Then for any prescribed integer* $N \geq 1$*, there are positive constants* c_N *and* B*, independent of* $(\mathbf{G},X)$*, such that*

$$\deg_{\mathscr{L}} \mathrm{Gal}_E S \geq c_N (\log D_{\mathbf{C}})^N \cdot \max\{1, \prod_{p\in\delta(\mathbf{C})} B|K_{\mathbf{C},p}^{\max}/K_{\mathbf{C},p}|\}$$

where $D_{\mathbf{C}}$ *is the absolute discriminant of the splitting field* $F_{\mathbf{C}}$ *of* $\mathbf{C}$, $\mathbf{C}$ *being the connected center of* $\mathbf{G}$.

The constants c_N *and* B *are level-free:* c_N *is determined by the representation* ρ *and* N, B *is determined by* ρ*; both are independent of the level* K*. Moreover* $B < 1$.

For convenience, we also write $t(S) = I_1(S) \cdot I_2(S)$ where $I_1(S) = c_N(\log D_{\mathbf{C}})^N$ and $I_2(S) = \prod_{p\in\delta(\mathbf{C})} B|K_{\mathbf{C},p}^{\max}/K_{\mathbf{C},p}|$.

Proof (Outline)**.** The proof is essentially the same as in [UY-1], but simpler because we estimate the Galois orbit of a component, i.e. a maximal special subvariety, and we only outline the main idea.

For any $x \in X$, we have the composition $\bar{x}: \mathbb{S} \xrightarrow{x} \mathbf{G}_{\mathbb{R}} \xrightarrow{\mathrm{pr}} \mathbf{T}$, which defines a Shimura datum $(\mathbf{T},\bar{x})$ independent of the choice of x. The reflex field $E(\mathbf{T},\bar{x})$ is contained in the common splitting field F of $\mathbf{C}$ and $\mathbf{T}$.

We have a morphism of Shimura varieties $\mathrm{pr}: M_K(\mathbf{G},X) \to M_{K_{\mathbf{T}}^{\max}}(\mathbf{T},\bar{x})$, $K_{\mathbf{T}}^{\max}$ being the maximal compact open subgroup of $\mathbf{T}(\mathbb{A}^f)$. The corresponding homomorphism of fundamental groups is

$$\mathrm{pr}: \pi_0(M_K(\mathbf{G},X)_{\mathbb{C}}) = \bar{\pi}_0\pi(\mathbf{G})/K \longrightarrow \pi_0(M_{K_{\mathbf{T}}}(\mathbf{T},\bar{x})_{\mathbb{C}}) = \bar{\pi}_0\pi(\mathbf{T})/K_{\mathbf{T}}^{\max}.$$

This map is $\mathrm{Gal}_{E(\mathbf{T},\bar{x})}$-equivariant, where $\mathrm{Gal}_{E(\mathbf{T},\bar{x})}$ acts on the left hand side via $\mathrm{Gal}_{E(\mathbf{T},\bar{x})} \hookrightarrow \mathrm{Gal}_E \xrightarrow{\mathrm{rec}_x} \bar{\pi}\pi_0(\mathbf{G})$. Say a connected component $S \in \bar{\pi}_0\pi(\mathbf{G})/K$ is mapped to $s \in M_{K_{\mathbf{T}}}(\mathbf{T},\bar{x})$. By putting $V = (\mathrm{Gal}_E S) \cap \mathrm{pr}^{-1}(s)$, we get

$$\deg_{\mathscr{L}} \mathrm{Gal}_E S \geq |\mathrm{Gal}_E s| \deg_{\mathscr{L}} V.$$

It remains to find constants c_N and B only dependent on $\mathbf{H} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$ such that:

- $|\mathrm{Gal}_E s| \geq I_1(S) = c_N(\log D_C)^N$;
- $\deg_{\mathscr{L}}(V) \geq I_2(S) = \max\{1, \prod_{p\in\delta(C)} B|K_{C,p}^{\max}/K_{C,p}|\}$.

(1) Estimation of $|\mathrm{Gal}_E s|$: it suffices to quote the same estimation in [Y-3], Theorem 2.15. In particular the constant c_N only involves the calculation with respect to the maximal compact open subgroup $K_{\mathbf{T}}^{\max}$ of $\mathbf{T}$, and it is independent from the level $K_{\mathbf{G}}$'s.

(2) Estimation of $\deg_{\mathscr{L}}(V)$: here we go over again the estimation in [UY-1] and keep trace of the constant B. We start with some estimations concerning homomorphisms between reductive groups.

(2-1) (cf. [UY-1] Lemma 2.3) *The kernel of the isogeny* $\mathbf{C} \to \mathbf{T}$ *is uniformly bounded by a constant that only depends on* $\dim \mathbf{H}$.

In fact the kernel is $\mathbf{C} \cap \mathbf{G}^{\mathrm{der}}$. It is a finite central $\mathbb{Q}$-subgroup of $\mathbf{G}^{\mathrm{der}}$, and it lifts to a finite central $\mathbb{Q}$-subgroup of $\tilde{\mathbf{G}}$ the simply connected covering of $\mathbf{G}^{\mathrm{der}}$. Consider the base change $\tilde{\mathbf{G}}_{\mathbb{C}}$, and $\Sigma_{\mathbf{G}}$ the set of simple factors of $\tilde{\mathbf{G}}_{\mathbb{C}}$. Then

$$\{\Sigma_{\mathbf{G}} : \mathbf{G} \text{ is a reductive } \mathbb{Q}-\text{subgroup of } \mathbf{H}\}$$

is finite by a simple dimension argument. Thus the order of the center $\mathbf{Z}_{\tilde{\mathbf{G}}}$ is bounded when $\mathbf{G}$ varying in the collection of reductive $\mathbb{Q}$-subgroup of $\mathbf{H}$: the supreme of these orders controls the order of $\mathbf{C} \cap \mathbf{G}^{\mathrm{der}}$.

We thus fix a positive integer h which kills all the kernel of $\mathbf{C}_{\mathbf{G}} \to \mathbf{T}_{\mathbf{G}}$ for any reductive $\mathbb{Q}$-subgroup $\mathbf{G}$ of $\mathbf{H}$ ($\mathbf{C}_{\mathbf{G}}$ being the connected center and $\mathbf{T}_{\mathbf{G}} = \mathbf{G}/\mathbf{G}^{\mathrm{der}}$).

As a consequence:

(cf. [KY] Lemma 7.2.3) *The cokernel of* $\pi_{\mathrm{der}} : \bar{\pi}\pi_0(\mathbf{G}) \to \bar{\pi}\pi_0(\mathbf{T})$ *is killed by an integer* $k > 0$, *independent of the choice of subdata* $(\mathbf{G},X) \subset (\mathbf{H},X_{\mathbf{H}})$.

The key point for this consequence is that $\bar{\pi}\pi_0(\mathbf{G})$ is an abelian group that "differs little" from $\bar{\pi}\pi_0(\mathbf{C})$. Recall that from the strong approximation theorem we know that $\bar{\pi}\pi_0(\tilde{\mathbf{G}}) = \{1\}$ is a single point, and the degree of the isogeny $\tilde{\mathbf{G}} \to \mathbf{G}^{\mathrm{der}}$ is bounded by some integer m independent of the choice of reductive $\mathbb{Q}$-subgroup $\mathbf{G} \subset \mathbf{H}$, following the same type of arguments as above.

We have seen that the kernel of $\bar{\pi}\pi_0(\mathbf{C}) \to \bar{\pi}\pi_0(\mathbf{T})$ is killedd by h. To prove the consequence it suffices to show that the cokernel of $\bar{\pi}\pi_0(\mathbf{C}) \to \bar{\pi}\pi_0(\mathbf{G})$ is a torsion abelian group killed by some integer q independent of $\mathbf{G} \subset \mathbf{H}$. Take $g \in \mathbf{G}(\mathbb{A}^f)$, then $g^h = cg_1$ for some $c \in \mathbf{C}(\mathbb{A}^f)$ and $g_1 \in \mathbf{G}^{der}(\mathbb{A}^f)$, because h kills the kernel of the isogeny $\mathbf{C} \times \mathbf{G}^{der} \twoheadrightarrow \mathbf{G}$. Now that $\tilde{\mathbf{G}} \to \mathbf{G}^{der}$ is killed by m, the cokernel of $\tilde{\mathbf{G}}(\mathbb{A}^f) \to \mathbf{G}^{der}(\mathbb{A}^f)$ is killed by m, and thus the class of g_1^m in $\bar{\pi}\pi_0(\mathbf{G})$ is trivial, namely the class of g^{hm} in $\bar{\pi}\pi_0(\mathbf{G})$ actually lies in the image of $\bar{\pi}\pi_0(\mathbf{C})$. This leads to a uniform bound of the cokernel of $\bar{\pi}\pi_0(\mathbf{C}) \to \bar{\pi}\pi_0(\mathbf{G})$, and thus the required consequence.

(2-2) *estimation of characters*

From $(\mathbf{G},\mathrm{X})$ we have the special datum $(\mathbf{T},\bar{x})$ where $\mathbf{T} = \mathbf{T}_{\mathbf{G}}$ and $\bar{x} : \mathbb{S} \to \mathbf{T}_{\mathbb{R}}$ is x modulo $\mathbf{G}^{der}_{\mathbb{R}}$ for any $x \in \mathrm{X}$. Now that F is the common splitting field of $\mathbf{C}$ and of $\mathbf{T}$, the cocharacter $\mu_x : \mathbb{G}_m \to \mathbf{T}_{\mathbb{C}}$ is defined over F, and we get the composition $r_{\bar{x}} : \mathbb{G}_m^F \longrightarrow \mathbf{T}^F \xrightarrow{\mathrm{Nm}} \mathbf{T}$ which is surjective. By (2-1) the h-th power of $r_{\bar{x}}$ lifts to an epimorphism $r : \mathbb{G}_m^F \to \mathbf{C}$, and this allows us to embed $\mathrm{X}_{\mathbf{C}}$ as a subgroup of $\mathrm{X}_{\mathbb{G}_m^F} = \oplus\mathbb{Z}\chi_\sigma$.

We now consider the characters χ of $\mathbf{C}$ that appears in the representation $\mathbf{C} \hookrightarrow \mathbf{G} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$.

• (cf. [UY-1] Lemma 2.4) *The coordinates of these χ's with respect to the basis χ_σ of $\mathrm{X}_{\mathbf{C}}$ are bounded by some integer C_1. In particular, the size of* $\mathrm{Tor}(\mathrm{X}_{\mathbb{G}_m^F}/\mathrm{X}_{\mathbf{C}})$ *is bounded by an integer C_2, say $\mathrm{C}_2 = \mathrm{C}_1^{\dim \mathbf{H}}$. Both C_1 and C_2 are independent of the choice of subdata in* $(\mathbf{H},\mathrm{X}_{\mathbf{H}})$.

The proof in [UY-1] is already independent of the starting level $\mathrm{K}_{\mathbf{G}}$.

Consequently the cokernel of the composition $\mathbf{rec}_{\bar{x}} : \mathrm{Gal}_F \twoheadrightarrow \bar{\pi}\pi_0(\mathbb{G}_m^F) \xrightarrow{r_{\bar{x}}} \bar{\pi}\pi_0(\mathbf{T})$ *is killed by h, independent of the choice of subdata* $(\mathbf{G},\mathrm{X}) \subset (\mathbf{H},\mathrm{X}_{\mathbf{H}})$.

(2-3) Take $x \in \mathrm{X}$ lifting $\bar{x}$, the cocharacter $\mu_x : \mathbb{G}_m \to \mathbf{G}_{\mathbb{C}}$ induces

$$\mathbf{rec}_x : \mathrm{Gal}_E \to \bar{\pi}\pi_0(\mathbb{G}_m^E) \to \bar{\pi}\pi_0(\mathbf{G}),$$

which describes the Galois action on the set of connected components of the Shimura scheme $\mathrm{M}(\mathbf{G},\mathrm{X})_{\mathbb{C}}$. Write $\pi_{der} : \bar{\pi}\pi_0(\mathbf{G}) \to \bar{\pi}\pi_0(\mathbf{T})$ for the homomorphism induced from $\mathbf{G} \twoheadrightarrow \mathbf{T}$.

Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{Gal}_{EF} \arrow[r, "\subset"] \arrow[d, "\mathbf{rec}_x"'] & \mathrm{Gal}_F \arrow[d, "\mathbf{rec}_{\bar{x}}"] \\ \bar{\pi}\pi_0(\mathbf{C}) \arrow[r] & \bar{\pi}\pi_0(\mathbf{G}) \arrow[r, "\pi_{der}"] & \bar{\pi}\pi_0(\mathbf{T}) \end{tikzcd}$$

(cf.[UY-1] Pro.2.5) *There is an integer* A, *independent of the choice of subdata* $(\mathbf{G},\mathrm{X}) \subset (\mathbf{H},\mathrm{X}_{\mathbf{H}})$, *such that for any $t \in \mathbf{C}(\mathbb{A}^f) \subset \mathbf{G}(\mathbb{A}^f)$, the image of t^{A} in $\bar{\pi}\pi_0(\mathbf{G})$ lies in* $\mathbf{rec}_x(\mathrm{Gal}_{EF})$.

Note that $[\mathrm{EF} : \mathrm{E}] \leq [\mathrm{F} : \mathbb{Q}]$ and $[\mathrm{F} : \mathbb{Q}]$ is already uniformly bounded, the size of $\mathbf{rec}_x(\mathrm{Gal}_E)$ and $\mathbf{rec}_x(\mathrm{Gal}_{EF})$ in $\bar{\pi}\pi_0(\mathbf{G})$ only differ by a uniform constant.

For $t \in \mathbf{C}(\mathbb{A}^f)$, the image $\pi_{der}(t^h)$ in $\bar{\pi}\pi_0(\mathbf{T})$ already equals $\mathbf{rec}_{\bar{x}}(\sigma)$ for some $\sigma \in \mathrm{Gal}_F$. We may even enlarge h by a uniform multiple so that σ comes from

$\mathrm{Gal}_{\mathrm{EF}}$, i.e. the class of $\pi_{\mathrm{der}}(t^h)$ falls in $\mathbf{rec}_{\bar{x}}(\mathrm{Gal}_{\mathrm{EF}}$. Obviously $\mathbf{rec}_{\bar{x}}(\sigma) = \pi_{\mathrm{der}}(\mathbf{rec}_x(\sigma))$ by construction. But according to (2-1), the kernel of $\pi_{\mathrm{der}} : \bar{\pi}\pi_0(\mathbf{G}) \to \bar{\pi}\pi_0(\mathbf{T})$ is killed by some constant k, thus $\dfrac{t^h}{\mathbf{rec}_x(\sigma)}$ in $\bar{\pi}\pi_0(\mathbf{G})$ is killed by k: therefore the class of t^{hk} equals $\mathbf{rec}_x(\sigma^k)$. We thus take $\mathrm{A} = hk$.

The above estimation is free from the level $\mathrm{K}_{\mathbf{G}}$. The level $\mathrm{K}_{\mathbf{G}}$ only enters in the following claim:

(2-4) (cf. [UY-1] Lemma 2.7) *Let* $\mathrm{K}_{\mathbf{G}}^m = \mathrm{K}_{\mathbf{C}}^{\max}\mathrm{K}_{\mathbf{G}}$ *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}^{\mathrm{f}})$. *Then* $\mathrm{pr} : \mathrm{M}_{\mathrm{K}_{\mathbf{G}}}(\mathbf{G},\mathrm{X}) \to \mathrm{M}_{\mathrm{K}_{\mathbf{G}}^m}(\mathbf{G},\mathrm{X})$ *is a finite étale covering of degree* $|\mathrm{K}_{\mathbf{G}}^m/\mathrm{K}_{\mathbf{G}}|$. ($\mathrm{K}_{\mathbf{G}}$ is assumed to be torsion free.)

The proof is the same as in [UY-1].

We then enter the estimation of $\deg_{\mathscr{L}}(\mathrm{V})$.

(2-5) Recall that V is a geometrically connected component of $\mathrm{M}_{\mathrm{K}_{\mathbf{G}}}(\mathbf{G},\mathrm{X})$. Write $\delta(\mathbf{C},\mathrm{K}_{\mathbf{G}})$ for the finite set of rational primes p such that $\mathrm{K}_{\mathbf{C},p} \subsetneq \mathrm{K}_{\mathbf{C},p}^{\max}$, and $i(\mathbf{C},\mathrm{K}_{\mathbf{G}})$ the cardinality of $\delta(\mathbf{C},\mathrm{K}_{\mathbf{G}})$

(cf. [UY-1] Prop.2.11) *There exists a constant* B, *independent of the choice of subdatum* $(\mathbf{G},\mathrm{X}) \subset (\mathbf{H},\mathrm{X}_{\mathbf{H}})$ *and the compact open subgroup* $\mathrm{K}_{\mathbf{G}}$, *such that the size of* $\pi_0(\mathrm{V}) = \pi_0(\mathrm{Gal}_{\mathrm{EF}}\,\mathrm{S} \cap \mathrm{pr}^{-1}\mathrm{pr}(\mathrm{S}))$ *is at least* $\mathrm{B}^{i(\mathbf{T})}$ *times the size of* $\pi_0(\mathrm{pr}^{-1}\mathrm{pr}(\mathrm{V}))$.

$\mathrm{K}_{\mathbf{C}}^{\max}$ preserves the fiber $\mathrm{pr}^{-1}\mathrm{pr}(\mathrm{S})$, and $\mathrm{K}_{\mathbf{C}}$ acts on it trivially, whence an action of $\mathrm{K}_{\mathbf{C}}^{\max}/\mathrm{K}_{\mathbf{C}}$ on the set of irreducible components of $\mathrm{pr}^{-1}\mathrm{pr}(\mathrm{S})$.

Write $\mathfrak{I}$ for the image of $\alpha : \mathrm{K}_{\mathbf{C}}^{\max}/\mathrm{K}_{\mathbf{C}} \to \mathrm{K}_{\mathbf{C}}^{\max}/\mathrm{K}_{\mathbf{C}}$ under the map $x \mapsto x^{\mathrm{A}}$. Then for $x \in \mathfrak{I}$, the action of x on $\pi_0(\mathrm{pr}^{-1}\mathrm{pr}(\mathrm{S}))$ is the same as a Galois conjugation $\mathbf{rec}_x(\sigma)$ for some $\sigma \in \mathrm{Gal}_{\mathrm{EF}}$. We thus have inequalities

$$\#\pi_0(\mathrm{pr}^{-1}\mathrm{pr}(\mathrm{S})) = \#\pi_0([\mathrm{K}_{\mathbf{C}}^{\max}/\mathrm{K}_{\mathbf{C}}]\mathrm{S}) \leq \#\mathrm{Coker}(\alpha)\cdot\#(\mathfrak{I}\cdot\mathrm{S})$$

$$\#\pi_0(\mathrm{V}) \geq \#(\mathfrak{I}\cdot\mathrm{S}).$$

Thus to prove the claim above it suffices to show that $\#\mathrm{Coker}(\alpha) \geq \mathrm{B}^{i(\mathbf{C},\mathrm{K}_{\mathbf{G}})}$ for some constant B. Clearly $\mathrm{Coker}(\alpha) = \prod_p \mathrm{Coker}(\alpha_p)$, α_p being the p-th component of α. Because $\mathrm{K}_{\mathbf{C},p} = \mathrm{K}_{\mathbf{C},p}^{\max}$ for all but finitely many p, the cokernel is trivial for p outside $\delta(\mathbf{C},\mathrm{K}_{\mathbf{G}})$.

We proceed to show that, for $p \in \delta(\mathbf{C},\mathrm{K}_{\mathbf{G}})$, $\mathrm{Coker}(\alpha_p)$ is uniformly bounded. Recall that F is the splitting field of $\mathbf{C}$, which is a Galois extension over $\mathbb{Q}$ of group Λ. $\mathrm{X}_{\mathbf{C}}$ is a $\mathbb{Z}[\Lambda]$-module of finite type, which allows a set of generators with cardinality not exceeding $d = \dim\mathbf{H}$. We thus have an epimorphism of $\mathbb{Z}[\Lambda]$-module $\mathbb{Z}[\Lambda]^d \twoheadrightarrow \mathrm{X}_{\mathbf{C}}$, and thus an embedding $\mathbf{C} \hookrightarrow (\mathbb{G}_{\mathrm{m}}^{\mathrm{F}})^d$, and $\mathrm{K}_{\mathbf{C}}^{\max}$ is embedded in $\mathrm{W}_p = [(\prod_{v|p}\mathcal{O}_v^\times)]^d$. Because $[\mathrm{F}:\mathbb{Q}]$ is uniformly bounded, W_p is a free $\mathbb{Z}_p$-module whose rank is bounded by a constant integer $r > 0$. Thus the finite quotient $\mathrm{K}_{\mathbf{C},p}^{\max}/\mathrm{K}_{\mathbf{C},p}$ is a product of at most r cyclic factors, and thus $\#\ker\alpha_p$ is at most A^r, and $\#\mathrm{Coker}\,\alpha_p$ is at least $\mathrm{B}|\mathrm{K}_{\mathbf{C},p}^{\max}/\mathrm{K}_{\mathbf{C},p}|$ with $\mathrm{B} = \mathrm{A}^{-r}$ and $p \in \delta(\mathbf{C},\mathrm{K}_{\mathbf{G}})$.

We remark that $\mathrm{B} = \mathrm{A}^{-r}$ is in general a positive real number less than 1.

(2-6) It remains to point out that $\mathscr{L}$ is ample, and $\deg_{\mathscr{L}}\mathrm{S}$ is always a positive integer. $\mathscr{L}$ is defined over E and $\deg_{\mathscr{L}}\mathrm{S}$ takes constant value when S varies in

$\pi_0(V)$. Combining the above results we get

$$\deg_{\mathscr{L}}(V) \geq \#\pi_0(V)\deg_{\mathscr{L}} S \geq \#\pi_0(V) \geq \prod_{p\in\delta(C,K_G}B|K^{\max}_{C,p}/K_{C,p}|.$$

Note that $\deg_{\mathscr{L}} V$ is always a positive integer, thus it is greater than 1 and the lower bound in the above inequality, hence the required estimation. ■

The canonical sheaf $\mathscr{L} = \mathscr{L}(K)$ is functorial in the following sense:

Lemma 4.1.2. *(cf [KY], Prop.4.2.2) Let $(\mathbf{G}_1,X_1) \subset (\mathbf{G},X)$ be a pair of Shimura data, $K \subset \mathbf{G}(\mathbb{A}^f)$ a neat compact open subgroup.*

(1) If $L \subset \mathbf{G}(\mathbb{A}^f)$ is a neat compact open subgroup such that $g^{-1}Lg \subset K$ for some $g \in \mathbf{G}(\mathbb{A}^f)$, then for the morphism $f : M_L(\mathbf{G},X) \to M_K(\mathbf{G},X)$, $[x, aL] \mapsto [x, agK]$, we have $f^*\mathscr{L}(K) \cong \mathscr{L}(L)$ canonically.

(2) Let $K_1 \subset \mathbf{G}_1(\mathbb{A}^f)$ be a compact open subgroup which is also contained in K, and $\phi : M_{K_1}(\mathbf{G}_1,X_1) \to M_K(\mathbf{G},X)$ the morphism induced by the inclusion $(\mathbf{G}_1,X_1) \hookrightarrow (\mathbf{G},X)$. Write $\mathscr{L}$ for the canonical line bundle on $M_K(\mathbf{G},X)$, and $\mathscr{L}_1$ the one on $M_{K_1}(\mathbf{G}_1,X_1)$. Then for any closed subvariety $Z \subset M_{K_1}(\mathbf{G}_1,X_1)$, we have $\deg_\Lambda Z \geq 0$, where $\Lambda = \phi^*\mathscr{L} \otimes \mathscr{L}_1^\vee$. In particular $\deg_{\phi^*\mathscr{L}} Z = \deg_\Lambda Z + \deg_{\mathscr{L}_1} Z \geq \deg_{\mathscr{L}_1} Z$.

From the part (2) of this lemma is deduced the estimation in [UY-1]:

Theorem 4.1.3. *(cf. [UY-1] Theorem 2.13) Let $(\mathbf{G}_1,X_1)$ be a subdatum of $(\mathbf{G},X)$ with $\mathbf{G}_1 = \mathbf{MT}(X_1)$, E the reflex field of $(\mathbf{G},X)$, $K \subset \mathbf{G}(\mathbb{A}^f)$ a torsion free compact open subgroup, and $K_1 = K \cap \mathbf{G}_1(\mathbb{A}^f)$. Let S be a special subvariety of $M_K(\mathbf{G},X)$ which is a geometrically connected component of the image of $f : M_{K_1}(\mathbf{G}_1,X_1) \to M_K(\mathbf{G},X)$. Then with respect to the canonical line bundle $\mathscr{L} = \mathscr{L}_K$ on $M_K(\mathbf{G},X)$ we have the estimation*

$$\deg_{\mathscr{L}}(\mathrm{Gal}_E S) \geq I_1(S)I_2(S) = c_N(\log D_{C_1})^N \max\{1, \prod_{p\in\delta(C_1,K)} B|K^{\max}_{C_1,p}/K_{C_1,p}|\}$$

as is found in Prop.4.1.1, where $\mathbf{C}_1$ denotes the connected center of $\mathbf{G}_1$, N is a prescribed positive integer, and c_N, B are constants determined by the given faithful representation of $\mathbf{G}$ on $\mathbf{M}$.

Proof. The cardinality of $\mathrm{Gal}_E \cdot S$ is at least that of $\mathrm{Gal}_{E_1} \cdot S$, where $E_1 \supset E$ is the reflex field of $(\mathbf{G}_1,X_1)$. Since K is assumed to be torsion free, $f : M_{K_1}(\mathbf{G}_1,X_1) \to M_K(\mathbf{G},X)$ is generically injective, and therefore $\deg_{\mathscr{L}} \mathrm{Gal}_{E_1} \cdot S$ is the same as $\deg_{f^*\mathscr{L}} \mathrm{Gal}_{E_1} \cdot Z$, where Z is a geometrically connected component of $M_{K_1}(\mathbf{G}_1,X_1)$. According to part (2) of the above lemma, we have

$$\deg_{\mathscr{L}} \mathrm{Gal}_E S \geq \deg_{\mathscr{L}} \mathrm{Gal}_{E_1} \cdot S = \deg_{f^*\mathscr{L}} \mathrm{Gal}_{E_1} Z \geq \deg_{\mathscr{L}_1} \mathrm{Gal}_E \cdot Z \geq I_1(Z)I_2(Z)$$

with $I_1(Z)$ and $I_2(Z)$ as was in the proposition. Here $\mathscr{L}_1$ is the canonical line bundle on $M_{K_1}(\mathbf{G}_1,X_1)$. The constants c_N and B in I_1 and I_2 are determined by $\mathbf{G} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$ as was shown in the lemma, which finishes the proof. ■

Apply the above results to the case of pure special subvarieties inside a mixed one, we have:

Theorem 4.1.4. *Let* $M \to S$ *be a fibration of a mixed Shimura variety over a pure section, given explicitly as*

$$\pi : M = M_K(\mathbf{P}, Y) \to S = M(0) = M_{K_{\mathbf{G}}}(\mathbf{G}, X)$$

where $\mathbf{P} = \mathbf{W} \rtimes \mathbf{G}$ *is a Levi decomposition,* $K = K_{\mathbf{W}} \rtimes K_{\mathbf{G}}$ *a compact open subgroup of* $\mathbf{P}(\mathbb{A}^f)$ *with* $K_{\mathbf{G}} = \prod_p K_p \subset \mathbf{G}(\mathbb{A}^f)$ *torsion free. Write* E *for the reflex field of* $(\mathbf{P}, Y)$*, and* $\mathscr{L}$ *the canonical line bundle on* S*. Suppose* M' *is a pure special subvariety contained in a special section* $M(w)$ *for some* $w \in \mathbf{W}(\mathbb{Q})$*, corresponding to a connected component of the Shimura variety given by the subdatum* $(w\mathbf{G}_1 w^{-1}, w \rtimes X)$*, where* $(\mathbf{G}_1, X_1) \subset (\mathbf{G}, X)$ *is a subdatum with* $\mathbf{MT}(X_1) = \mathbf{G}_1$*. Put* $\mathbf{C}_1$ *to be the connected center of* $\mathbf{G}_1$*.*

Fix $\mathscr{L}$ *the canonical line bundle on* S*, then we have*

$$\deg_{\pi^*\mathscr{L}} \mathrm{Gal}_E M' \geq I_1(M') I_2^w(M')$$

with $I_1(M'), I_2^w(M')$ *defined via constants* c_N *and* B *as follows*

- $I_1(M') = c_N(\log D_{\mathbf{C}_1})^N$ *for the prescribed integer* $N > 0$*,* $D_{\mathbf{C}_1}$ *the absolute discriminant of the splitting field of* $\mathbf{C}_1$*;*
- $I_2^w(M') = \max\{1. \prod_{p \in \Delta_w(\mathbf{C}_1)} B|K^{\max}_{\mathbf{C}_1,p}/K_{\mathbf{C}_1,p}(w)|\}$*, where* $K_{\mathbf{C}_1} = K \cap \mathbf{C}_1(\mathbb{A}^f) = \prod_p K_{\mathbf{C}_1,p}$*,* $K^{\max}_{\mathbf{C}_1} = \prod_p K^{\max}_{\mathbf{C}_1,p}$ *the maximal compact open subgroup of* $\mathbf{C}_1(\mathbb{A}^f)$*,* $K_{\mathbf{C}_1}(w) = \{g \in K_{\mathbf{C}_1} : wgw^{-1}g^{-1} \in K_{\mathbf{W}}\} = \prod_p K_{\mathbf{C}_1}(w)_p$*, and* $\Delta_w(\mathbf{C}_1) := \delta(\mathbf{C}_1, K_{\mathbf{G}}(w))$ *is the set of rational primes* p *such that* $K_{\mathbf{C}_1}(w)_p \subsetneq K^{\max}_{\mathbf{C}_1,p}$*.*

Moreover c_N *and* B *only depends on the representation* $\mathbf{G} \to \mathbf{GL}_{\mathbb{Q}}(\mathbf{M})$*, as was in the case of Theorem 4.1.3*

Proof. We put $K_{\mathbf{G}}(w) = \{g \in K_{\mathbf{G}} : wgw^{-1}g^{-1} \in K_{\mathbf{W}}\} = \prod_p K_{\mathbf{G}}(w)_p$. It is clear that $w\mathbf{G}(\mathbb{A}^f)w^{-1} \cap K_{\mathbf{W}} \rtimes K_{\mathbf{G}} = wK_{\mathbf{G}}(w)w^{-1}$. And comjugation by w^{-1} gives the commutative diagram

$$\begin{tikzcd} M(w) \arrow[r, "i_w"] \arrow[d, "w^{-1}"', "\cong"] \arrow[dr, "\pi_w"] & M \arrow[d, "\pi"] \\ S_w \arrow[r, "\mathrm{pr}_w"'] & S \end{tikzcd}$$

where $S_w = M_{K_{\mathbf{G}}(w)}(\mathbf{G}, X)$ and $\mathrm{pr}_w : S_w \to S$ is the projection induced by the inclusion $K_{\mathbf{G}}(w) \subset K_{\mathbf{G}}$.

Similarly, for a pure special subvariety M' of M contained in $M(w)$, we may assume that it is is a connected component of some pure Shimura subvariety given by a datum of the form $(w\mathbf{G}'w^{-1}, w \rtimes X_1)$ at level $wK_{\mathbf{G}'}(w)w^{-1}$ for some pure Shimura subdatum $(\mathbf{G}_1, X_1) \subset (\mathbf{G}, X)$ with finite level $K_{\mathbf{G}'}$. Conjugation by w^{-1} sends M' bijectively onto a special subvariety S' in S_w, which is a connected component of the subvariety in S_w associated to $(\mathbf{G}_1, X_1)$ at level $K_{\mathbf{G}'}(w)$.

We calculate $\deg_{\pi^*\mathscr{L}}(\mathrm{Gal_E\,M'})$. By definition of deg it is equal to $\deg_{i_w^*\pi^*\mathscr{L}}(\mathrm{Gal_E\,M'}) = \deg_{\pi_w^*\mathscr{L}}(\mathrm{Gal_E\,M'})$. Conjugation by w^{-1} shows that it is equal to $\deg_{\mathrm{pr}_w^*\mathscr{L}}(\mathrm{Gal_E\,S'})$. We have seen that $\mathrm{pr}_w^*\mathscr{L}$ is isomorphic to $\mathscr{L}_{\mathrm{S}_w}$, therefore by applying the estimation of E.Ullmo and A.Yafaev to $\mathrm{S}' \subset \mathrm{S}_w$, we obtain

$$\deg_{\pi^*\mathscr{L}}(\mathrm{Gal_E\,M'}) = \deg_{\mathscr{L}_{\mathrm{S}_w}}(\mathrm{Gal_E\,S'}) \geq c_{\mathrm{N}}(\log \mathrm{D_{C'}})^{\mathrm{N}} \cdot \max\{1, \prod_{p\in\Delta_w(\mathbf{C}')} \mathrm{B}|\mathrm{K}^{\max}_{\mathrm{C}',p}/\mathrm{K_{C'}}(w)_p|\}$$

where $\mathbf{C}'$ is the connected center of $\mathbf{G}'$, $\mathrm{D_{C'}}$ is the absolute discriminant of $\mathbf{C}'$, $\mathrm{K_{C'}}(w) = \mathbf{C}'(\mathbb{A}^{\mathrm{f}}) \cap \mathrm{K_G}(w) = w^{-1}(w\mathbf{C}'w^{-1}(\mathbb{A}^{\mathrm{f}}) \cap \mathrm{K_W} \rtimes \mathrm{K_G})w$, and $\Delta_w(\mathbf{C}')$ is the finite set of rational prime p over which $\mathrm{K_{C'}}(w)_p \subsetneq \mathrm{K}^{\max}_{\mathrm{C}',p}$. c_{N} and B remain the same constants, independent of w and $\mathrm{K_G}$.

It suffices to take $\mathrm{I_1(M')} = c_{\mathrm{N}}(\log \mathrm{D_{C'}})$ and

$$\mathrm{I}_2^w(\mathrm{M'}) = \max\{1, \prod_{p\in\delta(\mathbf{C}',w)} \mathrm{B}|\mathrm{K}^{\max}_{\mathrm{C}',p}/\mathrm{K_{C'}}(w)_p|\} \qquad \blacksquare$$

4.2 On the factor $\mathrm{I}_2^w(\mathrm{M}')$

We fix the diagram in the last section:

$$\begin{tikzcd} \mathrm{M}(w) \arrow[r, "i_w"] \arrow[d, "w^{-1}"', "\cong"] \arrow[dr, "\pi_w"] & \mathrm{M} \arrow[d, "\pi"] \\ \mathrm{S}_w \arrow[r, "\mathrm{pr}_w"'] & \mathrm{S} \end{tikzcd}$$

where $\mathrm{S}' \subset \mathrm{S}_w$ is a connected component of the Shimura subvariety given by a subdatum $(\mathbf{G}',\mathrm{X}')$ and $\mathrm{M}' \subset \mathrm{M}(w)$ is isomorphic to S' under the conjugation by w. Then

$$\deg_{\pi^*\mathscr{L}}(\mathrm{Gal_E\,M'}) = \deg_{\mathscr{L}_{\mathrm{S}_w}} \mathrm{Gal_E\,S'} \geq \mathrm{I_1(S')}\mathrm{I}_2^w(\mathrm{S}')$$

with $\mathrm{I_1} = c_{\mathrm{N}}(\log \mathrm{D_{C'}})^{\mathrm{N}}$ and

$$\mathrm{I}_2^w(\mathrm{S}') = \max\{1, \prod_{p\in\Delta_w(\mathbf{C}')} \mathrm{B}|\mathrm{K}^{\max}_{\mathrm{C}',p}/\mathrm{K_{C'}}(w)_p|\}$$

where $\Delta_w(\mathbf{C}') = \delta(\mathbf{C}', \mathrm{K_G}(w))$ is the set of rational primes that $\mathrm{K_{C'}}(w)_p \subsetneq \mathrm{K}^{\max}_{\mathrm{C}',p}$.

Note that both of these two factors are invariant when S' runs over the set of $\mathbf{C}'$-special subvarieties in S_w. Moreover $\mathrm{I_1(S')}$ is level free, while $\mathrm{I}_2^w(\mathrm{S}')$ varies when w moves in $\mathbf{W}(\mathbb{Q})$.

Write $\mathrm{J}_2^w(\mathrm{S}') = \prod_{p\in\Delta_w(\mathrm{C}')} \lambda_p$ with $\lambda_p = \mathrm{B}|\mathrm{K}^{\max}_{\mathrm{C}',p}/\mathrm{K_{C'}}(w)_p|$. Recall that $\mathrm{K_{C'}}(w)_p = \{g \in \mathrm{K_{C'}} : wgw^{-1}g^{-1} \in \mathrm{K_W}\}$. Let $\mathrm{K_W}[w]_p$ be the subgroup of $\mathbf{W}(\mathbb{Q}_p)$ generated by $\mathrm{K}_{\mathrm{W},p}$ and w. Note that in the general case, this group is not necessarily commutative.

Lemma 4.2.1. *(1) $K_{\mathbf{W}}[w]_p$ is a compact open subgroup in $\mathbf{W}(\mathbb{Q}_p)$;*

(2) With respect to the action of $K_{\mathbf{C}',p}$ on $\mathbf{W}(\mathbb{Q}_p)/K_{\mathbf{W},p}$, $K_{\mathbf{C}'}(w)_p$ is the isotropy subgroup in $K_{\mathbf{C}',p}$ of $(w^{-1} \mod K_{\mathbf{W},p})$, contained in the stabilizer of $K_{\mathbf{W}}[w]_p/K_{\mathbf{W},p}$, where $K_{\mathbf{W}}[w]_p$ is the subgroup of $\mathbf{W}(\mathbb{Q}_p)$ generated by $K_{\mathbf{W},p}$ and w ;

Proof. (1) If $\mathbf{W}$ is abelian, i.e. $\mathbf{W}$ equals $\mathbf{U}$ or $\mathbf{V}$, then clearly the reduction modulo $K_{\mathbf{W},p}$ of w generated a finite torsion subgroup of $\mathbf{W}(\mathbb{Q}_p)/K_{\mathbf{W},p}$.

In general $\mathbf{W}$ is non-commutative, and it is determined by the alternating bi-linear map $\psi : \mathbf{V}\times\mathbf{V} \to \mathbf{U}$ which comes from the Lie bracket of $\mathbf{Lie W}$, and the group laws is written as $(u_1, v_1)(u_2, v_2) = (u_1 + u_2 + \psi(v_1, v_2), v_1 + v_2)$, where the the notion of bracket (u, v) identifies $\mathbf{W}$ with the product of $\mathbb{Q}$-varieties $\mathbf{U}\times\mathbf{V}$ (not as $\mathbb{Q}$-groups). We equip $\mathbf{U}(\mathbb{Q}_p)\times\mathbf{V}(\mathbb{Q}_p)$ with the metric defined by the p-adic norms on $\mathbf{U}(\mathbb{Q}_p)$ and $\mathbf{V}(\mathbb{Q}_p)$ respectively.

Assume $w = (u, v) \in \mathbf{W}(\mathbb{Q}_p)$. To show that $K_{\mathbf{W}}[w]_p$ is compact, it suffices to give show that there exists compact open subgroups $K'_{\mathbf{U}}$ resp. $K'_{\mathbf{V}}$ in $\mathbf{U}(\mathbb{Q}_p)$ resp. $\mathbf{V}(\mathbb{Q}_p)$ such that any $(u', v') \in K_{\mathbf{W}}[w]_p$ is given by some $u' \in K'_{\mathbf{U}}$ and $v' \in K'_{\mathbf{V}}$.

First consider the reduction modulo $\mathbf{U}(\mathbb{Q}_p)$ of $K_{\mathbf{W}}[w]_p$. Then its image in $\mathbf{V}(\mathbb{Q}_p)$ is $K_{\mathbf{V}}[v]_p$, which is compact because its reduction modulo $K_{\mathbf{V},p}$ is a finite torsion group. We thus get a upper bound A for the upper bound of the p-adic norm of $v' \in \mathbf{V}(\mathbb{Q}_p)$ for $(u', v') \in K_{\mathbf{W}}[w]_p$. It remains to take $K'_{\mathbf{V}}$ to be the elements in $\mathbf{V}(\mathbb{Q}_p)$ of p-adic norm at most A.

Then we consider $K' = \mathbf{U}\cap K_{\mathbf{W}}[w]_p$, and we show that $K'/K_{\mathbf{U},p}$ is finite. Consider $(u + u_i, v + v_i) \in K_{\mathbf{W}}[w]_p$ for $i = 1,2$, with $w = (u, v)$ and $(u_i, v_i) \in K_{\mathbf{W}} = K_{\mathbf{U}}\times K_{\mathbf{V}}$. Then by the definition of the group law,

$$(u+u_1, v+v_1)(u+u_2, v+v_2) = (2u+u_1+u_2+\psi(v+v_1, v+v_2), 2v+v_1+v_2).$$

Since $\psi(v+v_1, v+v_2) = \psi(v, v_2 - v_1) + \psi(v_1 + v_2)$ by the anti-commutativity of ψ, and that $\psi(v_1, v_2) \in K_{\mathbf{U}}$ for any $v_1, v_2 \in K_{\mathbf{V}}$, we deduce that $K'/K_{\mathbf{U}}$ is generated by elements of the form $mu + \psi(nv, v') \mod K_{\mathbf{U}}$ with $v' \in K_{\mathbf{V}}$ and $m, n \in \mathbb{Z}$. $v \mod K_{\mathbf{V}}$ is a torsion element in $\mathbf{V}(\mathbb{Q}_p)/K_{\mathbf{V}}$, thus $\{\psi(nv, v') \mod K_{\mathbf{U}} : n \in \mathbb{Z},\ v' \in K_{\mathbf{V}}\}$ is finite, hence the finiteness of $K'/K_{\mathbf{U}}$ and thus the compactness of K', and we simply take $K'_{\mathbf{U}} = K'$.

Combine the two we get the compactness of $K_{\mathbf{W}}[w]_p$.

(2) By definition, for any $g \in K_{\mathbf{C}'}(w)_p$ we have $wgw^{-1}g^{-1} \in K_{\mathbf{W},p}$, thus $g(w^{-1}) = gw^{-1}g^{-1} \in w^{-1}K_{\mathbf{W},p}$, namely g fixes the class $(w^{-1} mod K_{\mathbf{W},p}$. In particular g stabilizes the compact open subgroup generated by $K_{\mathbf{W},p}$ and w (or equivalently, by $K_{\mathbf{W},p}$ and w^{-1}).

Similarly, it is easy to show that $K^{\max}_{\mathbf{C}'}(w)_p$ is the isotropy subgroup in the stabilizer of $K_{\mathbf{W}}[w]_p$ with respect to the action of $K^{\max}_{\mathbf{C}',p}$ on $\mathbf{W}(\mathbb{Q}_p)$. ■

Lemma 4.2.2. *For $J^w_2(S') = \prod_{p\in\Delta_w}\lambda_p$ as mentioned above, there exists a uniform constant c, independent of w and $\mathbf{C}'$, such that if $p \in \Delta_w(\mathbf{C}')$, then*

$$\lambda_p = B|K^{\max}_{\mathbf{C}',p}/K_{\mathbf{C}'}(w)_p| \geq cp$$

and thus $J_2^w(S') \geq \prod_{p\in\Delta_w(C')} cp$ *and*

$$I_2^w(S') \geq \max\{1, \prod_{p\in\Delta_w(C')} cp.$$

Proof. (i) We first consider the commutative case, where **W** equals either **U** or **V** or a direct sum $\mathbf{U}\oplus\mathbf{V}$ given by the trivial extension. In this case the quotient $\mathbf{W}(\mathbb{Q}_p)/K_{\mathbf{W},p}$ is a commutative torsion group.

Write $V = \mathbf{W}(\mathbb{Q}_p)$, then V is a finite dimensional $\mathbb{Q}$-vector space of dimension d, containing $L = K_{\mathbf{W}_p}$ as a lattice (i.e. compact open subgroup). $L_1 = L[w]$ is the lattice generated by L and w, and L_1/L is a finite p-group, because it is a finite subgroup of $V/L \cong (\mathbb{Q}_p/\mathbb{Z}_p)^d$. $\mathbf{H} = \mathbf{C}'_{\mathbb{Q}_p}$ is a $\mathbb{Q}_p$-torus acting on V, $H^{\max} = K^{\max}_{\mathbf{C}',p}$ the maximal compact open subgroup of $\mathbf{H}(\mathbb{Q}_p)$, $H = K_{\mathbf{C}',p}$ an open subgroup of $H^{\max}$ stabilizing L, and $H(w) = \{h \in H : w - g(w) \in L\}$ is the isotropy subgroup of $(w \mod L)$ with respect to the action of H on V/L.

Write $\mathrm{Stab}_H(L_1)$ for the stabilizer in H of L_1 with respect to the action of H on V. Then $H(w) \subset \mathrm{Stab}_H(L') \subset H$. We want to show that for $p \in \Delta_n$, $|H^{\max}/H(w)| \geq p-1$.

If $p \in \delta_n$, i.e. $H \subsetneq H^{\max}$, then the Proposition 4.3.9 of [EY] shows that $|H^{\max}/H(w)| \geq |H^{\max}/H| \geq p-1$.

If p is in $\Delta_n - \delta_n$ such that $H(w) \subsetneq \mathrm{Stab}_H(L_1)$, then the orbit of $(w \mod K_{\mathbf{W},p})$ in L_1/L under $\mathrm{Stab}_H(L_1)$ is a nontrivial subgroup, at least of order p because in this case L_1/L is a non-trivial finite p-group.

Finally, it remains the case where p is in $\Delta_n-\delta_n$ such that $H(w) = \mathrm{Stab}_H(L_1) \subsetneq H = H^{\max}$. Because $L_1 \subset V$ is a lattice, again we apply 4.3.9 of [EY] and we see that $|H^{\max}/H(w)| \geq |H/\mathrm{Stab}_H(L_1)| \geq p-1$.

Combining the three cases we see that $\lambda_p = B|H^{\max}/H(w)| \geq B(p-1) \geq \frac{B}{2}p$, and $c = \frac{1}{2}B(<1)$ suffices for the lemma.

(ii) We then consider the non-commutative case, where **W** is an extension of **V** by **U** via some non-trivial anti-symmetric bi-linear map $\psi : \mathbf{V}\times\mathbf{V} \to \mathbf{U}$. We write $w = (u, v)$ for the fixed identification $\mathbf{W} = \mathbf{U}\times\mathbf{V}$ (as $\mathbb{Q}$-varieties instead of $\mathbb{Q}$-groups). Note that for $g \in \mathbf{G}$ we have $g((u, v)) = (g(u), g(v))$.

Write $W = \mathbf{W}(\mathbb{Q}_p)$, $V = \mathbf{V}(\mathbb{Q}_p)$, and $U = \mathbf{U}(\mathbb{Q}_p)$. We have lattices $L_V = K_{\mathbf{V},p} \subset V$, $L_U = K_{\mathbf{U},p} \subset U$, and compact open subgroup $L_W = K_{\mathbf{W},p} = L_U L_V$. Adding w we get compact open subgroup $L_W[w] = K_{\mathbf{W}}[w]_p$, lattices $L_V[w]$ which is generated by L and $w \mod U$, $L_U[w] := U \cap K_{\mathbf{W}}[w]_p$, and we have $L_W[w] = L_U[w]L_V[w]$. We also write $H^{\max} = K^{\max}_{\mathbf{C}',p}$, $H = K_{\mathbf{C}',p}$, $H(w) = K_{\mathbf{C}'}(w)_p$.

We put

$$H_V(\bar{w}) := \{g \in H : \bar{w} g \bar{w}^{-1} g^{-1} \in L_V\}$$

where $\bar{w} = v = (w \mod U) \in V$. Similar to (i), we see that $H_V(w)$ is the isotropy subgroup of $(\bar{w} \mod L_V)$ with respect to the action of H on V/L_V, and we have the chain $H(w) \subset H_V(\bar{w}) \subset H \subset H^{\max}$.

For $p \in \delta_n$, it is known that $|\mathrm{H}^{\max}/\mathrm{H}(w)| \geq |\mathrm{H}^{\max}/\mathrm{H}| \geq p-1$. Therefore in the remaining part we may assume that $p \in \Delta_n - \delta_n$, i.e. $\mathrm{H}(w) \subsetneq \mathrm{H} = \mathrm{H}^{\max}$

- If $\mathrm{H}_\mathrm{V}(w) \subsetneq \mathrm{H}$, then by the same arguments as in (i) we conclude that $|\mathrm{H}^{\max}/\mathrm{H}(w)| \geq |\mathrm{H}^{\max}/\mathrm{H}_\mathrm{V}(w)| \geq p-1$.
- If $\mathrm{H}(w) \subsetneq \mathrm{H}_\mathrm{V}(w) = \mathrm{H}$, then H fixes $(\bar{w} \mod \mathrm{L}_\mathrm{V})$ in $\mathrm{V}/\mathrm{L}_\mathrm{V}$, and $g(w) = g(u,v) = (g(u), g(v)) = (g(u), v) \mod \mathrm{L}_\mathrm{W}$. It suffices to estimate the orbit of $u \mod \mathrm{L}_\mathrm{U}$ in $\mathrm{U}/\mathrm{L}_\mathrm{U}$ under H. This is at least $p-1$, as is again reduced to (i). We thus conclude that $|\mathrm{H}^{\max}/\mathrm{H}(w)| \geq p-1$.

Hence in the general case we still have $\lambda_p \geq cp$ by putting $c = \frac{1}{2}\mathrm{B}(<1)$. ■

This lemma leads us to the following

Proposition 4.2.3. *Fix $\pi : \mathrm{M} \to \mathrm{S}$ as above. Let $(\mathrm{M}_n)_n$ be a sequence of pure special subvarieties in M with $\mathrm{M}_n \subset \mathrm{M}(w_n)$ for some $w_n \in \mathbf{W}(\mathbb{Q})$, such that $\deg_{\pi^*\mathscr{L}} \mathrm{Gal}_\mathrm{E} \mathrm{M}_n \leq \mathrm{C}$ for some constant $\mathrm{C} > 0$ independent of n. Then*

(1) the sequence $(\mathrm{M}_n)_n$ is weakly homogeneous, and consequently, the Zariski closure of the union $\bigcup_n \mathrm{M}_n$ is weakly special;

(2) the sequence of test invariants $\tau_n(w_n)$ is finite.

Proof. (1) We may assume that $\mathrm{M}_n \subset \mathrm{M}(w_n)$ is the connected component of a pure Shimura subvariety given by a pure subdatum of the form $(w_n \mathbf{G}_n w_n^{-1}, w_n \rtimes \mathrm{X}_n)$. Then $\pi(\mathrm{M}_n) = \mathrm{S}_n \subset \mathrm{S}$ is a connected component of the pure Shimura variety given by the subdatum $(\mathbf{G}_n, \mathrm{X}_n)$. Write $\mathbf{C}_n$ for the connected center of $\mathbf{G}_n$. Then by putting $b_n = \deg_{\pi^*\mathscr{L}}(\mathrm{Gal}_\mathrm{E} \mathrm{M}_n)$ and $a_n = \deg_{\pi^*\mathscr{L}}(\mathrm{Gal}_\mathrm{E} \mathrm{S}_n) = \deg_{\mathscr{L}}(\mathrm{Gal}_\mathrm{E} \mathrm{S}_n)$ we have

$$b_n \geq \tau_n(w_n) = c_\mathrm{N}(\log \mathrm{D}_n)^\mathrm{N} \cdot \max\{1, \prod_{p \in \Delta_n} \mathrm{B}|\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p|\}$$

$$a_n \geq \tau_n(0) = c_\mathrm{N}(\log \mathrm{D}_n)^\mathrm{N} \max\{1, \prod_{p \in \delta_n} \mathrm{B}|\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n,p}|\}$$

where D_n is the absolute discriminant of the splitting field of $\mathbf{C}_n$, δ_n the set of rational primes p such that $\mathrm{K}_{\mathbf{C}_n,p} \subsetneq \mathrm{K}^{\max}_{\mathbf{C}_n,p}$ and $\Delta_n(w_n)$ the set of rational primes p such that $\mathrm{K}_{\mathbf{C}_n}(w_n)_p \subsetneq \mathrm{K}^{\max}_{\mathbf{C}_n,p}$.

Clearly we have $\delta_n \subset \Delta_n$. Note that

$$|\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p| = |\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n,p}||\mathrm{K}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p|.$$

Consider the intermediate quantities $\mathrm{J}_n(w_n) = |\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p|$ and $\mathrm{J}_n(0) = |\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n,p}|$, then

$$\frac{\mathrm{J}_n(w_n)}{\mathrm{J}_n(0)} = \prod_{p \in \delta_n} |\mathrm{K}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p| \times \prod_{p \in \Delta_n - \delta_n} \mathrm{B}|\mathrm{K}^{\max}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p|.$$

Since for $p \in \Delta_n - \delta_n$ we have $\mathrm{K}_{\mathbf{C}_n,p} = \mathrm{K}^{\max}_{\mathbf{C}_n,p}$, the quotient is also written as

$$\frac{\mathrm{J}_n(w_n)}{\mathrm{J}_n(0)} = \prod_{p \in \delta_n} |\mathrm{K}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p| \times \prod_{p \in \Delta_n - \delta_n} \mathrm{B}|\mathrm{K}_{\mathbf{C}_n,p}/\mathrm{K}_{\mathbf{C}_n}(w_n)_p|.$$

Because the constant B is taken from $]0,1[$, $J_n(w_n)$ might fail to exceed $J_n(0)$.

We follow the strategy of E.Ullmo and A.Yafaev. Suppose that the sequence (b_n) is bounded. Since $b_n \geq c_N(\log D_n)^N$, we deduce that $(\log D_n)$ is bounded. Therefore $\{D_n\}$ is finite, and only finitely many number fields occur as the splitting field of the $\mathbf{C}_n$'s. We thus assume for simplicity that the $\mathbf{C}_n$'s are of common splitting field F, and $\log D_n$ is constant.

Hence the sequence $c_n = J_2^{w_n}(M_n) = \prod_{p\in\Delta_n} B|K^{\max}_{\mathbf{C}_n,p}/K_{\mathbf{C}_n}(w_n)_p|$ is bounded, and therefore the sequence

$$d_n \prod_{p\in\Delta_n} cp \leq c_n$$

is bounded. We thus deduce that $\bigcup_n \Delta_n$ is finite. In particular $\bigcup_n(\Delta_n - \delta_n)$ is finite. Assume $\bigcup_n(\Delta_n - \delta_n)$ is of cardinal m (independent of n). Then $\frac{J_n(w_n)}{J_n(0)} \geq B^m$, and $J_n(0) \leq B^{-m} J_n(w_n)$ for all n. Consequently the sequence of test invariants $(\tau_n(0))_n$ for $(S_n \subset S)_n$ is bounded, and the theorem of E.Ullmo and A.Yafaev show that $(S_n)_n$ is weakly homogeneous. Because the notion of weak homogeneity only depends on the image under $\pi : M \to S$, we conclude that the original sequence $(M_n)_n$ is weakly homogeneous. Thus the Zariski closure of $\bigcup_n M_n$ is weakly special, essentially reduced to the ergodic arguments.

(2) Since $\tau_n(w_n)$ is bounded, the sequence

$$\tau'_n(w_n) = c_N(\log D_n)^N \prod_{p\in\Delta_n} B|K^{\max}_{\mathbf{C}_n,p}/K_{\mathbf{C}_n}(w_n)_p|$$

where $\Delta_n = \Delta_{w_n}(\mathbf{C}_n)$, is bounded from above. As we have seen, only finitely many number fields occur as the splitting fields of the $\mathbb{Q}$-tori $\mathbf{C}_n$'s, and the union $\bigcup_n \Delta_n$ is bounded. We may thus assume that the $\mathbf{C}_n$'s have the common splitting field F, and we write $\Delta = \bigcup_n \Delta_n$ for the finite set of primes, of cardinality d.

We show that the sequence $(\tau'_n(w_n))_n$ is finite. As F is the common splitting field, the first factor $c_N(\log D_n)^N$ is fixed all through, while the second factor $\prod_{p\in\Delta} B|K^{\max}_{\mathbf{C}_n,p}/K_{\mathbf{C}_n}(w_n)_p|$ is a positive rational number varying in $\mathbb{Z}\cdot(B^d)$, because $\frac{1}{B}$ is known to be a positive integer. The upper bound for $\tau_n(w_n)$ implies that the $\tau'_n(w_n)$'s, and the second factors, are bounded from above. Hence the second factors are finite, because they are a priori bounded from below by positivity, and are discrete in the set $\mathbb{Z}(B)^d$.

This leads to the finiteness of $(\tau'_n(w_n))_n$, hence that of $(\tau_n(w_n))_n$, because the later also lies in $\mathbb{Z}(B)^d$. ■

Remark 4.2.4. Note that from the uniform bound of $\deg_{\pi^*\mathscr{L}}(\mathrm{Gal}_E M_n)$ we have not deduced the finiteness of $\{w_n \mod K_W : n \in \mathbb{N}\} \subset \mathbf{W}(\mathbb{A}^f)/K_W$. For example, it might happens that the $\mathbb{Q}$-torus $\mathbf{C}_n$ happens to fix w_n via the action $\mathbf{C}_n \hookrightarrow \mathbf{G} \to \mathbf{Aut}_{\mathbb{Q}}(\mathbf{W})$. It seems hopeful to establish such results under additional conditions on the action of $\mathbf{G}$ on $\mathbf{W}$ and the distribution of w_n in $\mathbf{W}(\mathbb{A}^f)$. However we are not yet ready for a detailed investigation in this direction, besides, it is not yet clear how to characterize the contribution of the relatively ample line bundle $\mathcal{T}$ to the degrees. Much remains to be done to reach an exact formulation so as to adapt the approach of B.Klingler and A.Yafaev into the mixed case.

4.3 The case of mixed special subvarieties

We have been talking about the test invariant of a pure special subvariety in a given mixed Shimura variety M. We proceed to adapt this notion to general special subvarieties, motivated by the approach to Manin-Mumford conjecture presented in [RU].

Recall that the Manin-Mumford conjecture studies the Zariski closure of a sequence of torsion subvarieties $T_n = a_n + A_n$ inside a given abelian variety A (of characteristic zero), where A_n are abelian subvarieties of A and a_n are torsion points. If the torsion orders of the a_n's are bounded when n varies, namely in the case where $\{a_n\}_n$ is finite, then we are reduced to the case where $T_n = A_n$, and a little harmonic analysis on $A(\mathbb{C})_{an}$ shows that the archimedean closure of $\bigcup_n A_n(\mathbb{C})_{an}$ is a closed complex subgroup of $A(\mathbb{C})_{an}$ which underlies some abelian subvariety, and thus the archimedean/Zariski closure of $\bigcup_n T_n$ is a finite union of torsion subvarieties.

We see that the equidistribution of a sequence of torsion subvarieties is immediate as long as the "minimal torsion orders" of these varieties are uniformly bounded. Rather, we could define the test invariant of a torsion subvariety T in a given abelian variety A to be the minimal torsion order of a, a running over the torsion points of A such that T can be written in the form $T = a + A'$ for some abelian subvariety $A' \subset A$. Then the above paragraph reads: if $(T_n)_n$ is a sequence of torsion subvarieties whose test invariant is uniformly bounded, then the closure of $\bigcup_n T_n$ is a finite union of torsion subvarieties.

Inspired by this phenomenon, we put the following:

Definition 4.3.1. Let $M \to S$ be a mixed Shimura variety fibred over a pure section, defined by data $(\mathbf{P}, Y) \to (\mathbf{G}, X)$ with compact open subgroups $K = K_V \rtimes K_G = \prod_p K_p$, and $\mathscr{L}$ the Baily-Borel line bundle on S. For a special subvariety M' in M, we define the test invariant of M' to be

$$\tau(M') = \inf\{\tau(S') : S' \text{ being any maximal pure special subvariety of } M'\}$$

where the phrase maximal pure special subvariety can be replaced by special sections of M'.

Proposition 4.3.2. *Let $(M_n)_n$ be a sequence of special subvarieties of M such that the sequence of test invariants $(\tau(M_n))_n$ is boudnded when n varies. Then $(M_n)_n$ is weakly homogeneous, namely there exists finitely many $\mathbb{Q}$-tori $\mathbf{C}_i$'s ($i = 1, \ldots, m$) such that each M_n is $\mathbf{C}_i$-special for some i. Consequently, the Zariski closure of $\bigcup_n M_n$ is a finite union of $\mathbf{C}_i$-special subvarieties.*

Proof. As is assumed, there exists a constant C such that for any n, $\inf_{S_n} \tau(S_n) < C$, S_n running through the maximal pure special subvarieties inside M_n. In particular we get a sequence of special subvarieties (S_n) (with $S_n \subset M_n$) such that $\tau(S_n) \leq C$ for all n. By Proposition 4.2.3, we see that the sequence $(S_n)_n$ is weakly homogeneous, i.e. there exists finitely many $\mathbb{Q}$-tori $\mathbf{C}_i$'s $(i = 1, \ldots, m)$ in $\mathbf{G}$ such that each S_n is $\mathbf{C}_i$-special for some i.

But S_n is a maximal pure special subvariety of M_n and $\pi(M_n) = \pi(S_n)$, π being the canonical projection $M \to S$ defined by $\mathbf{P} \to \mathbf{G}$. In particular, M_n is also $\mathbf{C}_i$-special just as S_n is. We deduce that $(M_n)_n$ is weakly homogeneous itself, and the closure of $\bigcup_n M_n$ is weakly special. ■

Chapter 5

Further perspectives

In this last part we discuss generalizations of the Manin-Mumford conjecture and their relations with the André-Oort conjecture.

5.1 Motivation

From the results in Chapter 3 we immediately get the following

Theorem 5.1.1 (cf. Theorem 3.5.3, Corollary 3.5.4). *Let* $(\mathbf{P}, \mathrm{Y})$ *be a mixed Shimura datum, with a pure section* $(\mathbf{G}, \mathrm{X})$. *Denote by* $\mathbf{C}$ *the connected center of* $\mathbf{G}$. *Write* $\mathrm{Y}_{\mathbb{R}}^{+}$ *for a fixed connected component of the real part of* Y, *and consider* $\mathcal{M} = \Gamma\backslash \mathrm{Y}_{\mathbb{R}}^{+}$ *a connected S-space associated to some torsion free arithmetic subgroup* $\Gamma \subset \mathbf{P}(\mathbb{R})^{+}$, *and* $\mathrm{M} = \Gamma\backslash \mathrm{Y}^{+}$ *the corresponding connected mixed Shimura variety.*

Let $(\mathbf{P}_n, \mathrm{Y}_n)_n$ *be a sequence of* $\mathbf{C}$*-special subdata,* $\mathcal{M}_n = \Gamma\backslash\Gamma \mathrm{Y}_n^{+}$ *resp.* $\mathrm{M}_n = \Gamma\backslash\Gamma \mathrm{Y}_n^{+}$ *the corresponding* $\mathbf{C}$*-special S-subspaces resp.* $\mathbf{C}$*-special subvarieties.*

(1) The archimedean closure of $\bigcup_n \mathcal{M}_n$ *is a finite union of* $\mathbf{C}$*-special subspaces. Moreover, if we denote by* μ_n *the canonical probability measure on* $\mathcal{M}$ *associated to* $\mathcal{M}_n$, *then* (μ_n) *always admits a convergent subsequence. If we assume further that* $(\mathcal{M}_n)_n$ *is strict, i.e.* $\mathcal{M}_n \not\subset \mathcal{M}'$ *for n large enough,* $\mathcal{M}' \subsetneq \mathcal{M}$ *being an arbitrary special S-subspace, then* $(\mu_n)_n$ *converges to the canonical probability measure* μ *on* $\mathcal{M}$.

(2) The Zariski closure of $\bigcup_n \mathrm{M}_n$ *is a finite union of* $\mathbf{C}$*-special subvarieties.*

We remark that the $\mathbf{C}$-special subvarieties are understood to be of positive dimensions.

Proof. From 3.5.3 and 3.5.4, it suffices to treat the case where $(\mathcal{M}_n)_n$ is strict. If $(\mu_n)_n$ does not converge to μ, then by the compactness of $\mathcal{H}_{\mathbf{C}}(\mathcal{M})$ (as is in 3.5.3), there is some convergent subsequence $(\mu_{n_m})_m$ whose limit is $\mu' \neq \mu$, with support $\mathcal{M}' \subsetneq \mathcal{M}$. In particular we have $\mathrm{Supp}\,\mu_{n_m} \subset \mathcal{M}'$ for m large enough, contradicting the assumption that (μ_n) is strict. Thus (μ_n) converges to μ. ∎

We then apply the theorem to special sections of a mixed Shimura variety of Kuga type. Recall that a mixed Shimura datum of Kuga type is of the form $(\mathbf{P},\mathrm{Y}) = (\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes \mathrm{X})$ with $(\mathbf{G},\mathrm{X})$ a pure section of connected component $\mathbf{C}$, such that the weight -2 unipotent part of $\mathbf{P}$ is trivial. Note that in this case the S-space $\mathcal{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$ is exactly the real analytic space underlying the complex locus of the mixed Shimura variety $\mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$. In particular, if $(\mathrm{M}_n)_n$ is a sequence of $\mathbf{C}$-special subvarieties in $\mathrm{M}_{\mathrm{K}}(\mathbf{P},\mathrm{Y})$, then the archimedean closure of $\bigcup_n \mathrm{M}_n(\mathbf{C})_{\mathrm{an}}$ is a finite union of the complex loci of $\mathbf{C}$-special subvarieties. It is understood here that S, hence each $\mathrm{M}(v_n)$, is of positive dimension.

We work with a connected mixed Shimura variety of Kuga type whose complex locus is of the form $\mathrm{M} = \Gamma\backslash\mathrm{Y}^+$, where $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$ for some (torsion free) arithmetic subgroups $\Gamma_{\mathbf{V}} \subset \mathbf{V}(\mathbb{Q})$ and $\Gamma_{\mathbf{G}} \subset \mathbf{G}(\mathbb{Q})_+$. Then the canonical morphism induced by the reduction modulo $\mathbf{V}$ $\pi : \mathrm{M} \to \mathrm{S} = \Gamma_{\mathbf{G}}\backslash\mathrm{X}^+$ is naturally an abelian S-scheme. Take $(v_n)_n$ a sequence in $\mathbf{V}(\mathbb{Q})$, we have $\mathbf{C}$-special subdata $(v_n\mathbf{G}v_n^{-1}, v_n \rtimes \mathrm{X})$, and the corresponding special sections $\mathrm{M}(v_n) = \Gamma\backslash\Gamma(v_n \rtimes \mathrm{X}^+)$. All of them are $\mathbf{C}$-special, and by the theorem the Zariski closure of $\bigcup_n \mathrm{M}(v_n)$ is a finite union of $\mathbf{C}$-special subvarieties. Let M' be one of the component, then the Mumford-Tate group $\mathbf{P}'$ of M' contains some $v_n\mathbf{G}v_n^{-1}$, and the reduction modulo $\mathbf{V}$ gives $\pi(\mathbf{P}') = \mathbf{G}$. That means $\mathbf{P}' = \mathbf{V}' \rtimes (v'\mathbf{G}v'^{-1})$ for some ($\mathbf{G}$-stable) $\mathbb{Q}$-vector subspace $\mathbf{V}' \subset \mathbf{V}$ and $v' \in \mathbf{V}(\mathbb{Q})$, and M' can be viewed as the abelian S-subscheme $\Gamma\backslash\Gamma\mathbf{V}'(\mathbb{R}) \rtimes \mathrm{X}$ "translated by" a special section $\mathrm{M}(v')$, which is analogue to the classical Manin-Mumford conjecture. Here we have abused the term "translated by": actually special sections are not sections, and we can neither add them nor translate by them.

We summarize the above discussion as the following:

Corollary 5.1.2. *Let* $\mathrm{M} = \Gamma\backslash\mathrm{Y}^+$ *be a connected mixed Shimura variety of Kuga type defined by some datum* $(\mathbf{P},\mathrm{Y}) = (\mathbf{V} \rtimes \mathbf{G}, \mathbf{V}(\mathbb{R}) \rtimes \mathrm{X})$, *with* $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$. *Then* $\pi : \mathrm{M} \to \mathrm{S} = \Gamma_{\mathbf{G}}\backslash\mathrm{X}^+$ *is an abelian* S*-scheme. Moreover if* $(\mathrm{M}(v_n))_n$ *is a sequence of special sections of* π *indexed by* $v_n \in \mathbf{V}(\mathbb{Q})$, *then the Zariski closure of* $\bigcup_n \mathrm{M}(v_n)$ *is a finite union of abelian* S*-subschemes "translated by" special sections, namely a finite union of special subvarieties of the form* $\Gamma\backslash\Gamma\mathrm{Y}'^+$, *where* Y' *comes from some subdatum of the form* $(\mathbf{P}',\mathrm{Y}') = (\mathbf{V}' \rtimes (v'\mathbf{G}v'^{-1}), (\mathbf{V}'(\mathbb{R}) + v') \rtimes \mathrm{X}^+)$.

And of course we can refine the Zariski closure by archimedean closure (of the complex loci).

In short we have shown a relative version of the Manin-Mumford conjecture: the Zariski closure of a sequence of (special) torsion sections is a finite union of abelian subschemes "translated by" special sections. In the following we establish an algebraic version of the above corollary, without restriction to the framework of mixed Shimura varieties. The main idea is to extend the Manin-Mumford conjecture over the generic fiber to the case of abelian schemes.

5.2 Prerequisites on abelian schemes

We collect here some standard materials on abelian schemes.

Definition 5.2.1. (cf. [GIT] Chap.6, Definition 6.1) (1) Over a base scheme S, an abelian S-scheme is a group S-scheme $f : A \to S$ which is proper, smooth, of connected geometric fibers. We write $e_S : S \to A$ for the neutral section, $i_S : A \to A$ for the inverse map, and $m_S : A \times_S A \to A$ for the multiplication.

Consequent to the rigidity lemma (cf. [GIT] Chap.6, Prop.6.1, Cor.6.5, Cor.6.6), an abelian S-scheme is a commutative S-group, and the S-group law is unique with respect to the neutral section. We thus always write the group law additively.

We assume for simplicity that an abelian S-scheme is of some fixed relative dimension $g > 0$.

(2) A homomorphism between abelian S-schemes $\psi : A_1 \to A_2$ is an isogeny if it is an epimorphism of S-group with finite kernel.

(3) The endomorphism algebra of an abelian S-scheme $f : A \to S$ is the set of homomorphism of S-groups $A \to A$, denoted as $\mathrm{End}_S(A)$. The isogeneous-endomorphism algebra is the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_S(A)$, denoted as $\mathrm{End}^\circ_S(A)$.

The endomorphism sheaf of an abelian S-scheme $A \to S$ is the étale sheaf $(U \to S) \mapsto \mathrm{End}_U(A_U)$, denoted as $\mathbf{End}_S(A)$. The isogeneous-endomorphism sheaf of $A \to S$ is the étale sheaf $(U \to S) \mapsto \mathrm{End}^\circ_U(A_U)$, denoted as $\mathbf{End}^\circ_S(A)$.

Definition-Proposition 5.2.2 (Torsion sections)**.** Fix $f : A \to S$ an abelian S-scheme of relative dimension $g > 0$.

(1) Let N be a positive integer, and $[N] : A \to A$ is raising to the N-th power $a \mapsto Na$ (written additively). Then $A[N] := \mathrm{Ker}[N]$ is a finite flat S-group.

A section $\alpha : S \to A$ is a torsion section of $f : A \to S$ if it is killed by some $N > 0$. Locally for the *fppf* topology, A[N] splits as copies of disjoint torsion sections of $A \to S$.

If moreover N is invertible over S, then A[N] is étale over S, locally free of rank N^{2g}.

(2) Let p be a rational prime. The p^∞-torsion subgroup of $f : A \to S$ is the union $\bigcup_n A[p^n]$. The integral Tate module of f at p is the $\mathbb{Z}_{p,S}$-module $\varprojlim_n A[p^n]$, denoted as $\mathbb{T}_p(A)$. The Tate module of f at p is $\mathbb{T}^\circ_p(A) := \mathbb{Q}_{p,S} \otimes_{\mathbb{Z}_{p,S}} \mathbb{T}_p(A)$.

Note that when p is invertible over S, $\mathbb{T}_p(A)$ is étale locally isomorphic to $\mathbb{Z}^{2g}_{p,S}$.

(3) The total torsion subgroup of f is the union $\bigcup_{N>0} A[N]$. The integral Tate module of f is the $\hat{\mathbb{Z}}_S$-module $\mathbb{T}(A) := \varprojlim_N A[N]$, and the adelic Tate module of f is $\mathbb{T}^\circ(A) := \mathbb{A}^f{}_S \otimes_{\hat{\mathbb{Z}}_S} \mathbb{T}(A)$.

If S is of characteristic zero, then every rational prime p is invertible over S, hence $\mathbb{T}(A)$ is étale locally isomorphic to $\hat{\mathbb{Z}}^{2g}_S$.

Remark 5.2.3. When N is not invertible, A[N] is in general not étale, and it might fail to be of order N^{2g}: further information is required such as the characteristic of the base S, the Newton polygon of the corresponding p-divisible group, etc. to study the p-divisible group $\varinjlim_n A[p^n]$ in detail, p being any rational prime that is not invertible on S.

Definition-Proposition 5.2.4 (The monodromy representation). We fix $f : A \to S$ an abelian S-scheme of relative dimension g, and $\bar{x} : \mathrm{Spec}(\bar{k}) \to S$ a geometric point of S. $\pi_1(S, \bar{x})$ denotes the (étale) fundamental group of S at $\bar{x}$.

Let $N \in \mathbb{N}$ be invertible over S. Then $f : A[N] \to S$ is a finite étale S-group. The action of $\pi_1(S, \bar{x})$ on $A[N]_{\bar{x}}$ i.e. the fiber of the covering, is continuous and it preserves the group law. As a result we get a continuous representation $\mathbf{mon}_N(f, \bar{x}) : \pi_1(S, \bar{x}) \to GL_{\mathbb{Z}/N}(A[N]_{\bar{x}})$ of the profinite group $\pi_1(S, \bar{x})$ on $A[N]_{\bar{x}}$.

If p is a rational prime invertible over S, then the inverse limit of $(\mathbf{mon}_{p^n}(f, \bar{x}))_n$ gives a continuous (integral) p-adic representation $\mathbf{mon}_{\mathbb{Z}_p}(f, \bar{x}) : \pi_1(S, \bar{x}) \to GL_{\mathbb{Z}_p}(\mathbb{T}_p(A)_{\bar{x}})$. If moreover S is of characteristic zero, then the inverse limit of $(\mathbf{mon}_N(f, \bar{x}))_N$ gives the total monodromy representation $\mathbf{mon}_{\hat{\mathbb{Z}}}(f, \bar{x}) : \pi_1(S, \bar{x}) \to GL_{\hat{\mathbb{Z}}}(\mathbb{T}(A)_{\bar{x}})$

From now on we assume that S is of characteristic zero, so as to deduce some properties for our studies of (mixed) Shimura varieties. (In this writing we won't be concerned with the reduction behavior of a mixed Shimura varieties at a finite prime, and no integral model is needed.) Moreover when speaking of an abelian S-scheme we assume that S is geometrically connected and that its fibers are of common dimension g, and $\bar{x}$ is assumed to be the algebraic closure of the generic point η of S.

One of the advantages of these assumptions is that the torsion S-groups A[N] can be viewed as finite torsion (abelian) sheaves on $S_{\text{ét}}$, and they are equivalently characterized by the corresponding finite discrete $\pi_1(S, \bar{x})$-module, namely the corresponding monodromy representations. The sheaf is constant if and only if the corresponding $\pi_1(S, \bar{x})$-module is trivial: in our situation A[N] being constant is equivalent to the fact that A[N] splits into N^{2g}-sections of $A \to S$, which is also equivalent to the triviality of the monodromy action $\mathbf{mon}_N(f, \bar{x})$.

The main tools in our studies of characteristic zero consists of

(a)(cf. [EGA IV3], Sect.8) the reduction techniques for a finitely presented morphism over a general base, as is presented in loc.cit, so that the studies of an abelian S-scheme $A \to S$ is often reduced to a model $A_0 \to S_0$, namely $A \cong S \times_{S_0} A_0$ with S_0 a finite type scheme over $\mathbb{Q}$.

(b)(cf. [SGA 4] Tome 3, Exposé XI) the GAGA principle: for X a proper $\mathbb{C}$-scheme, the canonical functor $\theta : X_{\text{an}}^{\sim} \to X_{\text{ét}}^{\sim}$ is an equivalence of topoi, where X_{an} is the small site of the complex analytic space associated to $X(\mathbb{C})$, $X_{\text{ét}}$ the small étale site of X, and the superscript $\sim$ stands for topos. This generalizes the classical GAGA à la Serre, which an equivalence between the categories of coherent sheaves on X_{an} and X_{Zar} respectively.

We start with

Lemma 5.2.5. *Let $f : A \to S$ be an abelian S-scheme of relative dimension g. Let $(N_n)_n$ be a sequence of integers that tends to infinity as $n \to \infty$. Then the union $\bigcup_n A[N_n]$ is Zariski dense in* A.

Proof. The question is purely topological, and we may assume that S is geometrically integral.

Step 0: the case where $S = \operatorname{Spec}\mathbb{C}$.

This is an easy consequence of the complex uniformization of an abelian variety $f : A \to \operatorname{Spec}\mathbb{C}$, i.e. A_{an} is isomrphic to the quotient of the tangent space at the origin $\operatorname{Tan}_0 A$ by the fundamental group $\pi_1(A_{an})$ which is identified with a lattice in $\operatorname{Tan}_0 A$. Moreover we can show that the averaged Dirac measure on $A[N_n]$ converges to the Haar measure on the compact complex Lie group A_{an}, which is among the first examples of equidistribution of special points.

Step 1: the case where $S = \operatorname{Spec} F$ for some field F of characteristic zero.

By our reduction (a), we may assume that F is of finite transcendental degree over $\mathbb{Q}$, hence we are reduced to the case that F is embedded in $\mathbb{C}$. The torsion points of A are all defined over $F^{ac} \subset \mathbb{C}$, and the archimedean density in $A_{\mathbb{C},an}$ implies the Zariski density in $A_{F^{ac}}$, hence the Zariski density in A.

Step 2: the case where S is geometrically integral.

Write $\eta = \operatorname{Spec} F$ for the generic point of S. Then $A[N_n]_\eta$ is Zariski dense in $A[N_n]$, and the Zariski $\bigcup_n A[N_n]_\eta$ is Zariski dense in $\bigcup_n [A_n]$. By Step 1 we see that $\bigcup_n A[N_n]_\eta$ is Zariski dense in A_η, therefore A_η is contained in $X = \overline{\bigcup_n A[N_n]}^{\mathrm{Zar}}$. Now that A_η is dense open in A because of the density of η in S, we have $X = A$.
■

5.3 Generalized Manin-Mumford conjecture: the uniform case

The Manin-Mumford conjecture was originally raised for abelian varieties over $\mathbb{C}$. The generalization to an arbitrary base of characteristic zero is immediate after the preliminaries in the last section.

We work with abelian S-schemes $f : A \to S$ over a base S of characteristic zero. Since we are mainly concerned with the Zariski closures (in the underlying topological space), we may assume that S is geometrically integral. Let $\bar{x}$ be a geometric point lying above the generic point η of S (e.g. the algebraic closure of η).

In characteristic zero we have the total monodromy representation

$$\mathbf{mon}_{\hat{\mathbb{Z}}}(f,\bar{x}) : \pi_1(S,\bar{x}) \to \mathrm{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}(A)_{\bar{x}}).$$

The kernel of $\mathbf{mon}_{\hat{\mathbb{Z}}}(f,\bar{x})$ is an invariant subgroup of $\pi_1(S,\bar{x})$, which corresponds to a connected étale covering $\hat{S} \to S$. Write $\hat{f} : \hat{A} \to \hat{S}$ for the base change $\hat{S} \times_S f$, $\mathrm{pr} : \hat{S} \to S$ the covering map, and $\hat{x}$ a lifting of $\bar{x}$ into $\hat{S}$.

It is clear that $\pi_1(\hat{S},\hat{x})$ acts on each $\hat{A}[N]_{\hat{x}}$ trivially, and that $\hat{A}[N]$ splits into N^{2g} copies of torsion sections, g being the relative dimension of $A \to S$. In fact it suffices to consider the commutative diagram, each square of whom is cartesian:

$$\begin{tikzcd}
\hat{A}[N] \arrow[r, "pr'"] \arrow[d, "\cap"'] & A_N[N] \arrow[r, "pr_N"] \arrow[d, "\cap"'] & A[N] \arrow[d, "\cap"] \\
\hat{A} \arrow[r, "pr'"] \arrow[d, "\hat{f}"'] & A_N \arrow[r, "pr_N"] \arrow[d, "f_N"'] & A \arrow[d, "f"] \\
\hat{S} \arrow[r, "pr'"] & S_N \arrow[r, "pr_N"] & S
\end{tikzcd}$$

where $pr_N : S_N \to S$ is the Galois covering corresponding to the kernel of $\mathbf{mon}_N(f,\bar{x}) : \pi_1(S,\bar{x}) \to GL_{(Z/N)}(A[N]_{\bar{x}})$, and $f_N : A_N \to S_N$ is the base change of f by pr_N. Because $S_N \to S$ is an Galois covering, S being geometrically connected, we deduce that S_N is also geometrically connected. Let $\bar{x}_N$ be a geometric point of S_N lifting $\bar{x}$, then $\pi_1(S_N,\bar{x}_N)$ is identified with $\mathrm{Ker}\,\mathbf{mon}_N(f,\bar{x})$ canonically. From the commutative diagram

$$\begin{tikzcd}
\pi_1(S,\bar{x}) \arrow[r, "="'] \arrow[d, "\mathbf{mon}_{\hat{Z}}(f{,}\bar{x})"'] & \pi_1(S,\bar{x}) \arrow[d, "\mathbf{mon}_N(f{,}\bar{x})"] \\
\mathbf{GL}_{\hat{Z}}(\mathbb{T}(A)_{\bar{x}}) \arrow[r, "\mathrm{mod}\,N"] & \mathbf{GL}_{Z/N}(A[N]_{\bar{x}})
\end{tikzcd}$$

we get a canonical epimorphism $\mathrm{Ker}(\mathbf{mon}_{\hat{Z}}(f,\bar{x})) \twoheadrightarrow \mathrm{Ker}(\mathbf{mon}_N(f,\bar{x}))$, which means $\hat{S} \to S$ factors through $pr' : \hat{S} \to S_N$. Consequently for any lifting $\hat{x}$ of $\bar{x}$ in $\hat{S}$, the action of $\pi(\hat{S},\hat{x})$ on $\hat{A}[N]_{\hat{x}}$ is trivial, and $\hat{A}[N]_{\hat{x}}$ is a disjoint union of N^{2g} copies of $\hat{S}$, just as the case for $A_N \to S_N$.

It is also clear from the construction that $\hat{S} \cong \varprojlim_N S_N$, where S'_N is the same as above, namely the Galois covering corresponding to $\mathrm{Ker}\,\mathbf{mon}_N(f,\bar{x})$. We have seen that each S_N is geometrically integral, and so it is with $\hat{S}$.

Definition 5.3.1. Let $f : A \to S$ be as above.

(1) A special section of A is the image of the composition $\hat{S} \xrightarrow{\hat{t}} \hat{A} \xrightarrow{pr} A$ for some torsion section $\hat{t}$ of $\hat{f}$. Note that we regard it as a closed S-subscheme of A. Write $S(\hat{t})$ for the image in A, then $pr^{-1}(S(\hat{t})) \subset \hat{A}$ is stable under the covering group $\mathrm{Aut}_S(\hat{S})(\cong \pi_1(S,\bar{x}))$.

(2) A special S-subscheme of f is the image of the composition $\hat{B} \hookrightarrow \hat{A} \xrightarrow{pr} A$ where $\hat{B} \subset \hat{A}$ is an $\hat{S}$-subscheme of the form $\hat{t}_1 + \hat{A}_1$, where $\hat{A}_1$ is some abelian $\hat{S}$-subscheme of $\hat{A}$, and $\hat{t}_1+$ means translation in $\hat{A}$ by a torsion section $\hat{t}_1$ of $\hat{f}$. Write B for the image in A of $\hat{B}$ under pr, then, similar to (1), we have $pr^{-1}(B)$ is stable under the covering group $\mathrm{Aut}_S(\hat{S})$.

(3) A weakly special S-subscheme of A is defined to be a finite union of special S-subschemes.

Remark 5.3.2. Although we define special sections via the image from the base change $\hat{A} \to \hat{S}$ by the universal covering, it is easy to see that we could have worked with some finite covering. In fact, consider $T \subset A$ a special section defined by a section $\hat{t}$ of $\hat{f} : \hat{A} \to \hat{S}$: let N be the torsion order of $\hat{t}$, then $\hat{t}$ is fixed by the open subgroup $\mathrm{Ker}\,\mathbf{mon}_N(f, \bar{x}) \subset \pi_1(S, \bar{x})$, and thus by faithfully flat descent $\hat{t}$ is defined over $S' = S'_N$ the Galois covering corresponding to $\mathrm{Ker}\,\mathbf{mon}_N(f, \bar{x})$, namely for some section t' of $f' : A' = S' \times_S A \to S'$ contained in $A'[N]$ we have $\hat{t} = \mathrm{pr}'^{*} t'$, where pr' is the canonical projection $\hat{S} \to S'$. Write $\mathrm{Gal}(S'/S)$ for the covering group of S' over S, then the $\mathrm{Gal}(S'/S)$-orbit of t' is $\pi_1(S, \bar{x})$-invariant, hence by descent theory it equals $S' \times_S T_N$ for some S-subscheme $T_N \subset S$. It is easy to show that T_N is just the special section T defined by $\hat{t}$, or equivelently, by t'. In particular, $T \to S$ is a finite étale covering.

We start with the case where the monodromy representation is trivial, i.e. all the torsion sections already exist over S.

Lemma 5.3.3. *Let $f : A \to S$ be an abelian S-scheme such that $S = \hat{S}$, i.e. the fundamental group acts trivially on every torsion S-subgroup of A. Write η for the generic point of S. For an S-subscheme $T \subset A$, we have its specialization at η: $\mathbf{sp}_\eta(T) := \eta \times_S T \subset A_\eta$.*

(1) The map $\mathbf{sp}_\eta : t \mapsto t_\eta$ establishes a bijection from $\mathrm{Tor}(A)$ to $\mathrm{Tor}(A_\eta)$, where $\mathrm{Tor}(A)$ is the set of torsion sections of $f : A \to S$, and $\mathrm{Tor}(A_\eta)$ the set of torsion sections (torsion η-points) of A_η. Moreover it preserves the group structure, hence an isomorphism of abelian groups, where $\mathrm{Tor}(A) \subset A(S)$ is endowed with the canonical group structure.

(2) Put $\mathscr{F}(A)$ to be the set of abelian S-subschemes of A, and $\mathscr{F}(A_\eta)$ the set of abelian subvarieties of A_η. Then $\mathbf{sp}_\eta : B \mapsto B_\eta$ establishes a bijection from $\mathscr{F}(A)$ to $\mathscr{F}(A_\eta)$.

(3) The étale sheaves $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$ and $\mathbf{End}^{\circ}_{\mathbb{A}^f_S}(\mathbb{T}^{\circ}(A))$, are constant sheaves on $S_{\acute{e}t}$. Consequently the specialization induces the following isomorphisms, which preserve the corresponding algebraic structures (modules, rings, etc.):

(3-1) Tate modules: $\mathbb{T}(A) \xrightarrow{\mathbf{sp}_\eta} \mathbb{T}(A_\eta)$ and $\mathbb{T}^{\circ}(A) \xrightarrow{\mathbf{sp}_\eta} \mathbb{T}^{\circ}(A_\eta)$;

(3-2) endomorphism of Tate modules: $\mathrm{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A)) \xrightarrow{\mathbf{sp}_\eta} \mathrm{End}_{\hat{\mathbb{Z}}}(\mathbb{T}(A_\eta))$ and $\mathrm{End}^{\circ}_{\mathbb{A}^f_S}(\mathbb{T}^{\circ}(A)) \xrightarrow{\mathbf{sp}_\eta} \mathrm{End}^{\circ}_{\mathbb{A}^f}(\mathbb{T}^{\circ}(A_\eta))$;

Proof. It is clear that the $\mathbf{sp}_\eta$ in (1) and in (2) are both injective, and that they both preserve the group structures. We then verify the surjectivity.

(1) Note that we have assume $S = \hat{S}$. Thus for every $0 < N \in \mathbb{N}$, $A[N]$ splits into a disjoint union of N^{2g} sections of $f : A \to S$. Specialized at η they become N^{2g} sections of $f_\eta : A_\eta \to \eta$. Note that the rank $A_\eta[N]$ over η equals N^{2g}, thus the injectivity of $\mathbf{sp}_\eta$ implies sujectivity.

Because N is arbitrary, we get the bijectivity of $\mathbf{sp}_\eta : \mathrm{Tor}(A) \to \mathrm{Tor}(A_\eta)$.

(2) Let $B' \in \mathscr{F}(A_\eta)$ be an abelian subvariety. We want to find some $A' \in \mathscr{F}(A)$ such that $A'_\eta = B'$.

Consider $\mathrm{Tor}(B') \subset \mathrm{Tor}(A_\eta)$. Set $T = \mathbf{sp}_\eta^{-1}(\mathrm{Tor}(B')) \subset \mathrm{Tor}(A)$, and put A' to be the Zariski (schematic) closure of T in A. This is a closed S-subscheme of A.

$\mathrm{Tor}(B')$ is stable under the group operations: neutral section, inverse, multiplication. Now that $\mathbf{sp}_\eta$ preserves the group structure, we see that $T \subset \mathrm{Tor}(A)$ is stable under the group operations. Taking Zariski closure we conclude that A' is a closed S-subgroup of $f : A \to S$.

We then show that the composition $f' : A' \hookrightarrow A \xrightarrow{f} S$ is smooth. It suffices to show that $f' : A' \to S$ is flat and smooth along each fibers.

The smoothness along fibers is immediate. In fact, let s be a geometric point of S, and the specialization at s gives $\mathrm{Tor}(A) \xrightarrow{\mathbf{sp}_s} \mathrm{Tor}(A_s)$ and $\mathscr{F}(A) \xrightarrow{\mathbf{sp}_s} \mathscr{F}(A_s)$. The triviality of the action of $\pi_1(S, s)$ shows that $\mathbf{sp}_s : \mathrm{Tor}(A) \to \mathrm{Tor}(A_s)$ is an isomorphism of abelian groups. Thus
$\mathbf{sp}_s(\mathbf{sp}_\eta^{-1}(\mathrm{Tor}(B')) \subset \mathrm{Tor}(A_s)$ is a subgroup, whose Zariske closure equals $A'_s = (\mathbf{sp}_\eta^{-1}(B'))_s = (\overline{\mathbf{sp}_\eta^{-1}\mathrm{Tor}(B')}^{\mathrm{Zar}})_s$. Now that an algebraic group over a field of characteristic zero is automatically smooth. Apply this fact to A'_s over s, we wee that $f' : A' \to S$ is smooth at s, s being an arbitrary geometric point of S.

It remains to show the flatness. We may apply the reduction (a) and reduce to the case where S is a finite type k-scheme for some field k of characteristic zero. Consider the commutative diagram

$$\begin{tikzcd} A' \arrow[rr, "f'"] \arrow[dr] & & S \arrow[dl] \\ & \mathrm{Spec}\, k & \end{tikzcd}$$

where A' and S are finite type over $\mathrm{Spec}\, k$. Recall the flatness criterion along the fibers: firstly A' and S are both flat over $\mathrm{Spec}\, k$; secondly, for any point $s \in S$, the fiber $A'_s \to s$ is flat because this is again a scheme over a field. We conclude that $A' \to S$ is flat, hence smooth because of the fiberwise smoothness.

(3) First consider the étale sheaf $U \mapsto A_U[N] = A[N]_U$ on $S_{\text{ét}}$. Note that $S = \hat{S}$ and $A[N]$ splits into N^{2g} disjoint copies of sections of $f : A \to S$. Thus for any étale morphism $U \to S$, $A_U[N]$ is a disjoint union of N^{2g} copies of sections of f_U, and $A[N]$ is therefore a constant sheaf, which is also a constant $(\mathbb{Z}/N)_S$-modules, where $(\mathbb{Z}/N)_S$ is the constant sheaf of value $\mathbb{Z}/N$. Taking limit we get constant sheaf of $\hat{\mathbb{Z}}_S$-modules $\mathbb{T}(A)$ resp. $\mathbb{A}^f{}_S$-modules $\mathbb{T}^\circ(A)$, and the corresponding sheaves of endomorphisms, namely $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$ resp. $\mathbf{End}^\circ_{\mathbb{A}^f{}_S}(\mathbb{T}^\circ(A))$, are automatically constant.

For a constant sheaf F on the geometrically connected base S, the restriction maps are isomorphisms $F(S) \to F(U)$, where $U \to S$ is an arbitrary connected étale map. In particular, let U vary over the connected open subsets of S, the the inductive limit leads us to the specialization at the generic point $\mathbf{sp}_\eta : F(S) \cong F(\eta)$. Replace F by the constant sheaves $\mathbb{T}(A)$, $\mathbb{T}^\circ(A)$, etc. we get the required isomorphisms in (3-1), and (3-2). ■

Corollary 5.3.4. *For $f : \mathrm{A} \to \mathrm{S} = \hat{\mathrm{S}}$ as above, the specialization $\mathbf{sp}_\eta : \mathrm{B} \to \mathrm{B}_\eta$ establishes a bijection $\mathcal{S}(\mathrm{A}) \to \mathcal{S}(\mathrm{A}_\eta)$, where $\mathcal{S}(\mathrm{A})$ resp. $\mathcal{S}(\mathrm{A}_\eta)$ denotes the set of special* S*-subschemes of* A*, resp. special subvarieties of A_η. It is compatible with translation by torsion sections: $\mathbf{sp}_\eta(t + \mathrm{B}) = t_\eta + \mathrm{B}_\eta$.*

Proof. The proof is a combination of (1) and (2) of the lemma. ■

In characteristic zero, the Manin-Mumford can "extend" over a general base, namely

Proposition 5.3.5. *Let $f : \mathrm{A} \to \mathrm{S}$ be an abelian* S*-scheme, and $(\mathrm{B}_n)_n$ a sequence of special* S*-subschemes of* A*. Then the Zariski closure of $\bigcup_n \mathrm{B}_n$ in* A *is weakly special.*

Proof. By the reduction from the beginning, we may assume that S is geometrically integral. By the density of the set of special sections in a special S-subschemes, we may assume that all the B_n's are special sections of f.

(1) The case where $\mathrm{S} = \hat{\mathrm{S}}$:

In this case special sections are the same as torsion sections. Let $(t_n)_n$ be the given set of torsion sections, and X the Zariski closure of $\bigcup_n t_n$.

Let $\eta = \mathrm{Spec}\,\mathrm{F}$ be the generic point of S (geometrically integral). In Corollary 1.3.4 we have seen the bijection $\mathbf{sp}_\eta : \mathcal{S}(\mathrm{A}) \to \mathcal{S}(\mathrm{A}_\eta)$. Let A' be the Zariski closure of $\bigcup_n t_n$ in A. Then $\mathrm{A}'_\eta = \mathbf{sp}_\eta(\mathrm{A}')$ equals the Zariski closure of $\bigcup_n t_{n,\eta}$. Apply the Manin-Mumford conjecture for abelian varieties over field of characteristic zero (actually a theorem), we conclude that A'_η is a weakly special subvariety of A_η, i.e. a finite union of torsion subvarieties of A_η. Let $\mathrm{A}' = \bigcup_i \mathrm{A}'_i$ be the decomposition into finitely many irreducible S-subschemes, with finiteness followed from the finite presentation of A' over S, then $\bigcup_i \mathrm{A}'_{i,\eta}$ is a decomposition into irreducible subvarieties of A'_η. A' being weakly special implies that each $\mathrm{A}'_{i,\eta}$ is special, hence $\mathrm{A}'_i = \mathbf{sp}_\eta^{-1}(\mathrm{A}'_{i,\eta})$ is special, and thus A' is weakly special.

(2) The general case:

In general we put $\bar{x} = \bar{\eta} = \mathrm{Spec}\,\mathrm{F}^{\mathrm{ac}}$ for η the generic point of S, and consider the étale Galois covering $\mathrm{pr} : \hat{\mathrm{S}} \to \mathrm{S}$ and its pull-back $\mathrm{pr} : \hat{\mathrm{A}} \to \mathrm{A}$. Then $\hat{f} = \mathrm{pr}^* f : \hat{\mathrm{A}} \to \hat{\mathrm{S}}$ is an abelian $\hat{\mathrm{S}}$-scheme with all the torsion sections defined over $\hat{\mathrm{S}}$.

Let $(t_n)_n$ be a sequence of special sections of $f : \mathrm{A} \to \mathrm{S}$. We may assume that $t_n = \mathrm{pr}(\hat{t}_n)$ for some special section $\hat{t}_n$ of $\hat{f} : \hat{\mathrm{A}} \to \hat{\mathrm{S}}$. The Zariski closure of $\bigcup_n \hat{t}_n$ in $\hat{\mathrm{A}}$, denoted by $\hat{\mathrm{A}}'$ is weakly special, as we have seen in (1). Write $\hat{\mathrm{A}}' = \bigcup_i \hat{\mathrm{A}}'_i$ a finite union of special $\hat{\mathrm{S}}$-subschemes. pr is a covering, and in particular a closed map, thus $\mathrm{pr}(\hat{\mathrm{A}}') = \mathrm{A}'$ is closed and equals the Zariski closure of $\bigcup_n t_n$. It is clear that $\mathrm{A}' = \bigcup_i \mathrm{pr}(\hat{\mathrm{A}}'_i)$ is weakly special. ■

Remark 5.3.6. (1) Note that in (2) of the proposition above, $\mathrm{pr}^{-1}(\mathrm{pr}(\hat{t}_n)$ is stable under $\pi_1(\mathrm{S}, \bar{x})$ (with $\bar{x} = \bar{\eta}$). Say $\hat{t}_n$ is of order N, i.e. N is the minimal positive integer such that $\hat{t}_n$ is killed by N. Then the subgroup $\mathrm{K(N)} := \mathrm{Ker}(\pi_1(\mathrm{S}, \bar{x}) \to \mathrm{GL}_{(\mathbb{Z}/\mathrm{N})_\mathrm{S}}(\hat{\mathrm{A}}[\mathrm{N}])$ fixes $\hat{t}_n$, and $\mathrm{pr}^{-1}(\mathrm{pr}(\hat{t}_n)$ is the finite set of the orbit of $\hat{t}_n$ under

$\pi_1(S,\bar{x})/K(N)$. The orbit descends to an S-subscheme of A[N], which equals $\mathrm{pr}(\hat{t}_n)$.

This phenomenon is well understood in the case where $S = \mathrm{Spec}\,F$ for some number field: say the Zariski closure of $\bigcup_n t_n$ is defined over F, then it contains the Gal_F-orbit of the torsion points. And then the diophantine techniques follows, as is presented in [Hind] and in [RU].

(2) By exactly the same arguments, we can deduce the Manin-Mumford conjecture for S-torus and semi-abelian S-schemes by reduction to the generic fiber, under the assumption that S is of characteristic zero.

We have shown the constancy of the étale sheaves $\mathbb{T}(A)$ and $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$ in the case where $S = \hat{S}$. One might thus expect the following approach to work: the sheaf of endormorphisms $\mathbf{End}_S(A)$ $(U \mapsto \mathrm{End}_U(A_U))$ is a constant subsheaf of $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$, and the specialization at η gives a bijection between $\mathrm{End}_S(A)$ and $\mathrm{End}_\eta(A_\eta)$. An abelian subvariety B' of A_η can be realized as the kernel of some endomorphism ψ of A_η. We lift ψ to some endomorphism Ψ of A defined over S, then the kernel of Ψ should be the unique abelian S-subscheme which specializes to $B' = \mathrm{Ker}\,\psi$.

However, as is kindly pointed out by Prof.M.Raynaud, the subsheaf $\mathbf{End}_S(A)$ of $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$ is not necessarily constant, even when $S = \hat{S}$. He also communicated to us the following remedy due to A.Grothendieck:

Proposition 5.3.7. *(1) (cf. [G] Theoreme) Let* S *be a locally noetherian integral scheme over a field of characteristic zero,* A, B *two abelian* S*-schemes,* ℓ *a fixed rational prime, and* $u_\ell : \mathbb{T}_\ell(A) \to \mathbb{T}_\ell(B)$ *a homomorphism of Tate modules. If for some point* $s \in S$ *the homomorphism* u_{ℓ_s} *comes from some homomorphism of abelian* $k(s)$*-schemes* $u(s) : A(s) \to B(s)$, *then* u_ℓ *comes from some homomorphism of abelian* S*-schemes* $u : A \to B$. *Here a homomorphism of Tate module* $\mathbb{T}_\ell(A) \to \mathbb{T}_\ell(B)$ *is said to come from a homomorphism of abelian* S*-schemes* $A \to B$ *if it lies in the image of the canonical homomorphism* $\mathrm{Hom}_S(A,B) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{T}_\ell(A), \mathbb{T}_\ell(B))$.

(2)(cf. [G] Corollaire 4.2) Let S *be a connected locally noetherian normal scheme of characteristic zero,* U *an open subscheme of* S, A *an abelian* U*-scheme, and* ℓ *a prime number. Then* A *extends to an abelian* S*-scheme* $\mathcal{A}$ *if and only if* $\mathbb{T}_\ell(A)$ *is unramified over* S, *in the sense that for every* $n \in \mathbb{N}$, $A[\ell^n]$ *extends to an étale covering of* S.

The approach we've just mentioned would work if the base is assumed to be normal. Here we would like to work with the étale covering $\hat{S}$ instead of S. In this case S being normal is equivalent to $\hat{S}$ being normal, as can be verified through the following observation:

Lemma 5.3.8. *Let* A *be a normal integral domain, on which a finite group* G *acts by automorphisms. Then the subring of invariants* A^G *is normal.*

(The lemma can be extended to more general schemes, but this version suffices for us.)

Proof. Write F for the fraction field of A, and E the fraction field of the integral domain $A^G (\subset A)$. We need to show that A^G is integrally closed in E. Take $a \in E$ integral over A^G, then it satisfies a equation of the form $a^N = c_0 + c_1 a + \cdots + c_{N-1} a^{N-1}$, with c_i all coming from A^G. Then a is integral over A, hence lies in A and invariant under G, thus it falls into A^G. ■

Apply the lemma an arbitrary finite Galois étale covering $S' \to S$, we see that S is normal if and only if S' is normal.

Lemma 5.3.9. *Let $f : A \to S$ be an abelian S-scheme, with S geometrically integral of generic point η, such that $S = \hat{S}$. Assume further that S is normal. Then* $\mathbf{End}_S(A)$ *is a constant subsheaf of* $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$.

Proof. It is clear that $\mathbf{End}_S(A)$ is a subsheaf of $\mathbf{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$: for any (étale) S-scheme U, $\mathrm{End}_U(A_U)$ is naturally embedded in $\mathrm{End}_{\hat{\mathbb{Z}}_U}(\mathbb{T}(A_U))$ because an endomorphism is determined by its restriction on the set of torsion sections.

We proceed to show the constancy. We first prove that for any geometric point s lying over the generic point η, the restriction map $\tau : \mathrm{End}_S(A) \to \mathrm{End}_s(A_s)$ is an isomorphism.

• surjectivity: Consider the commutative diagram

$$\begin{tikzcd} \mathrm{End}_S(A) \arrow[r] \arrow[d] & \mathrm{End}_s(A_s) \arrow[d] \\ \mathrm{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A)) \arrow[r] & \mathrm{End}_{\hat{\mathbb{Z}}}(\mathbb{T}(A_s)) \end{tikzcd}$$

The vertical arrows are inclusions: an endomorphism of abelian scheme is determined by its restriction on the torsion sections. The horizontal arrow on the bottom is an isomorphism, due to the constancy we have seen in 5.3.3 (3). Then by diagram chasing, the theorem of A.Grothendieck, as is quoted in 5.3.7 (1), affirms the surjectivity of the horizontal arrow on the top, which is exactly the τ we've mentioned.

• injectivity: Suppose $a \in \mathrm{Ker}(\tau)$. Then $B := \mathrm{Im}(a)$ is a closed S-subgroup of A whose generic fiber is trivial. Its neutral component B° is a connected S-subgroup of A. By the same arguments we've applied in 5.3.3 (2), we find that B° is an abelian S-subscheme of A, whose generic fiber is trivial. Apply 5.3.7 (2) to $\eta \hookrightarrow S$ with S normal, we see that B° is trivial itself. Thus the fibers of B are of finite type over the residue field with trivial neutral component, hence is finite and of torsion along each fibers. Consequently B is a torsion S-subgroup of A which is generically trivial. From the bijectivity of $A[N] \to A_s[N]$ for all $N \in \mathbb{N}_{>0}$ we see that B is trivial itself. Now that $B = \mathrm{Im}(a)$, we conclude that $a = 0$, hence the injectivity of $\tau : \mathrm{End}_S(A) \to \mathrm{End}_s(A_s)$

For any connected étale S-scheme $U \to S$, we take s to be some geometric point lying over the generic point of U (and also over η). Since $S = \hat{S}$ (with respect to the abelian S-scheme $A \to S$), it is easy to see that $U = \hat{U}$ with respect to the abelian U-scheme $A_U \to U$, and thus the arguments above implies the bijectivity of the canonical map $\mathrm{End}_U(A_U) \to \mathrm{End}_s(A_s)$, and by functoriality the transition map $\mathrm{End}_S(A) \to \mathrm{End}_U(A_U)$ is an isomorphism for any U. This leads to the constancy of $\mathbf{End}_S(A)$. ■

The lemma will allow us to realize an abelian S-subscheme of $A \to S$ as the kernel of some endomorphism $\Psi \in \mathrm{End}_S(A)$. Recall the following

Theorem 5.3.10. *(Poincaré splitting theorem, cf.[RU] Prop.2.1) Let k be a field, and X an abelian variety over k. For any abelian subvariety $Y \subset X$, there exists an abelian subvariety $Z \subset X$ such that the product map $Y \times Z \to X$ is an isogeny.*

Let N be the degree of the isogeny $Y \times Z \to X$, then $[N] : X \to X$ factors through $Y \times Z$, which is seen in the composition below

$$X \xrightarrow{(p_Y,p_Z)} Y \times Z \xrightarrow{(i_Y,i_Z)} X.$$

Then $Y = \mathrm{Ker}\,\psi$ for $\psi = i_Z \circ p_Z \in \mathrm{End}_S(X)$.

We thus give an alternative proof of 5.3.3 (2), i.e. the bijectivity of $\mathbf{sp}_\eta : \mathscr{F}(A) \to \mathscr{F}(A_\eta)$, under the assumption that $S = \hat{S}$ is geometrically integral and normal:

5.3.7 (2) has implied that $\mathbf{sp}_\eta : \mathscr{F}(A) \to \mathscr{F}(A_\eta)$ is injective: an S-subgroup of A is trivial as long as it is generically trivial. It remains to show the surjectivity: for any $B \in \mathscr{F}(A_\eta)$, find some $A' \in \mathscr{F}(A)$ such that $A'_\eta = B$. According to 5.2.10, $B' = \mathrm{Ker}\,\psi$ for some $\psi \in \mathrm{End}_\eta(A_\eta)$. Inverting the isomorphic transition map we have $\mathrm{End}_\eta(A_\eta) \cong \mathrm{End}_S(A)$ which sends ψ to Ψ. Then $\Psi_\eta = \psi$, and by putting $A' = \mathrm{Ker}\,\Psi$ we get an abelian S-subscheme A' of A such that $A'_\eta = B$. Therefore $\mathbf{sp}_\eta$ is surjective.

Remark 5.3.11. From the abelian S-scheme $A \to S$ we can also consider the functor $\mathscr{F}_A : \mathbf{Sch}_{/S}^{op} \to \mathbf{Ens}$, which associates to each S-scheme T the set of abelian T-subschemes of A_T. This functor is actually representable. For the proof we refer the interested readers to the criterion of representability presented in [Mu].

5.4 Generalized Manin-Mumford conjecture: the non-uniform case

Up to now we have considered the "uniform" Manin-Mumford conjecture over a general base (of characteristic zero). It is also reasonable to raise the following "non-uniform" Manin-Mumford conjecture:

Question 5.4.1. *Let $f : A \to S$ be an abelian S-scheme, $(S_n)_n$ a sequence of closed subschemes of S, and T_n a special S_n-subscheme of the abelian S_n-scheme $A_n = S_n \times_S A$. Assume that $\bigcup_n S_n$ is Zariski dense in S. Then under what condition on $(S_n)_n$ and T_n can one conclude that the Zariski closure of $\bigcup_n T_n$ in A is a (weakly) special S-subscheme?*

The André-Oort conjecture for abelian schemes given by mixed Shimura varieties can be seen as a particular case of the above question: for $\pi : M = M_K(\mathbf{P},Y) \to S = M_{K_G}(\mathbf{G},X)$ with $(\mathbf{G},X)$ a pure Shimura datum, $(\mathbf{P} = \mathbf{V} \rtimes \mathbf{G}, Y = \mathbf{V}(\mathbb{R}) \rtimes X)$ a mixed one where $\mathbf{G}$ acts on the vector space $\mathbf{V}$ such that any $x \in X$ induces on $\mathbf{V}$ a rational Hodge structure of type $\{(-1,0),(0,-1)\}$, $K_{\mathbf{G}}$ resp. $K_{\mathbf{V}}$ is a compact open subgroup of $\mathbf{G}(\mathbb{A}^f)$ resp. of $\mathbf{V}(\mathbb{A}^f)$, and $K = K_{\mathbf{V}} \rtimes K_{\mathbf{G}}$. In this case $\pi : M \to S$ is an abelian S-scheme. A special subvariety of M is in general a connected component M' of a mixed Shimura subvariety given by some datum of the form $(\mathbf{V}' \rtimes v\mathbf{G}'v^{-1}, (\mathbf{V}'(\mathbb{R}) + v) \rtimes X')$, whose image under π is a connected component S' of the pure subvariety of the datum $(\mathbf{G}',X') \subset (\mathbf{G},X)$. Then M' is a special S'-subscheme in $S' \times_S M$. Assume that we have a sequence $(S_n)_n$ of pure special subvarieties of S with $\bigcup_n S_n$ Zariski dense in S, and $T_n \subset S_n \times_S M$ a sequence of special S_n-subschemes. If the André-Oort conjecture is true, then the Zariski closure of $\bigcup_n T_n$ is weakly special, say a finite union $\bigcup_i M_i$. If $\pi(M_i) = S$ for all i, then $\bigcup_i M_i$ is a weakly special S-subscheme.

Definition 5.4.2. Let $f : A \to S$ be an abelian S-scheme. A quasi-special subscheme of A is defined to be a finite union $\bigcup_i T_i$ where T_i is a special S_i-subscheme of the abelian S_i-scheme $S_i \times_S A$ for some subscheme SI $\subset$ S.

And the question becomes: for $(T_n)_n$ a sequence of quasi-special subschemes of A, does the Zariski closure of $\bigcup_n T_n$ remains quasi-special?

Proposition 5.4.3. *Let $f : A \to S$ be an abelian S-scheme of relative dimension g, where S is geometrically integral of generic point η. If $T \subset A$ is a closed S-subscheme, flat over S, with image $f(T)$ a normal subscheme of S, such that for some $2 \leq N \in \mathbb{N}$ we have $[N]T \subset T$, then T is quasi-special.*

Proof. $f : A \to S$ is closed, thus $f(T) = S' \subset S$ is closed, and it suffices to show that T is quasi-special in $S' \times_S A$. Thus we may assume that $f(T) = S$. In this case S is geometrically integral and normal, and every abelian variety over η extends to an abelian S-scheme.

It suffices to treat each irreducible component of T, and we assume for simplicity that T is irreducible itself.

As usual, we put $\pi_1(S,\bar{x})$ to be the fundamental group with respect to $\bar{x} = \bar{\eta}$, and $\hat{S}$ is the Galois étale covering of S corresponding to the kernel of the monodromy representation $\pi_1(S,\bar{x}) \to GL_{\hat{\mathbb{Z}}_S}(\mathbb{T}(A))$.

Note that $[N] : A \to A$ is finite hence closed, and $[N]T$ remains a closed subscheme of A. As we have assumed T to be irreducible, $[N]T \subset T$ implies $[N]T = T$ because of the finiteness of $[N]$.

(1) The case where $S = \hat{S}$:

T is irreducible and faithfully flat over S. Take generic fiber we get $T_\eta = \mathbf{sp}_\eta(T) \subset A_\eta$. Clearly T_η remains irreducible, and $T_\eta = [N]T_\eta$. Apply [RU] Lemme 3.2 to T_η we see that T_η is weakly special.

Because T_η is already irreducible, we conclude that it is special, because of the assumption $S = \hat{S}$.

Let $B \subset A$ be the special S-subscheme corresponding to T_η under $\mathbf{sp}_\eta$. Clearly $T \subset B$. Taking Zariski closure of the equality $T_\eta = B_\eta$ we get $T = B$ because T is assumed to be closed in A.

(2) The general case:

We have the cartesian diagram

$$\begin{tikzcd} \hat{A} \arrow[r, "\hat{f}"] \arrow[d, "pr"'] & \hat{S} \arrow[d, "pr"] \\ A \arrow[r, "f"] & S \end{tikzcd} .$$

To show that T is special, it suffices to show that $\hat{T} := pr^{-1}(T)$ is weakly special in $\hat{A}$. Now that $\hat{T}$ is faithfully flat over $\hat{S}$ and that $\hat{T} = [N]\hat{T}$ as it is with T, we conclude from (1) that $\hat{T}$ is weakly special, which ends the proof. ■

Pure Shimura varieties are always normal. Therefore, by the proposition above, the André-Oort conjecture for abelian schemes $M \xrightarrow{\pi} S$ defined by mixed Shimura varieties, viewed as a special case of the general question 5.2.9, can be established if we can prove that the closure in question is stable under the homothety by some integer $N > 1$.

(To check that a pure Shimura variety is always normal, it suffices to verify that a pure Shimura variety with torsion free level is normal, and then take quotient by a finite group we get the general case.)

5.4.4 Products and fibrations: further remarks

We mention, quite informally, a few problems that arise in the study of the André-Oort-Pink conjecture.

(1) The product structure remains one of the main difficulties in understanding the André-Oort conjecture: suppose that the André-Oort conjecture is proved for two pure Shimura varieties S_1 and S_2, what can one say about $S = S_1 \times S_2$? It is possible that S contains special subvarieties that are not of the form $S'_1 \times S'_2$ for special subvarieties $S'_i \subset S_i$, $i = 1, 2$. Moreover, the Galois orbit of a special point $s = (s_1, s_2)$ in S given by special points $s_i \in S_i$, $i = 1, 2$, does not behave like the product of the two Galois orbits of s_1 and s_2 respectively: rather it looks like a diagonal part of the product of $\mathrm{Gal}_E\, s_1 \times \mathrm{Gal}_E\, s_2$. The consequent estimation of the intersection degree produces weaker results than waht we had expected.

(2) Another difficulty arises in the fibration of a mixed Shimura variety M over a pure section S. Say we are in the case of an abelian scheme defined by $\pi : M \to S$. Even if the André-Oort conjecture is known for S, modulo the GRH, with the Manin-Mumford conjecture proved for all the fibers of π, the arguments of Klingler-Ullmo-Yafaev does not immediately generalize to the mixed case. The proof in [KY] via estimation of the degrees of Hecke correspondences

can be carried over to each special sections $M(w)$ of $M \to S$, but they are different from the restriction of a globally defined Hecke correspondence. This is the main reason why we proposed our question for quasi-special subschemes of an abelian S-subscheme: we want to emphasize that, for the mixed Shimura varieties $M \to S$ as above, to show speciality of a subvariety $T \subset M$ can be reduced to show that T is stable under some central homothety $[N]$, which reduces the problem to the uniform Manin-Mumford conjecture.

(3) However there also remains a great deal to be refined in our formulation of the general question 5.2.9. We can easily raise counter-examples as follows. Consider the product of two abelian varieties $B = A \times A$ over $\mathbb{C}$, then the second projection $\mathrm{pr} : B \to A$ makes B an abelian A-scheme. Meantime $B \to \mathrm{Spec}\,\mathbb{C}$ is also an abelian variety. Consider the diagonal embedding $A \hookrightarrow B = A \times A$ and write D for the image. Then D is an abelian subvariety of B (as an abelian $\mathbb{C}$-scheme), and equals the Zariski closure of the torsion points $(t_n)_n$ in it. Note that each t_n is quasi-special in B the abelian A-scheme, but D is in no way quasi-special in B, when B is viewed as an abelian A-scheme

To exclude the above phenomenon, it seems necessary to impose more conditions on the subschemes $S_n \subset S$ in the formulation of our question. Say $f : A \to S$ is principally polarized of relative dimension $g > 0$ with level-m structure for some integer $m > 6$. Then f corresponds to a morphism $S \to \mathcal{A}_{g,m}$, and we have a cartesian diagram

$$\begin{tikzcd} A \arrow[r, "h'"] \arrow[d, "f"'] & A_{g,m} \arrow[d, "\mathrm{pr}"] \\ S \arrow[r, "h"] & \mathcal{A}_{g,m} \end{tikzcd}$$

where pr is the universal abelian scheme over $\mathcal{A}_{g,m}$ corresponding to the identity map of $\mathcal{A}_{g,m}$. Let $T_n \subset S_n \times_S A$ be special S_n-subschemes as in Question 5.2.9. Assume further that $\bigcup_n h(S_n)$ is Zariski dense in $h(S)$, it remains open what conditions we should impose on T_n so as to produce a pulled-back version of the André-Oort conjecture of $A_{g,m} \to \mathcal{A}_{g,m}$.

Bibliography

[André-1] Y.ANDRÉ, *G-functions and geometry*, Aspects of Mathematics Vol.13. Vieweg 1989

[André-2] Y.ANDRÉ, *Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part*, Compositio Mathematica, 1992, Vol.82(1), pp.1-24

[André-3] Y.ANDRÉ, *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non-modulaire*, Journal für die reine und angewandte Mathematik, 1998, Vol.505, pp.203-208

[André-4] Y.ANDRÉ, *Shimura varieties, subvarieties, and CM points* Six lectures at the Franco-Taiwan Arithmetic Festival, 2001. c.f. http://www.math.umd.edu/ yu/notes.shtml

[B-HC] A.BOREL AND HARISH-CHANDRA, *Arithmetic subgroups of algebraic groups*, Bulletin of the American mathematics society, 1961, Vol.67(6), pp.579-583

[Bo-1] M.V.BOROVOI, *The Langlands conjecture on the conjugation of Shimura varieties*, Functional Analysis and Applications, 1982, Vol.16, pp.292-294

[Bo-2] M.V.BOROVOI, *Conjugation of Shimura varieties*, Proceedings of the International Congress of Mathematicians, Berkeley 1986, Part.I, pp.783-790

[Ch-1] K.CHEN, *On the equidistribution of special subvarieties in a mixed Shimura variety*, preprint

[Ch-2] K.CHEN, *Equidistribution and the Manin-Mumford conjecture*, draft

[CU-1] L.CLOZEL AND E.ULLMO, *Correspondances modulaires et mesures invariants*, Journal für die reine und angewandte Mathematik, 2003, Vol.558, pp47-83

[CU-2] L.CLOZEL AND E.ULLMO, *Équidistribution des points de Hecke*, in *Contribution to automorphic forms, geometry, and number theory*, pp. 193-254, John Hopkins University Press, 2004

[CU-3] L.CLOZEL AND E.ULLMO, *Équidistribution de sous-variétés spéciales*, Annals of Mathematics, 161(3), pp1571-1588, 2005

[CU-4] L.CLOZEL AND E.ULLMO, *Rigidité de l'action des opérateurs de Hecke sur les espaces de réseaux*, Mathematischen Annalen, 2003, Vol.326, No.2, pp.209-236

[CU-5] L.CLOZEL AND E.ULLMO, *Équidistribution adélique des tores et équidistribution des points CM*, Documenta Mathematica, 2006, Volume in honor of J.Coates, pp.233-260

[DM] S.DANI AND G.MARGULIS, *Asymptotic behaviors of trajectories of unipotent flows on homogeneous spaces*, Proceedings of Indian academy of sciences (mathematical sciences), 1991, Vol.101(1), pp.1-17

[D-1] P.DELIGNE, *Travaux de Shimura* Séminaire bourbaki 1971, Exposé 389, Lecture Notes in Mathematics, Vol.255, pp.123-165 Springer Verlag 1972

[D-2] P.DELIGNE, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques* in *Automorphic forms, representations, and L-functions* edited by A.Borel and W.Casselman, Proceedings of Symposia in Pure Mathematics, Vol.33, Part 2, pp.247-289. American Mathematics Society 1979

[Duke-1] W.DUKE, *Hyperbolic distribution problems and half-integral weight Maass forms*, Inventiones Mathematica, 1988, Vol.92, no.2, pp.385-401

[Duke-2] W.DUKE, *Modular functions and the uniform distributions of CM points*, Mathematische Annalen, 2006, Vol.334, pp. 241-252

[E-1] B.EDIXHOVEN, *Special points on the product of two modular curves*, Compositio Mathematica, 1998, Vol.114(3), pp.315-328

[E-2] B.EDIXHOVEN, *On the André-Oort conjecture for Hilbert modular surfaces*, in *Moduli of abelian varieties (Texel Island 1999)* edited by G.van der Geer, C.Faber, F.Oort, Progress in Mathematics Vol.199, pp.133-155, Birkhäuser 2001

[E-3] B.EDIXHOVEN, *Special points on products of modular curves* Duke Mathematics Journal, 1998, Vol.126(2), pp.621-645

[EY] B.EDIXHOVEN AND A.YAFAEV, *Subvarieties of Shimura varieties*, Annals of Mathematics, 2003, Vol.157(2), pp.621-645

[EGA IV_3] J.DIEUDONNÉ AND A.GROTHENDIECK, *Éléments de géométrie algébrique: IV. études locales des schémas et des morphismes des schémas, troisième partie*, Publications mathématiques de l'I.H.E.S, 1966, tome 28, pp.5-255

[E-M-S] A.ESKIN, S.MOZES, N.SHAH, *Non-divergence of translates of certain algebraic measures*, Geometric analysis and functional analysis, 1990, Vol.7, pp.48-80

[Hind] M.HINDRY, *Autour d'une conjecture de Serge Lang*, Inventiones Mathematica, 1988, vol.94, pp.575-603

[G] A.GROTHENDIECK, *Un théorème sur les homomorphismes de schémas abéliens*, Inventiones Mathematica, 1966, vol.2, pp.59-78

[GIT] J.FOGARTY AND D.MUMFORD, *Geometric invariant theory*, Springer 1982

[JLZ] D.JIANG, J.LI, AND S.ZHANG, *Periods and distributions of cycles on Hilbert modular varieties*, Pure and applied mathematics quaterly, 2006, Vol.2, No.1, pp.219-277

[Ku] M.KUGA, *Fiber varieties over a symmetric space whose fibers are abelian varieties*, Vol.I and Vol.II, The University of Chicago, 1963-1964

[KY] B.KLINGLER AND A.YAFAEV, *On the André-Oort conjecture* preprint 2008

[LNM900] P.DELIGNE, J.MILNE, A.OGUS, AND K.SHIH, *Hodge cycles, motives, and Shimura varieties*, Lecture notes in mathematics Vol.900, 1982, Springer Verlag

[Milne-0] J.MILNE, *Canonical models of Shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties, and L-functions Vol.I, edited by L.Clozel and J.Milne, Academeic Press 1990

[Milne-1] J.MILNE, *Introduction to Shimura varieties*, Harmonic analysis, Trace formula, and Shimura varieties, Clay Institute and American Mathematical Society

[Milne-2] J.MILNE, *Algebraic groups and arithmetic groups*, course notes, cf. http://www.jmilne.org/math

[Mo] B.MOONEN, *Models of Shimura varieties in mixed characteristics* in *Galois representations in arithmetic algebraic geometry (Durham 1996)* edited by A.J.Scholl and R.Taylor, London Mathematics Society Lecture Notes Series Vol.254, pp267-350. Cambridge University Press, 1998

[MS] S.MOZES AND N.SHAH, *On the space of ergodic invariant measures of unipotent flows*, Ergodic theory and dynamic systems, 1995, Vol.15, pp.149-159

[Mum] D.MUMFORD, *Abelian varieties*, Tata Institute,

[Mu] J-P.MURRE, *Representation of unramified functors. applications (according to unpublished results of A.Grothendieck)*, Séminaire Bourbaki, Exposé 294, 1964-1965

[Nori] M.NORI, *On subgroups of* $\mathrm{GL}_n(\mathbf{F}_p)$, Inventiones Mathematica, 1987, Vol.88(2), pp.257-275

[Noot] R.NOOT, *Correspondances de Hecke, actions de Galois, et la conjecture d'André-Oort*, Séminaire Bourbaki, Exposé 942, 2004-2005

[Oort-1] F.OORT, *Some questions in algebraic geometry*, 1995. c.f. http://www.math.uu.nl/people/oort

[Oort-2] F.OORT, *Canonical liftings and dense sets of CM-points*, in *Arithmetic geometry (Cortona 1994)*, Symposia Mathematica, Vol.37, pp.228-234, Cambridge University Press 1997

[Pink-0] R.PINK, *Arithmetical compactifications of mixed shimura varieties*, Ph.D Thesis,

[Pink-1] R.PINK, *A combination of the conjecture of Mordell-Lang and André-Oort* in *Geometric methods in algebra and number theory*, Progress in Mathematics, Vol.235, pp.251-282. Birkhäuser, 2005

[PR] V.PLATONOV AND A.RAPINCHUK, *Algebraic groups and number theory*, Pure and applied mathematics, Academic Press, 1990

[Ragh] M.S.RAUGHUNATHAN, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematic und ihrer Grenzgebiete, Band 68, Springer Verlag 1972

[Rat-1] M.RATNER, *On Raghunathan's measure conjecture*, Annals of mathematics, 1991, Vol.194, pp.545-607

[Rat-2] M.RATNER, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke mathematics journal, 1991, Vol.63, pp.235-280

[R-1] M.RAYNAUD, *Courbes sur une variété abélienne et points de torsion*, Inventiones Mathematica, 1983, Vol.71(1), pp.207-233

[R-2] M.RAYNAUD, *Sous-variétés dúne variété abélienne et points de torsion*, in *Arithmetic and Geometry, Vol.I*, edited by M.Artin and J.Tate, Progress in Mathematics, Vol.35, pp.327-352, Birkhäuser 1983

[RT] D.ROCKMORE AND K.TAN, *A note on the order of finite subgroups of* $\mathbf{GL}_n(\mathbb{Z})$, Archiv der Mathematik, 1995, vol.64(4), pp.283-288

[RU] N.RATAZZI AND E.ULLMO, *Galois + Équidistribution = Manin-Mumford*, notes of lectures at Göttingen Summer School of Arithmetic Geometry, 2006, cf. http://www.math.u-psud.fr/ ullmo

[S] N.SHAH, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Mathematischen Annalen, Vol.289(1991), pp.315-334

[Sh] G.SHIMURA, *Introduction to the arithmetic theory of automorphic functions*, Kanô Memorial Lectures I, Princeton University Press, 1971

[SGA 4] M.ARTIN, A.GROTHENDIECK, J-L.VERDIER, *Séminaire de géométrie algébrique du Bois-Marie: Théorie des topos et cohomologie étale des schémas*, Tome 3, Lecture notes in mathematics Vol.305, 1970, Springer Verlag

[Spr] T.A.SPRINGER, *Linear algebraic groups*, Second edition, Progress in Mathematics, Vol.9, Birkhäuser

[SUZ] L.SZPIRO, E.ULLMO, AND S.ZHANG, *Equidistribution des petits points*, Inventiones Mathematica, 1997, Vol.127, pp.337-347

[U-1] E.ULLMO, *Théorie ergodique et géométrie arithmétique* in *Proceedings of the International Congress of Mathematicians (Beijing 2002)* Vol.II, pp.197-206, Higher Education Press 2002.

[U-2] E.ULLMO, *Points rationnels des variétés de Shimura*, Internatianl Mathematical Research Notices, 2004, Vol.76, pp.4109-4125

[U-3] E.ULLMO, *Équidistribution de sous-variétés spéciales II*, Journal für die reine und angewandte Mathematik, 2007, Vol.606, pp.193-216

[U-3] E.ULLMO, *Manin-Mumford, André-Oort: the equidistribution point of view*, notes of a course at the Summer Scool "Equidistribution in Number theory"

[U-4] E.ULLMO, *Autour de la conjecture d'André-Oort*, notes of a course at Paris-Sud University, 2007.

[UY-1] E.ULLMO AND A.YAFAEV, *Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture*, preprint 2006, with an appendix by P.Gille and L.Morel-Bailly entitled *Actions algébriques des groupes arithmétiques*, cf. http://www.math.u-psud.fr/ ullmo

[UY-2] E.ULLMO AND A.YAFAEV, *The André-Oort conjecture for products of modular curves*, notes of lectures at Göttingen Summer School of Arithmetic Geometry, 2006, cf. http://www.math.u-psud.fr/ ullmo

[UY-3] E.ULLMO AND A.YAFAEV, *Points rationnels des variétés de Shimura: un principe du tout ou rien*, preprint 2007, cf. http://www.math.u-psud.fr/ ullmo

[Y-1] A.YAFAEV, *Special points on products of two Shimura curves*, Manuscripta Mathematica, 2001, Vol.104(2), pp/163-171

[Y-2] A.YAFAEV, *On a result of Moonen on the moduli space of principally polarized abelian varieties*, Compositio Mathematica, 2005, Vol.141(5), pp.1103-1108

[Y-3] A.YAFAEV, *A conjecture of Yves André*, Duke Mathematics Journal, 2006, Vol.132(3), pp.393-407

[Zh] S.ZHANG, *Equidistribution of small points on abelian varieties*, Annals of Mathematics, 1998, Vol.147(1), 159-165

[Zim] R.ZIMMER, *Ergodic theory and semisimple groups*, Monographs in Mathematics, Vol.81, Birkhäuser 1984